

Hoja 5

1) An alien race has three genders: male, female, and emale. A *married triple* consists of three persons, one from each gender, who all like each other. Any person is allowed to belong to at most one married triple. A special feature of this race is that feelings are always mutual – if x likes y , then y likes x . The race is sending an expedition to colonize a planet. The expedition has n males, n females, and n emales. It is known that every expedition member likes at least k persons of each of the two other genders. The problem is to create as many married triples as possible to produce healthy offspring so the colony could grow and prosper.

1. Show that if n is even and $k = \frac{n}{2}$, then it might be impossible to create even one married triple.
2. Show that if $k \geq \frac{3n}{4}$, then it is always possible to create disjoint married triples, thus marrying all of the expedition members.

2) Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $\forall x, y \in \mathbb{R} :$

$$\begin{aligned} f(x + y) &= f(x) + f(y) \\ f(xy) &= f(x)f(y) \end{aligned}$$

3) Define a set to be self-referenced if its number of elements is its element. For instance, $\{1, 3, 4\}$ is self-referenced and $\{1, 2, 4\}$ is not. Find the number of subsets of $\{1, 2, \dots, n\}$ that are minimal self-referenced sets, which means that they are self-referenced and do not have any smaller self-referenced subset.

4) Let $f : [0, \infty) \mapsto \mathbb{R}$ be a bounded differentiable function satisfying:

$$f(x)f'(x) \geq \sin(x).$$

Does it exist $\lim_{x \rightarrow \infty} f(x)$?

5) Let V be an infinite dimensional space with a scalar product and $S \subset V$ be an infinite dimensional subspace. Let $x \in V \setminus S$ and W be the space generated by x . Determine the dimension of $(S \oplus W) \cap S^\perp$.