Hoja 3

Probability

1) Let k be a positive integer. Suppose that the integers $1, 2, 3, \dots, 3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials.

Number Theory

2) Players $1, 2, 3, \dots, n$ are seated around a table, and each has a single penny. Player 1 passes a penny to player 2, who then passes two pennies to player 3. Player 3 then passes one penny to Player 4, who passes two pennies to Player 5, and so on, players alternately passing one penny or two to the next player who still has some pennies. A player who runs out of pennies drops out of the game and leaves the table. Find an infinite set of numbers n for which some player ends up with all n pennies.

Algebra

3) Let G be a finite group with the unity e. Given any subsets U, V, W of G, we denote by N_{UVW} the number of triples $(x, y, z) \in U \times V \times W$ satisfying $xyz = e$. Suppose that G is partitioned into three sets A, B and C (that is, sets A, B, C are pairwise disjoint and $G = A \cup B \cup C$). Show that $N_{ABC} = N_{CBA}$.

Analysis

4) Let $a < b < c < d$, and let $p(x) = (x-a)(x-b)(x-c)(x-d)$. Prove that

$$\int_a^b \frac{dx}{\sqrt{|p(x)|}} = \int_c^d \frac{dx}{\sqrt{|p(x)|}} \,.$$

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5) Let V be a convex polygon with n vertices.

(1) Prove that if n is divisible by 3 then V can be triangulated (i.e. dissected into non-overlapping triangles whose vertices are vertices of V) so that each vertex of V is the vertex of an odd number of triangles.

(2) Prove that if n is not divisible by 3 then V can be triangulated so that there are exactly two vertices that are the vertices of an even number of the triangles.
