# HOJA 1

**1.** A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2021, 2021)$?

*Solution.* The answer is 578.

Each hop corresponds to adding one of the 12 vectors $(0, \pm 5)$, $(\pm 5, 0)$, $(\pm 3, \pm 4)$, $(\pm 4, \pm 3)$ to the position of the grasshopper. Since $(2021, 2021) = 288(3, 4) + 288(4, 3) + (0, 5) + (5, 0)$, the grasshopper can reach $(2021, 2021)$ in $288 + 288 + 1 + 1 = 578$ hops.

On the other hand, let $z = x+y$ denote the sum of the $x$ and $y$ coordinates of the grasshopper, so that it starts at $z = 0$ and ends at $z = 4042$. Each hop changes the sum of the $x$ and $y$ coordinates of the grasshopper by at most 7, and $4042 > 577 \times 7$; it follows immediately that the grasshopper must take more than 577 hops to get from $(0, 0)$ to $(2021, 2021)$.

**Remark.** This solution implicitly uses the distance function

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

on the plane, variously called the *taxicab metric*, the *Manhattan metric*, or the $L^1$-*norm* (or $\ell_1$-norm). □

**2.** Let $\{a_n\}_{n=1}^{\infty}$ be a nonincreasing sequence such that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{n\to\infty} n a_n = 0.$$

*Solution.* Notice that $a_n \geq 0$ for all $n$. Also, since the series converges, it is Cauchy, and therefore

$$\lim_{n\to\infty} (a_{\lfloor n/2 \rfloor+1} + a_{\lfloor n/2 \rfloor+2} + \cdots + a_n) = 0.$$

Using that $a_n \geq a_{n+1}$ for all $n$, we have

$$a_{\lfloor n/2 \rfloor+1} + a_{\lfloor n/2 \rfloor+2} + \cdots + a_n \geq \left\lceil \frac{n}{2} \right\rceil a_n \geq \frac{n}{2} a_n.$$

Hence $\lim_{n\to\infty} \frac{n}{2} a_n = 0$, and the statement follows. □

**3.** Let $n$ and $k$ be integers with $2 \leq k \leq n$. Let $a_1, \ldots, a_k$ be distinct elements of the set $\{1, \ldots, n\}$ such that $n$ divides $a_i(a_{i+1} - 1)$ for all $i \in \{1, \ldots, k-1\}$. Prove that $n$ does not divide $a_k(a_1 - 1)$.

*Solution.* Since $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$, we have the following list of congruences

$$a_1 a_2 \equiv a_1 \text{ (mod. } n),$$

$$a_2 a_3 \equiv a_2 \text{ (mod. } n),$$

$$\vdots$$

$$a_{k-1}a_k \equiv a_{k-1} \text{ (mod. } n).$$

Suppose that $a_k a_1 \equiv a_k$ (mod. $n$). Multiplying this congruence by $a_{k-1}$ and using the last congruence of the list we obtain

$$a_{k-1}a_1 \equiv a_{k-1} \text{ (mod. } n).$$

Now multiplying this congruence by $a_{k-2}$ we obtain

$$a_{k-2}a_1 \equiv a_{k-2} \text{ (mod. } n).$$

And so on, we finally get

$$a_2 a_1 \equiv a_2 \text{ (mod. } n).$$

Comparing this with the first congruence of the list we obtain that $a_1 \equiv a_2$ (mod. $n$). But since $a_1, a_2 \in \{1, \ldots, n\}$, we have $a_1 = a_2$. But they were distinct by assumption. This contradiction shows that $a_k a_1 \not\equiv a_k$ (mod. $n$), i.e., $n$ does not divide $a_k(a_1 - 1)$. □

**4.** Let $D$ be the closed unit disc in the plane and $p_1, \ldots, p_n$ be fixed points in $D$. Show that there is a point $p \in D$ such that

$$\sum_{i=1}^{n} \operatorname{dist}(p, p_i) \geq n.$$

(Here $\operatorname{dist}(p, p_i)$ denotes the Euclidean distance between the points $p$ and $p_i$.)

*Solution.* We write $p_i$ as complex numbers and put $q = -\sum p_i$. First assume $q \neq 0$. We put $p = q/|q|$. Then, by the triangle inequality,

$$\sum_{i=1}^{n} |p - p_i| \geq \left| np - \sum_{i=1} p_i \right| = |nq/|q| + q| = n + |q| \geq n.$$

If $q = 0$, we put $p = 1$ and then we can use the triangle inequality in the same way as above. □

**5.** Let $V$ be a finite dimensional vector space and let $A$ and $B$ be two linear transformations of $V$ into itself such that $A^2 = B^2 = 0$ and $AB + BA = I$. Prove that

(a) $\ker A = A \ker B$ and $\ker B = B \ker A$.
(b) $\dim V$ is even.

*Solution.* (a) By symmetry in $A$ and $B$, it is enough to check that $\ker A = A \ker B$. Since $A^2 = 0$, we have $AV \subset \ker A$. (Analogously, $BV \subset \ker B$.) In particular, $A \ker B \subset \ker A$. Now let us see that $\ker A \subset A \ker B$. Take $u \in \ker A$, that is, $Au = 0$. Then obviously $BAu = 0$. Since $AB + BA = I$, we have

$$u = (AB + BA)u = ABu \in ABV \subset A \ker B.$$

Hence $\ker A \subset A \ker B$, as we wanted to prove.

(b) Since $\ker A = A \ker B$, we have $\dim \ker A \leq \dim \ker B$. Analogously, $\dim \ker B \leq \dim \ker A$, so $\dim \ker A = \dim \ker B$. Now let us see that

$$V = \ker A \oplus \ker B. \tag{$*$}$$

The statement will follow from this, since we would have $\dim V = 2 \dim \ker A$. For any $v \in V$ we have

$$v = (AB + BA)v = ABv + BAv \in \ker A + \ker B,$$

since $ABV \subset \ker A$ and $BAV \subset \ker B$. Hence $V = \ker A + \ker B$. Let us check that this sum is direct, that is, $\ker A \cap \ker B = \{0\}$. Indeed, take $u \in \ker A \cap \ker B$. Since $\ker A = A \ker B$, for some $b \in \ker B$ we have $u = Ab$. Now using that $u \in \ker B$ we get

$$0 = Bu = BAb = b - ABb = b.$$

Hence $u = A0 = 0$, as we wanted to prove. □

**6.** Let $x, y, z > 1$. Prove that

$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2} \geq 48.$$

**Quedó para la siguiente reunión.**