Curso Avanzado de Análisis
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Problem sheet 1

1) Let T be a (bounded) linear operator on a Hilbert space.

a) Prove that there are selfadjoint operators A, B such that $T{=}A+iB$.

b) Prove that this decomposition is unique.

c) What condition on A and B guarantees that T is normal?

d) Prove that both A and B are compact if and only if T is compact.

2) Let $\{d_n\}$ be a complex sequence. Define a diagonal operator T on $\ell^2 = \ell^2(\mathbb{N})$ by

$$T\{x_n\}_{n=1}^{\infty} = \{d_n x_n\}_{n=1}^{\infty}.$$

For which sequences $\{d_n\}$, is T **(a)** bounded; **(b)** unitary; **(c)** normal; **(d)** compact?

3) Let T be a bounded diagonal operator on ℓ^2 as above.

a) Calculate the point spectrum of T.

b) Calculate the approximate point spectrum of T.

c) Calculate the spectrum of T.

4) Let T, S be bounded operators on a Banach space X. Suppose that ST is compact. Does it follow that either T or S is compact?
HINT: Think of the operators from the exercise 2).

5) We define *the shift operator* on $\ell^2(\mathbb{Z}_+)$ by

$$S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$$

(here $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$).

a) Calculate $\|S\|$. Calculate S^*.

b) Is S a compact operator?

c) Calculate $\sigma_p(S)$ y $\sigma_p(S^*)$. For each $\lambda \in \mathbb{C}$, calculate $\dim\ker(S-\lambda)$ and $\dim\ker(S^*-\bar{\lambda})$.

d) Calculate $\sigma_{ap}(S)$, $\sigma_{ap}(S^*)$, $\sigma_{comp}(S)$, $\sigma_{comp}(S^*)$.

e) Does S have a left inverse on $\ell^2(\mathbb{Z}_+)$? Is it unique? Does S have a right inverse?

6) Let $T \in L(X)$, where X is a Banach space.

a) Prove that the approximate point spectrum $\sigma_{ap}(T)$ is closed.

b) Prove that $\sigma_{comp}(T) = \{\bar{z} : z \in \sigma_p(T^*)\}$.

7) Let H_1, H_2 be Hilbert spaces. An operator $U : H_1 \to H_2$ is called *isometric isomorphism* if $U^*U = I_{H_1}$ and $UU^* = I_{H_2}$. If $U : H \to H$ is an isometric isomorphism, we say that U is *unitary.*

a) Let $U : H_1 \to H_2$. Check that $U^*U = I_{H_1}$ if and only if U is an isometry, that is, $\|Uh\| = \|h\|$ for all $h \in H_1$.

b) Check that the operator S on $\ell^2(\mathbb{Z}_+)$ from the previous exercise is an isometry, but not is unitary. What is its image $S\ell^2(\mathbb{Z}_+)$?

8) Given an isometry $S : H_1 \to H_2$, check that the following properties are equivalent:

a) $S(H_1) = H_2$;

b) S has a (two-sided) inverse;

c) S is an isometric isomorphism (unitary in case $H_1 = H_2$).

9) Let A, B be bounded linear operators on a Hilbert space H. Prove or disprove the following assertions.

a) A, B are self-adjoint $\implies$ AB is self-adjoint;

b) A, B are unitary $\implies$ AB is unitary;

c) A, B are normal $\implies$ AB is normal.

10) Answer the same three questions for $A + B$, instead of AB.

11) Which answers in the above two exercises change if A and B commute?

12) Let N be a normal operator on a Hilbert space. Prove that for all $\lambda \in \mathbb{C}$, $\dim\ker(N - \lambda) = \dim\ker(N^* - \bar{\lambda})$.

HINT: Given a vector h, consider $\|(N - \lambda)h\|^2$ and use the normality of N.

13) What is the analogue of the above equality for compact operators? Are there exceptional values of λ, for which this analogue can fail? Justify your answer.

14) Let A be a Banach algebra without unity. Consider the vector space

$$A_1 = \{(x, a) : x \in A, a \in \mathbb{C}\}$$

and define the multiplication and the norm on A_1 by

$$(x, a)(y, b) = (xy + ay + bx, ab),$$
$$\|(x, a)\| = \|x\| + |a|.$$

Prove the following.

a) A_1 is an algebra and $(0, 1)$ is its unit;

b) The map $x \mapsto (x, 0)$ is an isometric isomorphism from A onto a two-sided closed ideal in A_1 of codimension 1.

15) The Banach inverse map theorem says that any bounded bijective map from one Banach space to another has a bounded inverse. Deduce it from the closed graph theorem.

16) Given $n \in \mathbb{N}$, consider the linear space $C^n[0, 1]$ of n times continuously differential complex functions on the interval $[0, 1]$, with the norm $\|f\|_{C^n[0,1]} = \|f\|_\infty + \|f^{(n)}\|_\infty$, $f \in C^n[0, 1]$.

a) Prove that the space $C^n[0, 1]$ with this norm is a Banach space.

b) Prove that there are constants M_k such that

$$\|f^{(k)}\|_\infty \le M_k\Big(\|f\|_\infty + \|f^{(n)}\|_\infty\Big)$$

for all $k = 1, 2, \ldots, n - 1$ and all $f \in C^n[0, 1]$.

c) Prove that $C^n[0, 1]$ satisfies all properties of Banach algebras, except for the submultiplicative property for the norm, instead of which the following weaker property holds: $\|fg\|_n \le$

$C_n\|f\|_n\|g\|_n$ for all $f, g \in C^n[0,1]$. Can one introduce an equivalent norm on $C^n[0,1]$, which makes it a Banach algebra?

17) We define the convolution of two finite (complex) Borel measures μ and ν on $\mathbb{R}$ by

$$(\mu * \nu)(B) = \iint_{\mathbb{R}^2} \chi_B(x+y)\, d\mu(x)\, d\nu(y).$$

where χ_B is the characteristic function of the set B.

a) Prove that $\mu * \nu$ is a finite Borel measure on $\mathbb{R}$.

b) Prove the formula

$$(\mu * \nu)(B) = \int_{\mathbb{R}} \mu(B-y)\, d\nu(y) = \int_{\mathbb{R}} \nu(B-x)\, d\mu(x).$$

c) Prove that $\mu * \nu$ is absolutely continuous if either μ or ν is absolutely continuous.

d) Prove that $\|\mu * \nu\| \leq \|\mu\|\,\|\nu\|$, where $\|\mu\|$ is *the total variation* of μ:

$$\|\mu\| = |\mu|(\mathbb{R}).$$

18) Prove that the space $M(\mathbb{R})$ of all finite (complex) Borel measures is a Banach algebra with respect to the convolution and the norm, defined as the total variation. Is this algebra commutative? Does this algebra have the unit?