Hoja 9

1) Prove that if α, β, γ are three different rational numbers, then the number

$$\frac{1}{(\alpha-\beta)^2}+\frac{1}{(\beta-\gamma)^2}+\frac{1}{(\gamma-\alpha)^2}$$

is of the form r^2, where r is a rational number.

2) *a)* Find all pairs of distinct natural numbers satisfying

$$x^y = y^x\,.$$

b) Find all solutions $x \neq y$ of the above equation, which are rational and positive (give a general formula for these solutions).

3) Is it possible to express a non-closed interval in the real line, such as $[0, 1)$, as a countable union of disjoint closed intervals?

4) Let K be the Cantor middle-third set. Let $K^* = K \times \{0\}$. Is there is a function F from $\mathbb{R}^2$ to $\mathbb{R}$ such that the following holds:

(i) for each $x \in \mathbb{R}$, the function $t \mapsto F(x, t)$ is continuous on $\mathbb{R}$,

(ii) for each $y \in \mathbb{R}$, the function $s \mapsto F(s, y)$ is continuous on $\mathbb{R}$, and

(iii) F is continuous on the complement of K^* and discontinuous on K^*?

5) Let $U = (u_{jk})$ be a 3×3 (real) orthogonal matrix with determinant 1. Prove that $u_{11} + u_{22} + u_{33} \geq -1$.

6) Let $\{a_n\}, \{b_n\}$ be decreasing sequences of positive real numbers, and put $c_n = \min(a_n, b_n)$.

(i) Can it happen that the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ diverge, but the series $\sum_{n=1}^{\infty} c_n$ converges?

(ii) What is the answer to this question if we take $b_n = 1/n$?