Hoja 10

1) The sequence $\{x_n\}_{n=1}^{\infty}$ is defined by $x_1 = 4$ and

$$x_{n+1} = 2^{n+1}\sqrt{2} \cdot \sqrt{1 - \sqrt{1 - \left(\frac{x_n}{2^{n+1}}\right)^2}}$$

for $n \geq 1$. Compute $\lim\limits_{n\to\infty} x_n$.

2) A deck consists of some cards. For each pair of them, a winner is defined (this winning relation is not supposed to be transitive). The deck is divided into two piles, you know the contents and orders of the cards in both piles. Each turn, one of the two cards at the top of the piles wins; you must move these two top cards to the bottom of the winning deck, but you may decide the order. Prove that you can collect all cards in a single pile.

3) Suppose that $\varphi \in C^2[0, +\infty)$; $\varphi(x) > 0$, $\varphi'(x) > 0$ and

$$\frac{\varphi(x)\varphi''(x)}{\big(\varphi'(x)\big)^2} \leq 2 \qquad \text{for all } x > 0.$$

Prove that $\lim\limits_{x\to+\infty} \dfrac{\varphi'(x)}{\big(\varphi(x)\big)^2} = 0.$

4) (i) Prove that if G_1 and G_2 are distinct subgroups of a finite group G such that $G = G_1 \cup G_2$, then either $G_1 = G$ or $G_2 = G$.

(ii) Give an example of a group H which has subgroups H_1, H_2 and H_3 such that every element of H is in at least one of H_1, H_2 or H_3 and $H_1 \neq H, H_2 \neq H, H_3 \neq H$.

5) Find all positive integers n for which

$$\frac{12n^3 - 5n^2 - 251n + 389}{6n^2 - 37n + 45}$$

is an integer.