Hoja 1 (completa)

1) Let n be a positive integer and $a_1, \ldots, a_{n+1}$ be $n+1$ distinct numbers among $\{1, 2, \cdots, 2n\}$. Show that one of them divides another one.

Show that if only n numbers $a_1, \ldots, a_n$ are selected, there is be a configuration with no coprime elements and another configuration with no element dividing another.

2) Let n be odd and $A_1 A_2 \ldots A_n$ be a regular n-gon whose circumradius equals 1. Find the value of

$$\prod_{k=1}^{n} \overline{B\,A_k}$$

where B is the diametrically opposite point of $A_{\frac{n+1}{2}}$ and $\overline{B\,A_j}$ denotes the length of the segment with endpoints B, A_j.

3) Can you prove that $\dfrac{\arctan(\frac{4}{3})}{\pi}$ is irrational? Whether you can or not, show that this number is irrational if and only if $\dfrac{\arctan(7)}{\pi}$ is irrational.

4) Let $a < b < c < d$, and let $p(x) = (x-a)(x-b)(x-c)(x-d)$. Prove that

$$\int_a^b \frac{dx}{\sqrt{|p(x)|}} = \int_c^d \frac{dx}{\sqrt{|p(x)|}}\,. \qquad (*)$$

5) Find all integers n such that $n-50$ and $n+50$ are both perfect squares.

6) Show that the quadratic forms $4x^2 + 6xy + 3y^2$ and $4x^2 + 2xy + y^2$ achieve the same values that $x^2 + 3y^2$ for integer values of x and y.

7) Show that for infinitely many natural values of n, there are integers x and y such that $x^2 + 12y^2 = n$ and there are no integers x and y such that $x^2 + 9y^2 = n$. What if we demand that x and y have to be coprime?