Hoja 2

1) Let Δ_{ABC} be a triangle in the plane. Find the point X, and show or refute its uniqueness, such that the sum of the squares of the distances from X to each vertex of the triangle is minimal. In other words, such that the following quantity is minimal:

$$\overline{XA}^2 + \overline{XB}^2 + \overline{XC}^2 .$$

2) Let $n \in \mathbb{Z}^+$ and f a function $f : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ such that $f(f(x)) = 1 - x,\ \forall x \in \mathbb{Z}_n$. Prove that $n \equiv 0, 1 \pmod 4$. *Note: $\mathbb{Z}_m$ denotes the ring $\mathbb{Z}/m\mathbb{Z}$ of integers module m.*

3) Let $A_0 A_1 ... A_{n-1}$ a regular n-gon whose circumradius equals 1.

- Compute the value of $\prod_{k=1}^{n-1} \overline{A_0 A_k}$.
- Find the maximum value of $\prod_{k=0}^{n} \overline{PA_k}$ as P ranges over the circumcircle.

4) Consider a non-empty real interval (a, b). Does there exist a function $f : (a, b) \to (0, \infty)$ with Riemann integral equal to 0?

5) Let $\{F_n\}_{n\geq 1}$ be the sequence of positive integers defined by the recurrence relation

$$F_1 := 1, \quad F_2 := 2, \quad F_{n+2} := F_n + F_{n+1} \quad (\forall n \geq 1).$$

Prove that every positive integer can be uniquely decomposed as a sum of distinct F_n's with no two consecutive terms appearing (that is, if F_n appears in the sum, then F_{n-1} and F_{n+1} cannot appear). For example,

$$10 = 8 + 2 = F_5 + F_2, \quad 20 = 13 + 5 + 2 = F_6 + F_4 + F_2.$$

6) Show that the series

$$\sum_{k=1}^{\infty} \frac{\sin k}{k}$$

converges.

7) The function f is continuous on the interval $[0, 1]$, is differentiable on $(0, 1)$ and satisfies the equalities $f(0) = f(1) = 0$. Prove that there exists some $x \in (0, 1)$ such that $f(x) = f'(x)$.

8) Suppose u and v are rational numbers such that the sum $\sqrt[3]{u} + \sqrt[3]{v} \neq 0$ is also rational. Prove that then the cubic roots $\sqrt[3]{u},\ \sqrt[3]{v}$ are both rational.

9) Let f be a positive continuously differentiable function on $[0, +\infty)$. Prove that

$$\int_0^{\infty} \frac{\sqrt{1 + (f')^2}}{f}\, dx = +\infty.$$