Hoja 5

Problem 1. a) Let A, B be real square matrices of size 2013 such that $AB = 0$. Prove that at least one of the matrices $A + A^T$, $B + B^T$ is singular.

b) Is this assertion true for complex matrices?

Problem 2. Consider $P(x) = (x - a_1)(x - a_2)\cdots(x - a_n)$, where $a_1 < a_2 < \cdots < a_n$ are integers. Prove that $P(x)^2 + 1$ is irreducible in $\mathbb{Z}[x]$ (i.e., prove that if $Q(x)$ and $R(x)$ are polynomials with integer coefficients such that $P(x)^2 + 1 = Q(x)R(x)$, then either $Q \equiv \pm 1$ or $R \equiv \pm 1$.

Problem 3. Let a, b, c be positive integers such that a divides b^3, b divides c^3, and c divides a^3. Prove that abc divides $(a + b + c)^{13}$.

Problem 4. Let f and g be (real-valued) functions defined on an open interval containing 0, with g nonzero and continuous at 0. If fg and f/g are differentiable at 0, must f be differentiable at 0?

Problem 5. Prove that for $n \geq 2$,

$$\overbrace{2^{2^{\cdots^{2}}}}^{n \text{ terms}} \equiv \overbrace{2^{2^{\cdots^{2}}}}^{n-1 \text{ terms}} \pmod{n}.$$

Problem 6. Let D_n denote the value of the $(n - 1) \times (n - 1)$ determinant

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

Is the set $\left\{ \dfrac{D_n}{n!} \right\}_{n \geq 2}$ bounded?