Hoja 3 (completa)

Problem 1. Find all positive functions $f : (0, +\infty) \to (0, +\infty)$ such that

$$f\big(f(x)\big) = 3x - f(x), \quad \forall x > 0.$$

Problem 2. Does there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is irrational for any rational x and $f(x)$ is rational for any irrational x?

Problem 3. Given an odd prime p, denote by $\mathbb{Z}/p\mathbb{Z}$ the set of residues modulo p (it is a field). How many elements has the set

$$\{x^2 : x \in \mathbb{Z}/p\mathbb{Z}\} \cap \{y^2 + 1 : y \in \mathbb{Z}/p\mathbb{Z}\}?$$

Problem 4. Suppose that functions f and g are defined on $\mathbb{R}$, are periodic (possibly with different periods) and satisfy

$$\lim_{x \to +\infty} f(x) - g(x) = 0.$$

Prove that $f(x) \equiv g(x)$.

Problem 5. Given any integers $n \geq m \geq 1$, prove that the number

$$\frac{\gcd(m, n)}{n} \binom{n}{m}$$

is an integer.

Problem 6. Let c be a positive constant. Describe all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = f(x^2 + c) \qquad \text{for all } x \in \mathbb{R}.$$

Problem 7. Can a 8×8 board be covered by fifteen 1×4 rectangles and one 2×2 square without overlapping?

Problem 8. Let p be a prime number. Define a sequence $\{a_n\}_{n \geq 0}$ by $a_0 = 0$, $a_1 = 1$ and $a_{k+2} = 2a_{k+1} - pa_k$. If one of the terms of the sequence is -1, then determine all possible value of p.