Hoja 4

1) Let $M, N \in M_2(\mathbb{C})$ be two nonzero matrices such that

$$M^2 = N^2 = 0_2 \text{ and } MN + NM = I_2$$

where O_2 is the 2×2 zero matrix and I_2 the 2×2 unit matrix. Prove that there is an invertible matrix $A \in M_2(\mathbb{C})$ such that

$$M = A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A^{-1} \text{ and } N = A \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} A^{-1}$$

2) Let $P \in \mathbb{Z}[x]$ be a polynomial, which has a rational root r. Prove that if $|P(n)| = |P(m)| = 1$ for two distinct integers n and m, then $r = (n+m)/2$.

3) Does there exist a continuous function f on $(0,1)$ such that

$$0 < \lim_{t \to 0^+} \frac{f(x+t) - f(x)}{t^2} < \infty$$

for any $x \in (0,1)$?

4) Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0, 1, 2, \ldots$ by the condition

$$a_m(0) = d/2^m, \quad a_m(j+1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.$$

Evaluate $\lim_{n \to \infty} a_n(n)$.

5) Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in decimal representation is k.

6)

1. Prove that an 8×8 chessboard cannot be covered without overlapping by fifteen 1×4 polyminos and the single polymino shown in Figure 1.
2. Prove that a 10×10 board cannot be covered without overlapping by the polyminos shown in Figure 2.
3. Prove that a 102×102 board cannot be covered without overlapping by 1×4 polyminos.

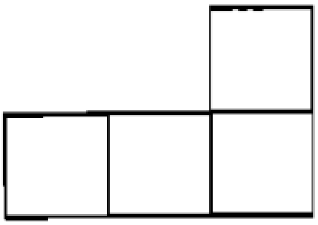

Figure 1

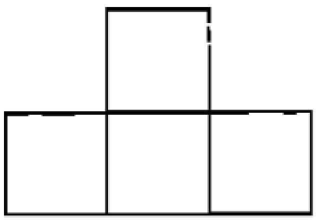

Figure 2