Hoja 6

1) Find the largest integer n such that n is divisible by all positive integers less than $\sqrt[3]{n}$.

2) Suppose that f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$.) Find

$$\int_0^1 f(x)\,dx.$$

3) The sequence

$$\{a_n\}_{n=1}^{\infty} = \{1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, \ldots\}$$

of positive integers is formed by taking one odd integer, then two even integers, then three odd integers, etc. Express a_n in closed form.

4) Let X and B_0 be real $n \times n$ matrices. Define by induction a sequence of matrices $B_k = B_{k-1}X - XB_{k-1}$. Prove that if $X = B_{n^2}$, then $X = 0$.

5) Show that one has $a^{b^a} > b^{a^b}$ if $a > b > 1$.

6) Let A be a countable subset of $\mathbb{R}^2$. Prove that it can be represented as $A_1 \cup A_2$, where the intersection of A_1 with any horisontal line, as well as the intersection of A_2 with any vertical line is a finite set.

7) There are n beetles situated in n different points of the circumference of a circular wheel. Each one moves by the circumference with a constant speed and makes one complete round in one hour. On hearing the initial signal, each of them selects a direction and starts moving immediately. When two beetles meet, both of them change directions and go without loss the speed.

Show that at a certain moment all beetles will return to their starting points.

¡Feliz Navidad!