Hoja 3

1. Prove the (well-known) inequality of arithmetic and geometric means: for any $n \geq 1$ and for any nonnegative real numbers $x_1, x_2, \ldots, x_n$,

$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n} .$$

Try to find several proofs.

2. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous and nowhere monotone function. Show that the set of points on which f attains local minima is dense in $[0,1]$. (A function is *nowhere monotone* if there exists no interval where the function is monotone. A set is *dense* if each non-empty open interval contains at least one element of the set.)

3. In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that $\operatorname{trace}(AB) = 0$ for all $A, B \in V$. (The *trace* of a matrix is the sum of the diagonal entries.)

4. Let m and n be nonnegative integers. Prove that

$$\frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}$$

is an integer. (Recall that $n! = n(n-1)\cdots 2 \cdot 1$ if n is a positive integer and $0! = 1$.)

5. Is it possible to plot 2023 points on a circle with radius 1 so that the distance between any two of them is a rational number (distances have to be measured by chords)?

6. Prove that, for any integers a, b, c, there exists a positive integer n such that $\sqrt{n^3 + an^2 + bn + c}$ is not an integer.

7. Is there a strictly increasing function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all x?