Hoja 3

1) Let $f : [0,1] \to \mathbb{R}$ a convex function such that $f(0) = 0$. Prove that,

$$\int_0^1 f(x)dx \geq 4 \int_0^{1/2} f(x)dx.$$

2) Determine the continuous functions $f : [1, \infty) \to \mathbb{R}$ such that:

$$\int_x^{x^n} f(t)dt = \int_1^x (t + t^2 + \cdots + t^{n-2} + t^{n-1}) f(t)dt$$

for all $x \in [1, \infty)$ and all $n \in \mathbb{N}$.

3) Prove that

$$(1 + 1/2)(1 + 1/4) \cdot \ldots \cdot (1 + 1/2^n) < 3,$$

for every natural number n.

4) Let $f : [0,1] \to [0,1]$ be a non-decreasing function. Is it true that there exists $x \in [0,1]$ such that $f(x) = x$? Is the same thing true if f is non-increasing?

5) The length of the sum of n unit vectors in the plane is greater or equal than 1. Prove that these vectors can be numbered as $v_1, v_2, \ldots, v_n$ so that

$$\|v_1 + v_2 + \cdots + v_k\| \geq 1, \qquad k = 1, \ldots, n,$$

where $\|v\|$ stands for the length of a vector v.

6) Given two positive rational numbers a, b such that $\sqrt[3]{a} + \sqrt[3]{b}$ is rational, prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are rational.