Hoja 5

1) Prove that every nonnegative integer can be represented in the form $a^2 + b^2 - c^2$, where a, b and c are positive integers with $a < b < c$.

2) Prove that there does not exist a non-constant polynomial with integral coefficients such that $P(0), P(1), P(2), \ldots$ are all prime numbers.

3) Consider the polynomial $f(X) := X^2 - 4$, with $a \in \mathbb{Z}$. Let $f^{(n)} = f \circ \cdots \circ f$ be its nth iterate and let $c_n = f^{(n)}(0)$. Prove that if p is an odd prime such that $p \mid (c_n + 2)$ for some n, then $p \nmid (c_m - 2)$ for all m.

4) Prove that the equation $p^2 + q^2 = r^2 + s^2 + t^2$ has no solution with primes p, q, r, s, t.

5) Call a real-valued function f *very convex* if

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + |x - y|$$

holds for all real numbers x and y. Prove that no very convex function exists.

6) Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

7) Given a function $f : [0, 1] \to \mathbb{R}$, we associate with it the set

$$E(f) = \{x \in [0, 1] : \lim_{t \to x} f(t) = +\infty\}.$$

Is it true that for any f, $E(f)$ is (at most) countable?

8) . Let $a_1, a_2, \ldots, a_n$ be positive numbers and let s be their sum. Prove that

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \leq 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots + \frac{s^n}{n!}.$$

9) Show that for any natural number n, there exist infinitely many positive Fibonacci numbers divisible by n.