Hoja 6

1) a) Show that for any $m \in \mathbb{N}$ there exists a real $m \times m$ matrix A such that $A^3 = A + I$, where I is the $m \times m$ identity matrix.

b) Show that $\det A > 0$ for every real matrix satisfying $A^3 = A + I$.

2) For any integer $n \geq 2$ and two $n \times n$ matrices with real entries A, B that satisfy the equation:

$$A^{-1} + B^{-1} = (A + B)^{-1},$$

prove that $\det(\mathrm{A}) = \det(\mathrm{B})$. Does the same conclusion holds for matrices of complex entries?

3) Find all the positive integer solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = x_1 x_2 x_3 x_4 x_5$$

4) We define a sequence $(a_n)_{n \geq 1}$ of rational numbers in two steps. First, let $a_1 = a_2 = 1$ and $a_3 = 3$. Then take the following inductive definition $\forall\, n \geq 3$:

$$a_{n+1} = \frac{2015 + a_n\, a_{n-1}}{a_{n-2}}$$

Prove that every a_n is an integer.

5) Prove that

$$\sum_{n=1}^{\infty} \frac{2^{\|\sqrt{n}\|} + 2^{-\|\sqrt{n}\|}}{2^n} = 3$$

where $\|x\|$ stands for the closest integer to x.

6) Let P be a polynomial that satisfies the identity

$$P(x)^2 - P(y)^2 = P(x + y) P(x - y)$$

for all $x, y \in \mathbb{R}$. Prove that there is a constant a such that $P(x) = ax$, $x \in \mathbb{R}$.