Hoja 6

1) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be positive real numbers. Prove the inequality

$$\sqrt[n]{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)} \geq \sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n}.$$

2) Find all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every rational number q, the number $f(q)$ is rational and has the same denominator as q. (The denominator of a rational number q is the unique positive integer b such that $q = a/b$ for some integer a with $\gcd(a, b) = 1$.) (Note: gcd means greatest common divisor.)

3) What is the smallest number of weighings on a balance scale needed to identify the individual weights of a set of objects known to weigh $1, 3, 3^2, \ldots, 3^{26}$ in some order? (The balance scale reports the weight of the objects in the left pan minus the weight of the objects in the right pan.)

4) Find all polynomials Q satisfying $Q(x^2 + 1) = Q(x)^2 + 1$.

5) **(a)** Solve in integers (positive or negative): $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6}$.

(b) Solve the diophantine equation $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{z}$ (write down a formula which gives all integer solutions).

6) A function $f : [0, \infty) \to \mathbb{R}$ is continuously differentiable.
(a) Suppose that $f(x) + f'(x) \to A$ as $x \to +\infty$. Show that $f(x) \to A$ as $x \to +\infty$.
(b) Now suppose that $f(x) - f'(x) \to A$ as $x \to +\infty$. Does it follow that $f(x) \to A$ as $x \to +\infty$?