Hoja 6

1) Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f\big(f(x) + y\big) = 2x + f\big(f(y) - x\big).$$

2) Let f be a continuous strictly increasing function on $(0, +\infty)$, satisfying

$$\lim_{x \to +\infty} \frac{f(x)}{x \log x} = 1.$$

Denote by $\varphi(x)$ the function, inverse to f (that is, it satisfies $\varphi(f(x)) = x$ for any positive x). Prove that

$$\lim_{x \to +\infty} \frac{\varphi(x)}{x/\log x} = 1.$$

3) Count the number of 3×3 matrices whose entries belong to $-1, 0, 1$ and whose determinant is of the form $3k + 1$, $k \in \mathbb{Z}$.

4) **a)** Does there exist a continuous function f on $\mathbb{R}$ such that $f(f(x)) = e^{-x}$, for all $x \in \mathbb{R}$.
b) The same question for the functional equation $f(f(x)) = e^{x}$, for all $x \in \mathbb{R}$.

5) In a real square matrix, all elements except the diagonal ones are given. Prove that one can place zeros and ones on the diagonal so that the resulting matrix is invertible.