Hoja 7

1) Let n be a positive integer and let $a_1, \dots, a_k$ ($k \geq 2$) be distinct integers from the set $\{1, \dots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for $i = 1, \dots, k-1$. Prove that n does not divide $a_k(a_1 - 1)$.

2) Find all the functions $f : (0, \infty) \to (0, \infty)$ such that

$$f\left(xf\left(\frac{1}{y}\right)\right) = xf\left(\frac{1}{x+y}\right),$$

for all $x, y > 0$.

3) Let $x_1, x_2, \dots, x_n$ be positive numbers, and let

$$s = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$(1 + x_1)(1 + x_2)\dots(1 + x_n) \leq 1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots + \frac{s^n}{n!}.$$

4) Prove that for any positive integer n

$$\sqrt{2}\,\sqrt[4]{4}\,\sqrt[8]{8}\dots\sqrt[2^n]{2^n} \leq n + 1.$$

5) Find all positive integers n that satisfy the following condition:
For all integers a, b relatively prime to n,

$$a \equiv b \mod n \iff ab \equiv 1 \mod n.$$