Hoja 3

1) At a party assume that no boy dances with all the girls, but each girl dances with at least one boy. Prove that there are two girl-boy couples (g, b) and (g', b') who dance, whereas b does not dance with g', and g does not dance with b'

2) Find all triples of real numbers (x, y, z) that are solutions to the system of equations

$$\frac{4x^2}{4x^2+1} = y, \quad \frac{4y^2}{4y^2+1} = z, \quad \frac{4z^2}{4z^2+1} = x.$$

3) Let $A_1 = 0$ and $A_2 = 1$. For $n > 2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3 = A_2A_1 = 10$, $A_4 = A_3A_2 = 101$, $A_5 = A_4A_3 = 10110$, and so forth. Determine all n such that 11 divides A_n.

4) Let f, g be nonconstant functions on an interval (a, b). It is known that $f + g \neq 0$ on (a, b) and that

$$f'(x)g(x) - f(x)g'(x) = 0. \qquad x \in (a, b).$$

Prove that g does not vanish on (a, b) and that $\dfrac{f}{g}$ is constant on (a, b).

5) The sequence $\{x_n\}$ is defined by $x_1 = 1/2$ and $x_{n+1} = x_n - x_n^2$. Prove that $\lim_{n\to\infty} nx_n = 1$.

6) Let $f : [0, 1] \to [0, 1]$ be a non-decreasing function. Is it true that there exists $x \in [0, 1]$ such that $f(x) = x$? Is the same thing true if f is non-increasing?