Hoja 1

1) Compute the following limit:

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right).$$

2) Show that for every natural number n there are infinitely many Fibonacci numbers which are divisible by n.

3) A function $f:\mathbb{R}\to\mathbb{R}$ satisfies $|f(a)-f(b)|<|a-b|$ for any $a\neq b$. Prove that $f(f(f(0)))=0$ implies that $f(0)=0$.

4) Let $n,p\in\mathbb{Z}^+$ with p a prime. Count the number of $n\times n$ matrices whose entries belong to $\mathbb{Z}/p\mathbb{Z}$, with determinant equal to 1 (in other words, congruent to 1 modulo p).
Indication: Consider first $p=3$.

5) Does there exist a continuous function f on $\mathbb{R}$ such that $f(f(x))=e^{-x}$, $x\in\mathbb{R}$?

6) Let $x_1,\dots,x_n$ be positive numbers. Prove that

$$\big(\max_{1\le j\le n} x_j\big)(x_1+2x_2+\cdots+nx_n)\ge\frac{1}{2}(x_1+x_2+\cdots+x_n)^2.$$

7) Find all integer solutions of $x^3+y^3+z^3=(x+y+z)^3$

8) Let $a\in\mathbb{R}$ and n a positive integer. Show that at least one of $a,2a,\cdots,(n-1)a$ is at distance at most $\frac{1}{n}$ of an integer

9) Let α, β, be positive irrational numbers such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$. This is, they are said to be conjugate.
Define $(a_n)_{n\in\mathbb{Z}^+},(b_n)_{n\in\mathbb{Z}^+}$ such that

$$a_n\le n\alpha<a_n+1\qquad\qquad b_n\le n\beta<b_n+1$$

Show that the sequences (a_n) and (b_n) form a partition of $\mathbb{Z}^+$.

10) Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane?

11) Twenty-three people of positive integral weights decide to play football. They select one person as referee and then split up into two 11-person teams of equal total weights. It turns out that no matter who is referee this can always be done. Prove that all 23 people have equal weights.