Hoja 2

1) Find all integer solutions of

$$x^2 + y^2 + z^2 = 7\,t^2$$

2) Find all integer solutions of

$$x^4 + y^4 = z^6$$

3) This problem is also motivated by the Last Fermat Theorem. Given an odd prime p, define a function $\nu_p : \mathbb{Z} \setminus \{0\} \longrightarrow \mathbb{Z}_0^+$ by

$$m \longmapsto \nu_p(m) = \alpha \quad \text{where} \quad p^\alpha \mid m \quad \text{and} \quad p^{\alpha+1} \nmid m.$$

(In other words, $\nu_p(m)$ equals to the largest exponent $\alpha \in \mathbb{Z}_0^+$ such that $p^\alpha \mid m$.)

Let $x, y \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ such that $p \mid x - y,\ p \nmid x,\ p \nmid y$. Prove that

$$\nu_p(x^n - y^n) = \nu_p(x - y) + \nu_p(n)$$

4) The sequence $\{x_n\}$ of positive real numbers decreases and satisfies $\sum_{n=1}^{\infty} x_n = \infty$. Prove that

$$\sum_{n=1}^{\infty} x_n \exp(-\frac{x_n}{x_{n+1}}) = \infty.$$