Hoja 5

1) Given n points in the plane, suppose there is a unique line that minimises the sum of the distances from the points to the line. Prove that this line contains at least 2 of the given points.

2) Let $f, g : \mathbb{R} \to \mathbb{R}$ be periodic functions such that $\lim_{x\to\infty}(f(x) - g(x)) = 0$. Prove that f and g are identical functions.

3) Given $n \geq 1$ an odd number, $a_1, \ldots, a_n$ a permutation of $1, \ldots, n$, show that $(1 - a_1)\cdots(n - a_n)$ is even.

4) Let $n > 2$ be an integer, and $T : \mathbb{Q}^n \longrightarrow \mathbb{Q}^n$ be the linear application carrying any

$$(x_1, \ldots, x_n) \longmapsto \left(\frac{x_1 + x_2}{2}, \frac{x_2 + x_3}{2}, \cdots, \frac{x_n + x_1}{2}\right)$$

Then show that, if the initial terms are pairwise different integers, then some image of $(x_1, \ldots, x_n)$ under $T, T^2, T^3, \ldots$ will consist of non-integer entries.

5) Solve in $\mathbb{C}$ the equation $(x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x$

6) Find all the functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(xf(y)) = yf(x)$$

for every positive real numbers x, y, and also $f(x) \to 0$ when $x \to +\infty$.

7) Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\sqrt{a^{1-a}\, b^{1-b}\, c^{1-c}} \leq \frac{1}{3}.$$