Hoja 7

1) Let $n \geq 2$ be an integer. Show that

$$\sum \frac{1}{pq} = \frac{1}{2},$$

where the sum is taken over all pairs of integers $p, q,$ which are coprime and satisfy $0 < p < q \leq n$, $p + q > n$.

2) Let $\alpha \in \mathbb{R}$. Calculate the integral $\displaystyle\int_0^{\infty} \frac{dx}{(1+x^2)(1+x^{\alpha})}$.

3) Consider the set $A := \{|2^m - 3^n| \text{ s.t. } m, n \in \mathbb{N}\}$. Determine the smallest prime number p such that $p \notin A$.

4) For a positive integer n, consider the equation:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}\,.$$

Prove that the equation has exactly 3 solutions in positive integers when n is a prime number and write them down.

How many solutions are there when n is composite?

5) Every cell of 100×100 table is colored black or white. Every cell on the border of the table is black. It is known that every 2×2 square contains cells of both colors. Prove that there exists a 2×2 square that is colored in chess order.

6) Is there an infinite sequence of natural numbers such that it strictly increases and the sum of every 2 different numbers is relatively prime with the sum of every 3 different numbers?

7) By an array we mean here a finite nondecreasing sequence of integers. Let $A(n)$ be the number of distinct arrays composed of two or more primes that sum to n. (For instance, $A(4) = A(5) = 1$, $A(6) = 2$.) Prove that $A(n+1) \geq A(n)$ for all n.