Hoja 1

1. A grasshopper starts at the origin in the coordinate plane and makes a sequence of hops. Each hop has length 5, and after each hop the grasshopper is at a point whose coordinates are both integers; thus, there are 12 possible locations for the grasshopper after the first hop. What is the smallest number of hops needed for the grasshopper to reach the point $(2023, 2023)$?

2. Let $\{a_n\}_{n=1}^{\infty}$ be a nonincreasing sequence such that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{n\to\infty} n a_n = 0.$$

3. Let n and k be integers with $2 \leq k \leq n$. Let $a_1, \ldots, a_k$ be distinct elements of the set $\{1, \ldots, n\}$ such that n divides $a_i(a_{i+1} - 1)$ for all $i \in \{1, \ldots, k-1\}$. Prove that n does not divide $a_k(a_1 - 1)$.

4. Let D be the closed unit disc in the plane and $p_1, \ldots, p_n$ be fixed points in D. Show that there is a point $p \in D$ such that

$$\sum_{i=1}^{n} \operatorname{dist}(p, p_i) \geq n.$$

(Here $\operatorname{dist}(p, p_i)$ denotes the Euclidean distance between the points p and p_i.)

5. Let V be a finite dimensional vector space and let A and B be two linear transformations of V into itself such that $A^2 = B^2 = 0$ and $AB + BA = I$. Prove that

(a) $\ker A = A \ker B$ and $\ker B = B \ker A$.

(b) $\dim V$ is even.

6. Let $x, y, z > 1$. Prove that

$$\frac{x^4}{(y-1)^2} + \frac{y^4}{(z-1)^2} + \frac{z^4}{(x-1)^2} \geq 48.$$