Hoja 2

Problem 1. Consider a doubly infinite table of positive integers:

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Suppose that each positive integer appear exactly 10 times in this table. Prove that for some m and n, $a_{m,n} > mn$.

Problem 2. Suppose that a given function $p(x)$ increases on $[0, c]$ and is differentiable on $(0, c]$, whereas $p'(x)$ decreases on $(0, c]$. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} p'\left(\frac{c}{n}\right) < +\infty.$$

Problem 3. A ball moves endlessly on a circular billiard table. When it hits the edge it is reflected. Show that if it passes through a point on the table three times, then it passes through it infinitely many times.

The same question for a billiard with an elliptic shape.

Problem 4. Determine all integers $n > 2$ with the property that there exists one of the numbers $1, 2, \ldots, n+1$ such that after its removal, the n numbers left can be arranged as $a_1, a_2, \ldots, a_n$ with no two of $|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{n-1} - a_n|, |a_n - a_1|$ being equal.

Problem 5. The sequence $\{x_n\}$ is defined by $x_1 = 1/2$ and $x_{n+1} = x_n - x_n^2$ for $n \geq 1$. Prove that

$$\lim_{n\to\infty} nx_n = 1.$$

Problem 6. One starts with three line segments of lengths $1, 2, 3$. Then the segment of length 3 is cut into $n \geq 2$ line segments. Prove that from these $n + 2$ segments, three segments can be chosen that can form a triangle.