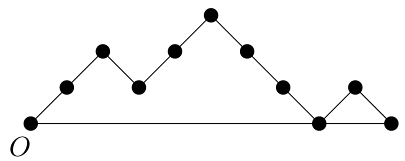
Hoja 4

1) By an upper n-path we mean a path in $\mathbb{Z}^2$ consisting of n upsteps $(1,1)$ and n downsteps $(1,-1)$ beginning in the origin $O=(0,0)$, which never goes lower the x-axis. By a return we mean a maximal sequence of contiguous downsteps ending on the x-axis. For instance, the upper 5-path given on the diagram below has two returns, one of length 3 and the other of length 1.



Find a one-to-one correspondence between upper $(n-1)$-paths and upper n-paths that have no returns of even length.

2) Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\operatorname{lcm}(a_n, a_{n+1})}$$

converges where lcm denotes the least common multiple.

3) Prove that any two numbers in the following sequence are relatively prime:

$$2+1, 2^2+1, 2^4+1, \ldots, 2^{2^n}+1, \ldots$$

4) Let N be an arbitrary natural number. Show that in the elementary equality

$$N = \frac{N}{2} + \frac{N}{4} + \frac{N}{8} + \cdots + \frac{N}{2^n} + \cdots$$

every fraction may be replaced by the nearest integer:

$$N = (\frac{N}{2}) + (\frac{N}{4}) + (\frac{N}{8}) + \cdots + (\frac{N}{2^n}) + \cdots.$$

We understand here that the nearest integer (x) to a real number x is taken to be $x+\frac{1}{2}$ in the case when x is a half-integer.

5) Suppose that a group G has the following properties:

- G has no element of order 2;
- $(xy)^2 = (yx)^2$, for all $x, y \in G$.

Prove that G is abelian.