Hoja de Verano-2019

1) Suppose that a sequence $a_1, a_2, \cdots$ of positive real numbers satisfies

$$a_{k+1} \geq \frac{k a_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \cdots + a_n \geq n$ for every $n \geq 2$.

2) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms.

(a) If the series diverges, prove that there exists a sequence $\{c_n\}$ monotonically decreasing to 0 such that $\sum_{n=1}^{\infty} a_n c_n$ also diverges.

(b) If the series converges, prove that there exists a sequence $\{c_n\}$ monotonically increasing to infinity such that $\sum_{n=1}^{\infty} a_n c_n$ also converges.

3) For any square matrix A, we can define $\sin A$ by the usual power series:

$$\sin A = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} A^{2n+1}.$$

Prove or disprove: there exists a 2×2 matrix A with real entries such that

$$\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}.$$

4) Prove that $(5^{125} - 1)/(5^{25} - 1)$ is composite.

5) For every positive integer k, find the smallest n such that $2^k | 5^n - 1$.

6) There are infinitely many powers of 2 in the sequence $\lfloor n\sqrt{2} \rfloor$.

7) Prove that there exist infinitely many triples of integers x, y, z such that the numbers $x(x+1), y(y+1), z(z+1)$ form an increasing arithmetic progression.

8) Evaluate the integral

$$\int_0^1 \int_0^1 \frac{1}{1-xy}\, dx\, dy.$$

9) What is the maximal value of

$$|ax + ay + bx - by|$$

over the 4-dimensional "cube" $|a| \leq 1$, $|b| \leq 1$, $|x| \leq 1$, $|y| \leq 1$? Consider separately the cases of real and of complex parameters.