Hoja 5

1) Prove that for any positive real numbers a, b, c such that $abc = 1$, one has

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

2) Let $n \geq 2$, $n \in \mathbb{Z}$, and the divisors of n are $1 \leq d_1 < d_2 \leq \cdots \leq d_k = n$. Prove that the sum $d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$ is always less than n^2. Find, for which values of n is this sum a divisor of n^2.

3) Suppose we have a disk D of radius 16 and an annulus A of inner radius 2 and outer radius 3. We choose a finite subset $X \subset D$ having 650 points. Show that it is always possible to move the annulus A and place it in such a way that it covers at least 10 points of X, regardless of how the set X was chosen.

4) Let A, B, C be pairwise commuting complex matrices. Prove that there exist real numbers α, β, γ, not all equal to zero, such that $\det(\alpha A + \beta B + \gamma C) = 0$.

5) Find all the functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$f(x, x) = x, \quad f(x, y) = f(y, x) \quad \text{and} \quad (x + y) f(x, y) = y f(x, x + y) \quad (\forall x, y \in \mathbb{N}).$$