Hoja 9 completa

1) Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

2) Let B be a subset of the set of vertices of the n-dimensional cube $[-1, 1]^n$, where $n > 3$ (so that all points of B are of the form $(\pm 1, \pm 1, \dots, \pm 1)$). Suppose that B has more than $2^{n+1}/n$ points. Show that one can find three distinct points of B, which are the vertices of an equilateral triangle.

3) Let n be an even positive integer. The numbers $1, 2, \dots, n^2$ are written in the subdivisions of an $n \times n$ square table, so that the k-th row, from left to right, is

$$(k-1)n + 1, (k-1)n + 2, \dots, (k-1)n + n.$$

The subdivisions are colored so that half of them in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red subdivisions is equal to the sum of the numbers on the black subdivisions.

4) Give an example of a reducible polynomial $f(X) \in \mathbb{Z}[X]$ which for m different positive integer values of x would give m different primes.

5) Find all prime numbers p such that $p, p+2, p+6, p+8, p+12, p+14$ are primes.

6) Is it true that for any polynomial $f(X) \in \mathbb{Z}[X]$ of positive degree, the congruence $f(X) \equiv 0 \pmod{p}$ is solvable for infinitely many primes?

7) Suppose that a sequence $\{x_n\}$ of real numbers satisfies

$$\lim_{n\to\infty} x_n + x_{P(n)} = 0,$$

whenever P is a quadratic polynomial with positive integer coefficients. Does it follow that $\lim x_n = 0$?

8) Let u be a continuously differentiable function on $[0, \infty)$ such that $u > 0$, $u' > 0$ and

$$\int_0^\infty \frac{dx}{u(x) + u'(x)} < \infty.$$

Prove that $\displaystyle\int_0^\infty \frac{dx}{u(x)} < \infty.$

9) Let φ and ψ be decreasing functions on $(0, +\infty)$, that are inverse to each other, such that

$$\int_0^\infty \varphi(x)\, dx = \int_0^\infty \psi(x)\, dx = a.$$

Prove that

$$\int_0^\infty \varphi^2(x)\, dx + \int_0^\infty \psi^2(x)\, dx \geq \frac{1}{2}\, a^{3/2}.$$

10) The sequence $\{a_n\}$ satisfies $a_1 > 0$, $a_2 > 0$, and

$$a_{n+1} = \frac{2}{a_{n-1} + a_n}, \quad n \geq 2.$$

Show that $\{a_n\}$ has a limit.

11) Prove that there exist infinitely many pairs of integers m, n such that:

(1) m, n have the same prime divisors;

2) $m + 1, n + 1$ have the same prime divisors.