

## Hoja 4

1) Let  $\phi$  and  $\psi$  be two continuous monotonically decreasing functions on  $(0, +\infty)$ , which are mutually inverse and satisfy

$$\int_0^\infty \phi(t) dt = \int_0^\infty \psi(s) ds = a.$$

Prove that

$$\int_0^\infty \phi^2(t) dt + \int_0^\infty \psi^2(s) ds \geq \frac{1}{2} a^{3/2}.$$

2) Let  $n$  be an odd positive integer. Prove that  $24 \mid n^n - n$ .

3) Is there a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(\mathbb{Q}) \subseteq \mathbb{R} \setminus \mathbb{Q}$  and  $f(\mathbb{R} \setminus \mathbb{Q}) \subseteq \mathbb{Q}$ ?

In case there is give an explicit construction.

4) Let  $n, k \in \mathbb{N}_*$  such that  $n \geq k$ , and let  $\mathcal{F}$  be a family of finite sets with the following properties:

1.  $\mathcal{F}$  contains at least  $\binom{n}{k} + 1$  distinct sets containing exactly  $k$  elements;
2. For any two sets  $A, B \in \mathcal{F}$ , their union  $A \cup B$  also belongs to  $\mathcal{F}$ .

Prove that  $\mathcal{F}$  contains at least three sets with at least  $n$  elements.

5) Find all continuous maps  $f : \mathbb{R} \rightarrow \mathbb{C} \setminus \{0\}$  satisfying the following functional equation:

$$\forall x, y \in \mathbb{R} \quad f(x + y) = f(x)f(y)$$

6) Can the plane be coloured in 2020 colours so that any nondegenerate circle contains points of all 2020 colors?