Hoja 8

1) If $ax^2 + by^2 = 1$, with nonzero a, b has a rational solution $(x_0, y_0) \in \mathbb{Q}^2$, then it has infinitely many rational solutions (and we can write down all of them explicitly).

2) Show that for every integer $n \geq 3$, the number 2^n admits a representation of the form $2^n = x^2 + 7y^2$, where x, y are odd integers.

3) Find the minimal value of $\left|5x^2 + 11xy - 5y^2\right|$, assuming that x, y run over entire numbers and are not equal to 0 simultaneously.

4) Given an integer $n \geq 2$, find the least constant C such that inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_i x_i \right)^4$$

is true for any real numbers $x_1, \ldots, x_n \geq 0$.

5) Prove that the set $\mathbb{N}$ of positive integers cannot be represented as a disjoint union of three nonempty subsets such that, for any x, y taken from two different subsets, the number $x^2 - xy + y^2$ will always belong to the third one.

6) Let μ be a fixed positive real number. Suppose that initially n fleas sit in some points on a horizontal line, not all at the same point. On each move, two fleas are chosen, at some points A and B, with A to the left of B, and the flea from A jumps to the point C to the right of B such that $BC/AB = \mu$.

Find all values of μ such that for any choice of a point M on the line and for any initial position of the n fleas, there exists a sequence of moves such that all the fleas finally get to positions to the right of M.