Hoja 1

1) Assume an 8×8 chessboard with the usual coloring. You may change the color of all squares in a row or column. Can you achieve a configuration with only one black square?

2) Five points with integer coordinates are chosen in the plane. Prove that you can always choose two of these points such that the segment joining them passes through another point with integer coordinates.

3) Which number is larger,

$$\prod_{n=1}^{25}\left(1-\frac{n}{365}\right) \quad \text{or} \quad \frac{1}{2}\,?$$

Do not use a calculator (except for "easy" computations).

4) Does there exist a continuous function f on $\mathbb{R}$ such that $f(f(x)) = e^{-x}$, $x \in \mathbb{R}$?

5) Consider the sequence $a_1 = 2$, $a_{k+1} = a_k^2 - a_k + 1$. Show that

$$\sum_{k=1}^{\infty} \frac{1}{a_k} = 1.$$

6) Characterize the positive integers having an odd number of divisors.

7) Let ABC be any triangle in the plane. Choose the (unique) point A_1 inside the triangle such that one has the following relations for angles:

$$\angle A_1CB = \frac{1}{3}\angle ACB, \quad \angle A_1BC = \frac{1}{3}\angle ABC.$$

Choose ponts B_1 and C_1 inside the triangle ABC in a symmetric way:

$$\angle B_1AC = \frac{1}{3}\angle BAC, \quad \angle B_1CA = \frac{1}{3}\angle BCA;$$
$$\angle C_1BA = \frac{1}{3}\angle CBA, \quad \angle C_1AB = \frac{1}{3}\angle CAB.$$

Prove that the points A_1, B_1 and C_1 are vertices of an equilateral triangle.