Hoja 8

1) Let A be a real orthogonal matrix without eigenvalue 1. Let B be obtained from A by replacing one of its rows or one of its columns by its negative. Show that B has 1 as an eigenvalue.

2) Let $n > 1$ be a positive integer, $c_1, c_2, \ldots, c_n$ non-zero real numbers and let $0 < a_1 < a_2 < \cdots < a_n$. Prove that the number of real roots of the equation

$$c_1 a_1^x + c_2 a_2^x + \ldots + c_n a_n^x = 0$$

is not larger than the number of negative elements of the sequence $\{c_1 c_2, c_2 c_3, \ldots, c_{n-1} c_n\}$.

3) Let $C(\alpha)$ be the coefficient of x^{2019} in the Taylor expansion of $(1+x)^\alpha$ at $x = 0$. Evaluate the sum

$$\sum_{i=1}^{2019} \int_0^1 \frac{C(-y-1)}{y+i}\, dy.$$

4) For a given positive integer m, find all triples of positive integers (n, x, y), with $\gcd(m, n) = 1$, that solve the equation:

$$(x^2 + y^2)^m = (xy)^n$$

5) Let $0 < x_1 < 1$. Define the sequence $\{x_n\}$ inductively by

$$x_{n+1} = x_n + \frac{x_n^2}{n^2}, \quad n \geq 1.$$

Prove that this sequence is bounded.