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LINEARLY SIMILAR SZÖKEFALVI-NAGY–FOIAŞ MODEL IN A DOMAIN

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§0. Introduction

In this paper we construct a new linearly similar functional model for linear operators and study its elementary properties. This model generalizes the Sz.-Nagy–Foiaş model for $C_{0\cdot}$-contractions and also for $C_{0\cdot}$-dissipative operators. We shall not restrict ourselves to the disk or the half-plane: the model will be constructed in a fairly arbitrary domain. The reduction of an operator to an "almost diagonal" model form will be written directly via the resolvent of the operator. Attention will be focused on the case of C_{00}-operators. The main results of this paper were announced in [76].

Let Γ be a union of finitely many piecewise-smooth contours in the complex plane $\mathbb{C}$. Suppose that the sets $\Omega_{\rm int}$, $\Omega_{\rm ext}$ are open and have empty intersection, $\mathbb{C} = \Omega_{\rm int}\cup\Gamma\cup\Omega_{\rm ext}$, and $\Gamma = \partial\Omega_{\rm int} = \partial\Omega_{\rm ext}$ (here we omit a certain technical condition on Γ). The main objects of this paper are model spaces $\mathcal{H}(\delta)$ that will be associated with operator-valued bounded analytic functions δ defined on $\Omega_{\rm int}$ and having some special properties.

Consider the special case where $\delta \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(R, R_*)\big)$, δ^{-1} is a meromorphic function on $\Omega_{\rm int}$, and $\|\delta^{-1}(\lambda)\| \le C$ a.e. on Γ. Here R and R_* are Hilbert spaces, and $\mathcal{L}(R, R_*)$ denotes the space of all bounded linear operators from R to R_*. We always consider only separable Hilbert spaces. Let $E^2(\Omega_{\rm ext}, R)$ denote the Smirnov class of functions on $\Omega_{\rm ext}$ whose values are vectors in the space R (see §2). Suppose that $\Omega_{\rm int}$ and $\Omega_{\rm ext}$ are unbounded. In this special case the model space $\mathcal{H}(\delta)$ consists of all functions f meromorphic in $\Omega_{\rm int}$, holomorphic in $\Omega_{\rm ext}$, and satisfying

$$f|\Omega_{\rm ext} \in E^2(\Omega_{\rm ext}, R), \quad \delta\cdot f|\Omega_{\rm int} \in E^2(\Omega_{\rm int}, R_*), \quad f_{\rm int} = f_{\rm ext} \text{ a.e. on } \Gamma.$$

This is a Hilbert space. The general definition of $\mathcal{H}(\delta)$ will be given in §2.

A pair (A, J) of linear operators (possibly, unbounded) will be called a 2-*system* if

1) A is a closed operator in a Hilbert space H with nonempty field of regularity $\rho(A) = \mathbb{C}\setminus\sigma(A)$ and with domain $\mathcal{D}(A)$ (here $\sigma(A)$ is the spectrum of the operator A);

2) $J : \mathcal{D}(J) \to R$, where $\mathcal{D}(J) = \mathcal{D}(A) \subset H$ and J is bounded in the graph norm $\|x\|_G \overset{\rm def}{=} \big(\|x\|^2 + \|Ax\|^2\big)^{1/2}$ in $\mathcal{D}(A)$. Here R is a Hilbert space.

With every 2-system (A, J), we associate a map $U_{A,J}$ from H to the space of functions analytic in $\rho(A)$; this map acts by the formula

$$U_{A,J}\, x(\lambda) = J(\lambda I - A)^{-1}x, \quad x \in H,\ \ \lambda \in \rho(A).$$

In §1 we consider general properties of this map and show that it diagonalizes the operator A in some sense.

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Our goal in this paper is a criterion for the operator $U_{A,J}$ to be an isomorphism of the space H onto the space $\mathcal{H}(\delta)$ for some function δ (this isomorphism is not assumed to be isometric.) If this is the case, δ will be called a *generalized characteristic function* of A. An efficient criterion will be obtained in §3. If $U_{A,J} : H \to \mathcal{H}(\delta)$ is an isomorphism, and the $*$-inner part of δ is a two-sided inner function, then A is similar to the operator of multiplication by z in the quotient space $E^2(\Omega_{\text{int}}, R_*)/\delta \cdot E^2(\Omega_{\text{int}}, R)$, which, in its turn, is similar to the restricted shift operator in the orthogonal complement $E^2(\Omega_{\text{int}}, R_*) \ominus \delta E^2(\Omega_{\text{int}}, R)$. The intimate relationship between our model and the model of Sz.-Nagy–Foiaş will be discussed in §§4, 5. In this paper, we make some modifications in the technique of Sz.-Nagy–Foiaş in order to extend the range of applications.

To operators A fitting in the above pattern, we can apply all the ideas and methods pertaining to the Sz.-Nagy–Foiaş model, such as description of the commutant, invariant subspaces, functional calculus, criteria for the eigenvectors to form a basis, etc.; see [13], [20], [60], [61].

In §6 we give a series of examples of operators A and their models obtained in this way. For contractions and dissipative operators, we show that under an appropriate choice of J we obtain their Sz.-Nagy–Foiaş models. Also, we consider generators of semigroups related to the dynamics of the growth of populations and the neutral linear systems with delay (for the latter, the model was constructed earlier by S. Lunel and the author in [56]). The calculation of the generalized characteristic functions δ is postponed until §10.

In §7 we establish the relationship between our approach and the exact controllability of linear systems.

In §8 we show that the generator of any c_0-group admits a linearly similar model of this form, and that an appropriate vertical strip can be taken as Ω_{int}. (In contrast to the symbols C_{00} and $C_{0\cdot}$, we use the symbol c_0 in the sense of the theory of strongly continuous semigroups of linear operators; see [23].) So, in particular, our approach covers arbitrary bounded perturbations of selfadjoint operators.

The method under consideration is applicable to unbounded perturbations of selfadjoint operators with discrete spectrum; then an appropriate parabolic domain should be taken as Ω_{int}. This will be discussed in a separate publication.

Often, a generalized characteristic function δ of the system (A, J) can be expected to be the matrix "denominator" of an expression for $U_{A,J}$. In these cases, it is easy to obtain the inclusion $U_{A,J}\, H \subset \mathcal{H}(\delta)$. In §9 an abstract method is presented, which allows us to obtain the identity $U_{A,J}\, H = \mathcal{H}(\delta)$.

An important feature of the theory is the duality of models. Namely, let $\overline{\Omega}_{\text{int}} = \{\bar{z} : z \in \Omega_{\text{int}}\}$, and let $\delta^{\mathbf{T}}(z) = \delta^*(\bar{z})$, $z \in \overline{\Omega}_{\text{int}}$. It turns out that the spaces $\mathcal{H}(\delta)$ and $\mathcal{H}(\delta^{\mathbf{T}})$ constructed by the domains Ω_{int} and $\overline{\Omega}_{\text{int}}$ are dual to each other with respect to the duality $\langle \cdot, \cdot \rangle_\delta$ defined in §4. A system (A^*, J_*) will be called *dual to a system* (A, J) *with respect to a function* δ if $U_{A,J}$ is an isomorphism of H onto $\mathcal{H}(\delta)$, U_{A^*,J_*} is an isomorphism of H onto $\mathcal{H}(\delta^{\mathbf{T}})$, and the following duality relation is fulfilled:

$$\langle U_{A,J}\, x, U_{A^*,J_*}\, y \rangle_\delta = \langle x, y \rangle, \quad x, y \in H. \tag{0.1}$$

For J and δ fixed, if $U_{A,J}$ is an isomorphism of H onto $\mathcal{H}(\delta)$, then the above conditions determine the operator $J_* : \mathcal{D}(A^*) \to R_*$ uniquely.

A triple of operators (A, J, J_*) will be called a 3-*system* if the operator A is closed and densely defined (which implies the existence of A^*), and (A, J), (A^*, J_*) are 2-systems. We define the *transfer function* Φ of a 3-system (A, J, J_*) by the formula

$$\Phi(\lambda) - \Phi(\mu) = J[(\lambda I - A)^{-1} - (\mu I - A)^{-1}]J_*^*. \tag{0.2}$$

This formula determines the function Φ up to an additive constant (see §9). A relationship between the generalized characteristic and transfer functions is established in

Theorems 9.5 and 9.6. Given a system (A, J), take a function δ, which is a candidate for the role of a generalized characteristic function. Then it is possible to construct a function Φ and the dual system (A^*, J_*). Under the assumption that these objects are known, the theorems mentioned above give criteria for δ to be in fact a generalized characteristic function; thus, they allow us to compute a model of A completely.

Function models of nondissipative operators have been studied quite thoroughly, in particular, by Naboko and his students [10, 17], by Vasyunin–Makarov [68], and by Kapustin [6]. Our approach differs from that of the papers mentioned; the classes of operators under consideration are also different.

An important difference of our method from the theory of Sz.-Nagy–Foiaş is the nonuniqueness of a generalized characteristic function δ, even if the domain Ω_{int} is fixed. Under our assumptions, the characteristic function in the sense of Sz.-Nagy–Foiaş will be $*$-inner. If a system admits a generalized characteristic function, then it also admits a $*$-inner characteristic function (if the components of the set Ω_{int} are simply connected). However, in many cases, precisely the abandonment of the condition for a generalized characteristic function to be inner makes it possible to write out such a function explicitly. The duality of model spaces introduced in the paper is adapted to these explicit procedures.

Note that, often, linearly similar models of operators are not uniquely determined. Even for finite-rank operators, there is no unique canonical way to reduce them to the Jordan form. This nonuniqueness of models under consideration will be discussed in §11.

The use of the Cauchy duality between the model of an operator and its adjoint is quite natural if an operator is not "attached" to the circle or the line (see the papers by Xia [72] and Clark [33], and also the earlier papers by the author [73]–[77], etc.). Duality and model spaces were used in a different form by Fuhrmann in [39, 40]. Despite the fact that the model function spaces are Hilbert, often we do not identify them with their duals. Due to this, we achieve simpler formulas, together with some symmetry between the operator and its adjoint. In contrast with [75], we view the conjugate space as a space of antilinear continuous functionals.

If A is an operator defined by some formula and $\mathcal{H}$ is a space of functions, then $A\lceil\mathcal{H}$ denotes the operator A acting on $\mathcal{H}$. If $A\lceil\mathcal{H}$ is unbounded, the domain $\mathcal{D}(A\lceil\mathcal{H})$ is a proper subset of $\mathcal{H}$ and will be indicated explicitly.

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Added in proof. Quite recently, a paper by A. S. Tikhonov [79] appeared, which is undoubtedly related to the subject of our research. In that paper, a model was constructed for an arbitrary trace class perturbation of a normal operator with spectrum on a smooth closed curve. This model is close to those of Sz.-Nagy–Foiaş and S. N. Naboko; in Tikhonov's paper the absolutely continuous and singular components are investigated for operators of that class.

§1. Construction of the resolvent model of a linear operator

1.1. The model operator. Let Ω_0 be a nonempty open set in $\mathbb{C}$, let R be a Banach space, and let $\mathcal{H}$ be a Banach space of analytic functions $f(z)$ in Ω_0 with values in R. We denote by $\mathrm{Hol}(\Omega_0, R)$ the topological vector space of all R-valued holomorphic functions on Ω_0 with the standard topology of uniform convergence on compact sets.

Definition. A Banach space $\mathcal{H}$ of functions is said to be *admissible* if

(1) $\mathcal{H}$ is continuously embedded in $\mathrm{Hol}(\Omega_0, R)$;

(2) $\mathcal{H}$ contains no nonzero constants;
(3) $\lambda \in \Omega_0$, $f \in \mathcal{H} \implies \frac{f(z)-f(\lambda)}{z-\lambda} \in \mathcal{H}$.

We shall assume that $\mathcal{H}$ is admissible. We define operators $M_z^{\mathbf{T}}$ and j (possibly, unbounded) in the following way. Put

$$\mathcal{D}(M_z^{\mathbf{T}}) = \mathcal{D}(M_z^{\mathbf{T}}\lceil\mathcal{H}) = \{f \in \mathcal{H} : \exists c \in R : zf - c \in \mathcal{H}\}. \tag{1.1}$$

For f in $\mathcal{D}(M_z^{\mathbf{T}})$, the constant c is unique. Therefore, the operators

$$j : \mathcal{D}(j) \stackrel{\text{def}}{=} \mathcal{D}(M_z^{\mathbf{T}}) \to R, \quad M_z^{\mathbf{T}} : \mathcal{D}(M_z^{\mathbf{T}}) \to \mathcal{H},$$
$$jf \stackrel{\text{def}}{=} c, \;\; (M_z^{\mathbf{T}} f)(z) \stackrel{\text{def}}{=} zf(z) - c, \quad f \in \mathcal{D}(M_z^{\mathbf{T}}), \tag{1.2}$$

are well defined. We shall call $M_z^{\mathbf{T}}$ the *model* operator.

With an admissible space $\mathcal{H}$, we associate the set

$$\Omega(\mathcal{H}) = \Big\{\lambda \in \mathbb{C} : \forall g \in \mathcal{H}\, \exists!\, c \in R : \frac{g(z) - c}{z - \lambda} \in \mathcal{H}\Big\}. \tag{1.3}$$

Obviously, $\Omega(\mathcal{H}) \supset \Omega_0$. We extend the functions in $\mathcal{H}$, which are defined initially only on Ω_0, to the set $\Omega(\mathcal{H})$ as follows: if $\lambda \in \Omega(\mathcal{H})$, then $g(\lambda) \stackrel{\text{def}}{=} c$, where c is determined by (1.3).

Proposition 1.1. 1) *The operator $(M_z^{\mathbf{T}} - \lambda I)^{-1}$ exists and is bounded if and only if $\lambda \in \Omega(\mathcal{H})$. Then*

$$(M_z^{\mathbf{T}} - \lambda I)^{-1} g(z) = \frac{g(z) - g(\lambda)}{z - \lambda}. \tag{1.4}$$

2) $\mathbb{C} \setminus \Omega(\mathcal{H}) = \sigma(M_z^{\mathbf{T}})$. *In particular, the set $\Omega(\mathcal{H})$ is open.*
3) *We have $g(\lambda) = j(\lambda I - M_z^{\mathbf{T}})^{-1} g$ for $g \in \mathcal{H}$, $\lambda \in \Omega(\mathcal{H})$.*
4) *For any $g \in \mathcal{H}$, the function $g(\lambda)$ is analytic in $\Omega(\mathcal{H})$.*

Proof. It is easily seen that (1.4) holds if $\lambda \in \Omega_0$. Hence the operator $(M_z^{\mathbf{T}} - \lambda I)^{-1}$ is bounded for $\lambda \in \Omega_0$, which implies that the operator $M_z^{\mathbf{T}}$ is closed.

Let $g \in \mathcal{H}$, and let $\lambda \in \mathbb{C}$. Then the relation $(M_z^{\mathbf{T}} - \lambda I)f = g$ is equivalent to the identities $f(z) = (z-\lambda)^{-1}\big(g(z) - c\big)$, $c = -jf$. The third relation follows from the second and the condition $f \in \mathcal{H}$. This implies 3). Applying the closed graph theorem, we obtain 1), whence 2) follows immediately. For $\mu \in \Omega(\mathcal{H})$ and λ close to μ, from statement 3) we deduce that

$$g(\lambda) = j(\mu I - M_z^{\mathbf{T}})^{-1}\big(I - (\mu - \lambda)(\mu I - M_z^{\mathbf{T}})^{-1}\big)^{-1} g.$$

Since the operator $j(\mu I - M_z^{\mathbf{T}})^{-1}$ is bounded, the function g is analytic at the point μ. This proves 4). ☐

Since $(M_z^{\mathbf{T}} - \lambda I)^{-1}$ is a bounded operator for some λ, the operator $M_z^{\mathbf{T}}$ is closed.

1.2. Modeling 2-systems. Let H, R be Banach spaces, (A, J) a 2-system, $\mathcal{D}(A) \subset H$, $A : \mathcal{D}(A) \to H$, $J : \mathcal{D}(A) \to R$. We say that the system (A, J) is *observable* if $U_{A,J}\, x \equiv 0$ implies $x = 0$.

Proposition 1.2. *Suppose a system (A, J) is observable. Consider the space $\mathcal{H} = U_{A,J}\, H$ of functions on $\rho(A)$ with norm induced from H. Then $U_{A,J}$ is a linear isomorphism of H onto $\mathcal{H}$. Moreover,*

1)

$$\big(U_{A,J}(A - \lambda I)^{-1}\, {U_{A,J}}^{-1} f\big)(z) = \frac{f(z) - f(\lambda)}{z - \lambda}, \quad f \in \mathcal{H}, \;\; \lambda \in \rho(A); \tag{1.5}$$

2) *the space $\mathcal{H}$ is admissible*;

3) $U_{A,J}\,\mathcal{D}(A) = \mathcal{D}(M_z^{\mathbf{T}}\lceil\mathcal{H})$, *and* $U_{A,J}$ *intertwines the* 2*-system* (A,J) *with the* 2*-system* $(M_z^{\mathbf{T}}, j)$:

$$U_{A,J}\,A\,U_{A,J}{}^{-1} = M_z^{\mathbf{T}}\lceil\mathcal{H}, \quad J\,U_{A,J}{}^{-1} = j; \tag{1.6}$$

4) $\Omega(\mathcal{H}) = \rho(A)$.

Proof. Assertion 1) follows from the Hilbert identity; this directly implies 2). By (1.4) and (1.5), we have

$$U_{A,J}(A-\lambda I)^{-1}\,U_{A,J}{}^{-1} = (M_z^{\mathbf{T}} - \lambda I)^{-1}, \tag{1.7}$$
$$U_{A,J}\,\mathcal{D}(A) = U_{A,J}(A-\lambda I)^{-1}H = (M_z^{\mathbf{T}} - \lambda I)^{-1}\mathcal{H} = \mathcal{D}(M_z^{\mathbf{T}}\lceil\mathcal{H}).$$

From (1.7) and assertions 1) and 2) of Proposition 1.1 we obtain 3) and 4). □

Definition. Let A and A_1 be operators in H and H_1, respectively, and let (A,J) and (A_1,J_1) be two 2-systems with the same space R. We shall say that an isomorphism $U : H \to H_1$ *intertwines* the operators A and A_1 and that the operators A and A_1 are *similar* if $U\mathcal{D}(A) = \mathcal{D}(A_1)$ and $UAU^{-1} = A_1$. If, moreover, $Jx = J_1Ux$ for all $x \in \mathcal{D}(A)$, we shall say that U intertwines the 2-system (A,J) with the 2-system (A_1,J_1) and that the 2-systems (A,J) and (A_1,J_1) are similar.

It is easy to check that 2-systems (A,J) and (A_1,J_1) are similar if and only if $\rho(A) = \rho(A_1)$ and the spaces $U_{A,J}\,H$ and $U_{A_1,J_1}H$ coincide as sets. Here we mention the paper [1], where in particular the week similarity of systems and the relationship of it with the transfer function were discussed, and [2], where the similarity of all minimal passive realizations of a given transfer function was treated.

We shall often use the following obvious fact (basically, it has been checked above). Let A and A_1 be closed operators and λ a fixed point in $\rho(A)$. An operator U intertwines A and A_1 if and only if $\lambda \in \rho(A_1)$ and $U(A-\lambda I)^{-1}U^{-1} = (A_1 - \lambda I)^{-1}$.

§2. The spaces $\mathcal{H}(\delta)$

Let $\Omega_{\rm int}$, $\Omega_{\rm ext}$ be two disjoint nonempty open sets in $\mathbb{C}$ such that $\Gamma = \partial\Omega_{\rm int} = \partial\Omega_{\rm ext}$ is a finite union of piecewise-smooth curves and $\mathbb{C} = \Omega_{\rm int}\cup\Gamma\cup\Omega_{\rm ext}$. We assume that each connected component of Γ is homeomorphic to the unit circle or the line; in the latter case both ends of this component are assumed to go to infinity. Then $\Omega_{\rm int}$ and $\Omega_{\rm ext}$ have finitely many connected components. We always assume that

$$(|z|+1)^{-1} \in L^2(\Gamma, |dz|). \tag{2.1}$$

If the domain $\Omega_{\rm int}$ is connected, then the Smirnov class $E^2(\Omega_{\rm int})$ consists of all functions f analytic in $\Omega_{\rm int}$ and such that $\sup_n \int_{\partial\Omega_n} |f|^2\,|dz| < \infty$ for some sequence of domains $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ with rectifiable boundary and with $\bigcup_n \Omega_n = \Omega_{\rm int}$. For the properties of the Smirnov classes E^p and the relevant notions, see [36, 16]. In the general case we define $E^2(\Omega_{\rm int})$ as the direct sum of the classes E^2 in the connected components of $\Omega_{\rm int}$. The functions in $E^2(\Omega_{\rm int})$ have nontangential boundary values almost everywhere on Γ. Equipped with the norm

$$\|f\|^2_{E^2(\Omega_{\rm int})} = \frac{1}{2\pi}\int_\Gamma |f(z)|^2\,|dz|,$$

the class $E^2(\Omega_{\rm int})$ is a Hilbert space. Sometimes, for simplicity, we assume that all components of $\Omega_{\rm int}$ are simply connected, although usually this assumption can be lifted.

Scalar or operator-valued functions on $\Omega_{\rm int}$ will be called *outer* or *inner* if they possess the corresponding properties on each component of $\Omega_{\rm int}$.

Let R be an auxiliary Hilbert space. We shall need the Hilbert spaces $L^2(\Gamma, R) = L^2(\Gamma, |dz|) \otimes R$, $E^2(\Omega_{\text{int}}, R) = E^2(\Omega_{\text{int}}) \otimes R$, and $E^2(\Omega_{\text{ext}}, R) = E^2(\Omega_{\text{ext}}) \otimes R$. The elements of the second and the third spaces are R-valued functions analytic in Ω_{int} (Ω_{ext}) and having nontangential boundary values a.e. [20]. The norm in $L^2(\Gamma, R)$, $E^2(\Omega_{\text{int}}, R)$, or $E^2(\Omega_{\text{ext}}, R)$ is given by the formula

$$\|f\|^2 = \frac{1}{2\pi} \int_\Gamma \|f(z)\|^2 \, |dz|.$$

The functions in $E^2(\Omega_{\text{int}}, R)$, $E^2(\Omega_{\text{ext}}, R)$ are identified with their boundary values on Γ. Thus, these two spaces can be regarded as closed subspaces of $L^2(\Gamma, R)$.

Let Ω be one of the domains Ω_{int}, Ω_{ext}. We put $\widehat{E}^2(\Omega, R) = E^2(\Omega, R)$ if $\mathbb{C} \setminus \Omega$ is an unbounded set, and

$$\widehat{E}^2(\Omega, R) = \{v \in E^2(\Omega, R) : v(\infty) = 0\}$$

if $\mathbb{C} \setminus \Omega$ is bounded. We orient the curves that constitute Γ in such a way that, under the movement along them, the domain Ω_{int} remain on the left. The boundary values on Γ of functions f defined on Ω_{int} (Ω_{ext}) will be denoted by f_i (f_e). If Ω_{int} is a disk or a half-plane, then the classes $\widehat{E}^2(\Omega_{\text{int}}, R)$ and $\widehat{E}^2(\Omega_{\text{int}}, R)$ coincide with the classical scalar or vector Hardy class H^2 in Ω_{int} [36].

The following assertion can easily be checked.

Proposition 2.1. *The space $L^2(\Gamma, R)$ splits into the direct sum*

$$L^2(\Gamma, R) = \widehat{E}^2(\Omega_{\text{int}}, R) \dotplus \widehat{E}^2(\Omega_{\text{ext}}, R). \tag{2.2}$$

The parallel projections onto the direct summands are the corresponding Cauchy integrals,

$$P_{\text{int}} f(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)\, dz}{z - \zeta}, \quad \zeta \in \Omega_{\text{int}},$$
$$P_{\text{ext}} f(\zeta) = -\frac{1}{2\pi i} \int_\Gamma \frac{f(z)\, dz}{z - \zeta}, \quad \zeta \in \Omega_{\text{ext}}$$

(*so that* $P_{\text{int}}^2 = P_{\text{int}}$, $P_{\text{ext}}^2 = P_{\text{ext}}$, $P_{\text{int}} + P_{\text{ext}} = I$).

Note that condition (2.1) guarantees the convergence of these integrals.

Consider also the domains $\overline{\Omega}_{\text{int}} = \{\bar z : z \in \Omega_{\text{int}}\}$, $\overline{\Omega}_{\text{ext}} = \{\bar z : z \in \Omega_{\text{ext}}\}$ and put $\overline{\Gamma} = \partial \overline{\Omega}_{\text{int}} = \partial \overline{\Omega}_{\text{ext}}$. We introduce the Cauchy duality between the Hilbert spaces $L^2(\Gamma, R)$ and $L^2(\overline{\Gamma}, R)$ by the formula

$$\langle f, g \rangle \overset{\text{def}}{=} \frac{1}{2\pi i} \int_\Gamma \langle f(z), g(\bar z) \rangle \, dz, \quad f \in L^2(\Gamma, R), \ \ g \in L^2(\overline{\Gamma}, R). \tag{2.3}$$

Proposition 2.2. *The decomposition*

$$L^2(\overline{\Gamma}, R) = \widehat{E}^2(\overline{\Omega}_{\text{ext}}, R) \dotplus \widehat{E}^2(\overline{\Omega}_{\text{int}}, R)$$

is dual to (2.2) *with respect to the pairing* (2.3). *In other words*, *for the annihilators of the spaces involved in* (2.2) *we have*

$$\widehat{E}^2(\Omega_{\text{int}}, R)^\perp = \widehat{E}^2(\overline{\Omega}_{\text{int}}, R), \quad \widehat{E}^2(\Omega_{\text{ext}}, R)^\perp = \widehat{E}^2(\overline{\Omega}_{\text{ext}}, R). \tag{2.4}$$

In accordance with this assertion, we may identify

$$\widehat{E}^2(\Omega_{\text{int}}, R)^* = L^2(\overline{\Gamma}, R) \Big/ \widehat{E}^2(\overline{\Omega}_{\text{int}}, R)^\perp = \widehat{E}^2(\overline{\Omega}_{\text{ext}}, R), \tag{2.5}$$

$$\widehat{E}^2(\Omega_{\text{ext}}, R)^* = \widehat{E}^2(\overline{\Omega}_{\text{int}}, R), \tag{2.6}$$

i.e., the pairs $\big(\widehat{E}^2(\Omega_{\text{int}}, R), \widehat{E}^2(\overline{\Omega}_{\text{ext}}, R)\big)$ and $\big(\widehat{E}^2(\Omega_{\text{ext}}, R), \widehat{E}^2(\overline{\Omega}_{\text{int}}, R)\big)$ are pairs of dual Hilbert spaces with respect to the duality (2.3).

We define the operator $M_z = M_{z,\mathrm{int}}$ of multiplication by the independent variable on $\widehat{E}^2(\Omega_{\mathrm{int}}, R)$ by

$$\big(M_z f\big)(z) = zf(z), \quad f \in \mathcal{D}(M_z) \subset \widehat{E}^2(\Omega_{\mathrm{int}}, R),$$
$$\mathcal{D}(M_z) \stackrel{\mathrm{def}}{=} \big\{f \in \widehat{E}^2(\Omega_{\mathrm{int}}, R) : zf \in \widehat{E}^2(\Omega_{\mathrm{int}}, R)\big\}.$$

In a similar way, the operator $M_{z,\overline{\mathrm{int}}}$ is defined on $\widehat{E}^2(\overline{\Omega}_{\mathrm{int}}, R)$. These operators are bounded if and only if the domain Ω_{int} is bounded.

Obviously, the spaces $E^2(\Omega_{\mathrm{ext}}, R)$ and $E^2(\overline{\Omega}_{\mathrm{ext}}, R)$ are admissible; the operators $M_z^{\mathbf{T}}$ corresponding to them will be denoted by $M_{z,\mathrm{out}}^{\mathbf{T}}$ and $M_{z,\overline{\mathrm{ext}}}^{\mathbf{T}}$. The operators $M_{z,\mathrm{int}}$, $M_{z,\overline{\mathrm{int}}}$, $M_{z,\mathrm{out}}^{\mathbf{T}}$, and $M_{z,\overline{\mathrm{ext}}}^{\mathbf{T}}$ have dense domains and are closed.

From the formulas

$$\big((M_{z,\overline{\mathrm{int}}} - \lambda I)^{-1}\big)^* = M^*_{(z-\lambda)^{-1},\overline{\mathrm{int}}} = \big(M_{z,\mathrm{out}}^{\mathbf{T}} - \lambda I\big)^{-1} \tag{2.7}$$

(which can easily be checked; see (1.4)) it follows that

$$M^*_{z,\overline{\mathrm{int}}} = M_{z,\mathrm{out}}^{\mathbf{T}}.$$

Remark. Suppose that the geometry of the domain Ω_{ext} is such that

$$\|(z-\lambda)^{-1}\|_{L^2(\Gamma)} \to 0 \quad \text{for } \lambda \in \Omega_{\mathrm{ext}},\ \mathrm{dist}(\lambda, \Gamma) \to \infty. \tag{2.8}$$

From the Cauchy formula it follows that $|f(\lambda)| \le \|f\|_{E^2(\Omega_{\mathrm{ext}},R)}\|(z-\lambda)^{-1}\|_{L^2(\Gamma)}$ for any $f \in \widehat{E}^2(\Omega_{\mathrm{ext}}, R)$. Hence,

$$\lim_{\substack{z\in\Omega_{\mathrm{ext}}\\ \mathrm{dist}(z,\Gamma)\to\infty}} |f(z)| = 0, \quad f \in \widehat{E}^2(\Omega_{\mathrm{ext}}, R).$$

Relation (1.1) applied to $\mathcal{H} = \widehat{E}^2(\Omega_{\mathrm{ext}}, R)$ yields

$$\mathcal{D}(M_{z,\mathrm{out}}^{\mathbf{T}}) = \big\{f \in \widehat{E}^2(\Omega_{\mathrm{ext}}, R) : \\ \exists \lim_{\substack{z\in\Omega_{\mathrm{ext}}\\ \mathrm{dist}(z,\Gamma)\to\infty}} zf =: jf \text{ and } M_{z,\mathrm{out}}^{\mathbf{T}} f = zf - jf \in \widehat{E}^2(\Omega_{\mathrm{ext}}, R)\big\}.$$

Note that condition (2.8) follows, for instance, from the Carleson condition

$$\mathrm{length}(\Gamma \cap B(\lambda, r)) \le Cr,$$

where C is an absolute constant. Therefore, usually (2.8) is fulfilled in applications.

Definitions. Let δ be a function belonging to $H^\infty\big(\Omega_{\mathrm{int}}, \mathcal{L}(R_1, R_2)\big)$.

1) We say that δ is *admissible* if there exists a constant $\epsilon > 0$ such that $\|\delta(\lambda)r\| \ge \epsilon\|r\|$ for all $r \in R_1$ and almost all $\lambda \in \Gamma$.

2) Consider the function $\delta^{\mathbf{T}} \in H^\infty\big(\overline{\Omega}_{\mathrm{int}}, \mathcal{L}(R_2, R_1)\big)$ defined by

$$\delta^{\mathbf{T}}(z) \stackrel{\mathrm{def}}{=} \delta^*(\bar z), \quad z \in \overline{\Omega}_{\mathrm{int}}.$$

We say that δ is $*$-*admissible* if the function $\delta^{\mathbf{T}} \in H^\infty\big(\overline{\Omega}_{\mathrm{int}}, \mathcal{L}(R_2, R_1)\big)$ is admissible, which is equivalent to the relation $\delta(\lambda)\delta^*(\lambda) \ge \epsilon I$ for some $\epsilon > 0$ and almost all $\lambda \in \Gamma$.

3) We say that δ is *two-sided admissible* if it is both admissible and $*$-admissible, which means that δ^{-1} exists a.e. on Γ and $\|\delta^{-1}\| \le C$ a.e. on Γ.

4) We say that δ is *contractive* if $\|\delta(z)\| \le 1$ for all $z \in \Omega_{\mathrm{int}}$.

Note that the functions in $H^\infty\big(\Omega_{\mathrm{int}}, \mathcal{L}(R_1, R_2)\big)$ have nontangential limits a.e. on Γ in the sense of strong operator convergence (see [20, §V.2]).

We shall say that the simply connected case occurs if all connected components of Ω_{int} are simply connected, and that the multiply connected case occurs if some of these components are multiply connected. We need the following result, which is a consequence of the theorem of Beurling–Lax–Halmos in the simply connected case, and of the theorem of Voichick–Hasumi [69, 41] in the multiply connected case.

Theorem A. *A subspace G of the space $\widehat{E}^2(\Omega_{\mathrm{int}}, R)$ is invariant under the operators $M_{(z-\lambda)^{-1}}$, $\lambda \in \Omega_{\mathrm{ext}}$, if and only if there exists a Hilbert space R_* and an admissible function $\varphi \in H^\infty\big(\overline{\Omega}_{\mathrm{int}}, \mathcal{L}(R_*, R)\big)$ such that*

$$G = \varphi \cdot \widehat{E}^2(\overline{\Omega}_{\mathrm{int}}, R_*). \tag{2.9}$$

In the simply connected case, it suffices to restrict (2.9) to the inner functions φ (see the definition in §5). However, (2.9) determines a closed invariant subspace for any φ that satisfies the assumptions of the theorem.

Let δ be a $*$-admissible function in $H^\infty\big(\Omega_{\mathrm{int}}, \mathcal{L}(R, R_*)\big)$. We set

$$\mathcal{H}(\delta) = \mathcal{H}(\delta, \Omega_{\mathrm{int}}) = \big\{ f \in \widehat{E}^2(\Omega_{\mathrm{ext}}, R) : \delta \cdot f|\Gamma \in \widehat{E}^2(\Omega_{\mathrm{int}}, R_*) \big\}.$$

The space $\mathcal{H}(\delta)$ is a closed subspace of $\widehat{E}^2(\Omega_{\mathrm{ext}}, R)$. It is admissible as a space of functions on Ω_{ext}. In accordance with §1, the operator $M_z^{\mathbf{T}}$ is defined on $\mathcal{H}(\delta)$. It will play the role of the main model operator. It is an analog of the Sz.-Nagy–Foiaş model operator (see §§3, 5). In the case of an unbounded domain Ω_{int}, the operator $M_z^{\mathbf{T}}$ may be unbounded.

Definition [13]. The set of all $\lambda \in \operatorname{clos} \Omega_{\mathrm{int}}$ such that $\delta^{-1} \notin H^\infty\big(\Omega_{\mathrm{int}} \cap \mathcal{W}, \mathcal{L}(R_*, R)\big)$ for any neighborhood $\mathcal{W}$ of λ will be called the *spectrum of the function* δ. It will be denoted by $\operatorname{spec} \delta$.

The spectrum of δ is contained in $\operatorname{clos} \Omega_{\mathrm{int}}$ and is closed. It may fill the entire set $\operatorname{clos} \Omega_{\mathrm{int}}$. Its intersection with Ω_{int} coincides with the set of $\lambda \in \Omega_{\mathrm{int}}$ for which the operator $\delta(\lambda)$ is not invertible.

Item 2) of the next assertion is a direct analog of the theorem in [20, VI.4.1] and can be deduced from it in the simply connected case.

Proposition 2.3. 1) $\Omega(\mathcal{H}(\delta)) = \mathbb{C} \setminus \operatorname{spec} \delta$;
2) $\sigma\big(M_z^{\mathbf{T}} \lceil \mathcal{H}(\delta)\big) = \operatorname{spec} \delta$;
3) $\sigma_p\big(M_z^{\mathbf{T}} \lceil \mathcal{H}(\delta)\big) = \{\lambda \in \Omega_{\mathrm{int}} : \operatorname{Ker} \delta(\lambda) \neq 0\}$. *Here $\sigma_p(A)$ is the point spectrum of A.*

For convenience, we postpone the proof of this proposition until §4. Here we only prove the following fact.

Lemma 2.4. $\Omega(\mathcal{H}(\delta)) \supset \mathbb{C} \setminus \operatorname{spec} \delta$.

Proof. Let $g_{\mathrm{ext}} \in \mathcal{H}(\delta)$; then $\delta \cdot g_{\mathrm{ext}}|\Gamma = g_{\mathrm{int}}$ for some $g_{\mathrm{int}} \in \widehat{E}^2(\Omega_{\mathrm{int}}, R_*)$. We identify the element g_{ext} with the function

$$g(z) = \begin{cases} g_{\mathrm{ext}}(z) & \text{if } z \in \Omega_{\mathrm{ext}}, \\ \delta^{-1}(z) g_{\mathrm{int}}(z) & \text{if } z \in \Omega_{\mathrm{int}} \setminus \operatorname{spec} \delta. \end{cases} \tag{2.10}$$

Under this agreement, the functions in $\mathcal{H}(\delta)$ are analytic in Ω_{ext} and in $\Omega_{\mathrm{int}} \setminus \operatorname{spec} \delta$ and satisfy $g_i = g_e$ on $\Gamma \setminus \operatorname{spec} \delta$. If $\lambda \in \Gamma \setminus \operatorname{spec} \delta$ and $\mathcal{W}$ is a small disk centered at λ, then the restriction of g to $\mathcal{W} \setminus \Gamma$ lies in $E^2(\Omega_{\mathrm{ext}} \cap \mathcal{W}, R)$ and in $E^2(\Omega_{\mathrm{int}} \cap \mathcal{W}, R)$; moreover, the values of these restrictions on Γ coincide a.e. From the Cauchy integral formula it follows that $g|\mathcal{W} \setminus \Gamma$ is a restriction of an analytic function defined on $\mathcal{W}$, which allows us to define $g(\lambda)$ also for $\lambda \in \Gamma \setminus \operatorname{spec} \delta$. We have “extended” the function g_{ext} up to an analytic function g on $\mathbb{C} \setminus \operatorname{spec} \delta$.

After this extension, obviously, we have $\frac{g(z)-g(\lambda)}{z-\lambda} \in \mathcal{H}(\delta)$ if $g \in \mathcal{H}(\delta)$ and $\lambda \in \mathbb{C} \setminus \operatorname{spec} \delta$, and the claim follows. □

In the sequel, when it is convenient, we shall use the extension (2.10) and view the elements of $\mathcal{H}(\delta)$ as analytic functions on $\mathbb{C} \setminus \operatorname{spec}\delta$. This extension will be called a *pseudocontinuation* (usually, this term is used in a narrower sense). If the set $\mathbb{C} \setminus \operatorname{spec}\delta$ is connected, then this is the usual analytic continuation.

Proposition 2.5. $\mathcal{H}(\delta) = \big(\delta^{\mathbf{T}} \widehat{E}^2(\overline{\Omega}_{\mathrm{int}}, R_*)\big)^{\perp}$ *with respect to the duality* (2.3).

Proof. This follows directly from (2.3) and (2.4). □

§3. A theorem on the model

Definitions. 1. Let (A, J) be a 2-system, where H and R are Hilbert spaces. Suppose that $\sigma(A) \subset \operatorname{clos}\Omega_{\mathrm{int}}$. The operator J is said to be *admissible* for A if

$$\| U_{A,J}\, x\|_{E^2(\Omega_{\mathrm{ext}},R)} \le C\|x\|, \quad x \in H.$$

2. The operator J is *exact* with respect to A if we have the two-sided estimate $\| U_{A,J}\, x\|_{E^2(\Omega_{\mathrm{ext}},R)} \asymp C\|x\|$, $x \in H$.

Note that if A is a generator of a C_0-semigroup $\{T(t)\}$ and $\Omega_{\mathrm{int}} = \{z : \operatorname{Re} z < 0\}$, then, by the Parseval identity,

$$\| U_{A,J}\, x_0\|^2_{E^2(\Omega_{\mathrm{ext}},R)} = \int_0^\infty \|JT(t)x_0\|^2\, dt$$

for all $x_0 \in \mathcal{D}(A)$. Therefore, we have $\|y\|_{L^2((0,\infty),R)} = \| U_{A,J}\, x_0\|$, where y is the output of the system $\dot{x}(t) = Ax(t)$, $y(t) = Jx(t)$ $(t \ge 0)$, $x(0) = x_0$.

In a similar way, in the case where $\Omega_{\mathrm{int}} = \mathbb{D} = \{z : |z| < 1\}$, the formula

$$\| U_{A,J}\, x\|^2_{E^2(\Omega_{\mathrm{ext}},R)} = \sum_{n=0}^\infty \|JA^n x\|^2$$

relates the terms introduced above to the theory of discrete systems. The properties of being admissible and exact for the unit disk and half-plane were introduced and analyzed, e.g., in [70, 71, 42, 43, 67, 51] from the point of view of system theory, in particular, in connection with the so-called Weiss hypothesis.

Obviously, if J is exact, then it is both admissible and observable.

Theorem 3.1. *Suppose that* Ω_{int}, Ω_{ext}, *and* Γ *satisfy the conditions of* §2.

1) *Let* (A, J) *be a* 2*-system, and let* $\sigma(A) \subset \operatorname{clos}\Omega_{\mathrm{int}}$. *If the operator* J *is exact with respect to* A, *then there exists a Hilbert space* R_* *and a* $*$*-admissible function* $\delta \in H^\infty(\Omega_{\mathrm{int}}, \mathcal{L}(R, R_*))$ *such that*

$$U_{A,J} \,:\, H \to \mathcal{H}(\delta)$$

is a linear isomorphism with $U_{A,J}\, \mathcal{D}(A) = \mathcal{D}\big(M_z^{\mathbf{T}} \lceil \mathcal{H}(\delta)\big)$ *and*

$$U_{A,J}\, A\, U_{A,J}{}^{-1} f = M_z^{\mathbf{T}} f, \quad f \in \mathcal{D}\big(M_z^{\mathbf{T}} \lceil \mathcal{H}(\delta)\big).$$

2) *Conversely, suppose that* $A : \mathcal{D}(A) \to H$, *where* $\mathcal{D}(A) \subset H$ *is a linear* (*possibly, unbounded*) *operator in* H, $\delta \in H^\infty\big(\Omega_{\mathrm{int}}, \mathcal{L}(R)\big)$ *is a* $*$*-admissible function, and* $W : H \to \mathcal{H}(\delta)$ *is a linear isomorphism intertwining* A *with* $M_z^{\mathbf{T}}$ *on* $\mathcal{H}(\delta)$. *Then there exists an operator* $J : \mathcal{D}(A) \to R$ *exact with respect to* A *and such that* $W = U_{A,J}$.

Thus, under the assumptions of item 1) of the theorem, the operator $M_z^{\mathbf{T}}$ on $\mathcal{H}(\delta)$ serves as a model for the operator A, and the 2-system (A, J) is similar to the 2-system $(M_z^{\mathbf{T}}, j)$ on $\mathcal{H}(\delta)$.

Proof of Theorem 3.1. We use Proposition 2.3. 1) Since the system (A, J) is exact, the image $U_{A,J}\,H$ is a closed subspace of $\widehat{E}^2(\Omega_{\text{ext}}, R)$. Fix $\lambda \in \Omega_{\text{ext}}$. By Proposition 1.2, $U_{A,J}\,H$ is invariant under the operator $(M_z^{\mathbf{T}} - \lambda I)^{-1} f(z) = \frac{f(z)-f(\lambda)}{z-\lambda}$. We shall think of $\widehat{E}^2(\overline{\Omega}_{\text{int}}, R)$ as of the space dual to $\widehat{E}^2(\Omega_{\text{ext}}, R)$ (see (2.3), (2.6)). From (2.7) it follows that the annihilator $\big(U_{A,J}\,H\big)^{\perp}$ (which is a subspace of $\widehat{E}^2(\Omega_{\text{int}}, R)$) is invariant under $M_{(z-\lambda)^{-1}}$, $\lambda \in \Omega_{\text{ext}}$. Consequently, there is a Hilbert space R_* and a function $\varphi \in H^\infty\big(\overline{\Omega}_{\text{int}}, \mathcal{L}(R_*, R)\big)$ satisfying the assumptions of Theorem A and such that

$$\big(U_{A,J}\,H\big)^{\perp} = \varphi \cdot \widehat{E}^2(\Omega_{\text{int}}, R_*).$$

Therefore, $U_{A,J}\,H = \big(\varphi \cdot \widehat{E}^2(\Omega_{\text{int}}, R_*)^{\perp}\big) = \mathcal{H}(\delta)$, where we have introduced the notation $\delta = \varphi^{\mathbf{T}} \in H^\infty(\Omega_{\text{int}}, \mathcal{L}(R, R_*))$.

Now we show that on $\Omega_{\text{int}} \setminus \sigma(A)$ the values of the functions $U_{A,J}\,x$, $x \in H$, coincide with the pseudoanalytic continuation of their values on Ω_{ext} (see §2). By Proposition 1.2, $U_{A,J}$ intertwines A with the operator $M_z^{\mathbf{T}}$. In particular, $\sigma(A) = \operatorname{spec}\delta$. Let $x \in H$ and $g = U_{A,J}\,x$. Formulas (1.5) and (1.6) imply that

$$g(\lambda) = j(\lambda - M_z^{\mathbf{T}})^{-1} g = J\,{U_{A,J}}^{-1}\,U_{A,J}(\lambda - A)^{-1}\,{U_{A,J}}^{-1}\,g = J(\lambda - A)^{-1}x = (U_{A,J}\,x)(\lambda)$$

for $\lambda \in \rho(A)$. Thus, the function $U_{A,J}\,x$ coincides with g also on $\Omega_{\text{int}} \setminus \sigma(A)$.

2) It suffices to set $J = jW$ and apply Proposition 2.3 and assertion 3) of Proposition 1.2. □

In the cases of contractions or dissipative operators with a special choice of the operator J, the model of Theorem 3.1 becomes the original Sz.-Nagy–Foiaş model (see §§5 and 6 below). Various presentations of this model can be found, e.g., in [20], [14], [32], [18], [60], [19], [30]. Note also that Theorem 3.1 is related to the work of A. V. Shtraus (see [22]) and to the paper [27] by Z. Arova.

In the case where Ω_{int} is a half-plane, Theorem 3.1 has close relationship with [80, Theorem 3.1]. The terminology and approach in [80] differs from ours.

Definitions. 1) Let an operator A be similar to $M_z^{\mathbf{T}}$ in the space $\mathcal{H}(\delta)$ constructed for the domain Ω_{int}. Then the function δ is called *a generalized characteristic function of A in the domain Ω_{int}*.

2) Let (A, J) be a 2-system and $U_{A,J}$ an isomorphism of H onto $\mathcal{H}(\delta)$. Then δ will be called *a generalized characteristic function of the system (A, J) in the domain Ω_{int}*.

Under the conditions of Definition 2), a pair (A, J) will be called a $C_{0\cdot}$-system in the domain Ω_{int}, and A will be called a $C_{0\cdot}$-operator. If δ is a two-sided admissible function, then (A, J) is called a C_{00}-system.

Our model allows us to define an $H^\infty(\Omega_{\text{int}})$-calculus for any $C_{0\cdot}$-operator in Ω_{int}. It is easily seen that this calculus is continuous in the sense of the weak topology in $H^\infty(\Omega_{\text{int}})$ and the weak operator topology in $\mathcal{L}(H)$. Hence, it does not depend on a specific choice of the model. We mention, e.g., the paper of McIntosh [57], where the relationship between the existence of an H^∞-calculus and some special quadratic estimates for generators of analytic semigroups was investigated.

Remarks. 1) If (A, J) is a C_{00}-system, then sometimes the domain Ω_{int} can be varied so that the generalized characteristic function δ will remain the same (see examples in §§6, 8). This property turns out to be important.

2) Consider the following conditions: (a) R is finite-dimensional and $\sigma(A) \neq \operatorname{clos}\Omega_{\text{int}}$; (b) $\sigma(A) \cap \Gamma$ has zero length; (c) δ admits a scalar multiple, which means that there exists a function $\delta_1 \in H^\infty\big(\Omega_{\text{int}}, \mathcal{L}(R_*, R)\big)$ and a scalar function $\psi \in H^\infty(\Omega_{\text{int}})$, $\psi \not\equiv 0$,

such that $\delta\delta_1 \equiv \psi I$ and $\delta_1\delta \equiv \psi I$. In each of these cases the generalized characteristic function δ is automatically two-sided admissible.

3) Note that in our construction it is possible to replace the spaces $L^2(\Gamma, R)$ by similar weighted spaces such that the assertions of Propositions 2.1 and 2.2 are fulfilled. Theorem 3.1 for the case of a half-plane is related to G. M. Gubreev's work on quasiexponentials (see [5]).

The nonuniqueness of the choice of a generalized characteristic function will be discussed in §11. The problem of calculation of the generalized characteristic function will be considered in §9. It should be emphasized that, by Theorem 3.1, the inclusion of an operator into any exact system (A, J) yields a model of A. In this connection, we observe the following two obvious properties.

(1) Let J be an admissible operator for A. Suppose that $K : R \to L$ is a bounded operator. Then KJ is admissible for A.

(2) If $J_k : \mathcal{D}(A) \to R_k$, $k = 1, 2$, and J_1, J_2 are admissible for A, then $J_1 \oplus J_2$ is also admissible for A. If J_2 is admissible and J_1 is exact (with respect to A), then $J_1 \oplus J_2$ is exact with respect to A.

§4. Proof of Proposition 2.3. Duality of model spaces

Let $\delta \in H^\infty(\Omega_{\text{int}}, \mathcal{L}(R, R_*))$ be a $*$-admissible function. Consider the quotient operator $M_{z,\delta^{\mathbf{T}}}$ of M_z acting in the quotient space $\widehat{E}^2(\Omega_{\text{int}}, R)/\delta^{\mathbf{T}} E^2(\overline{\Omega}_{\text{int}}, R_*)$:

$$\mathcal{D}(M_{z,\delta^{\mathbf{T}}}) \overset{\text{def}}{=} \big\{\hat{f} \in \widehat{E}^2(\Omega_{\text{int}}, R)/\delta^{\mathbf{T}} \widehat{E}^2(\Omega_{\text{int}}, R_*) : \exists f \in \hat{f} : zf \in \widehat{E}^2(\Omega_{\text{int}}, R_*)\big\},$$

$$M_{z,\delta^{\mathbf{T}}} \hat{f} = \widehat{zf}, \quad \hat{f} \in \mathcal{D}(M_{z,\delta^{\mathbf{T}}}); \tag{4.1}$$

here $\hat{f} = f + \delta^{\mathbf{T}} E^2(\overline{\Omega}_{\text{int}}, R_*)$ is the coset of f. It is easily seen that $\big(M_{z,\delta^{\mathbf{T}}} - \lambda\big)^{-1} \hat{f} = \big((z-\lambda)^{-1} f\big)\widehat{\ }$, e.g., for all $\lambda \in \overline{\Omega}_{\text{ext}}$.

From Proposition 2.5 it follows that the Cauchy duality (2.3) determines a duality between $\mathcal{H}(\delta)$ and the quotient space $E^2(\Omega_{\text{int}}, R)/\delta^{\mathbf{T}} E^2(\overline{\Omega}_{\text{int}}, R_*)$. By (2.7), we have

$$\big(M_z \lceil \mathcal{H}(\delta)\big)^* = M_{z,\delta^{\mathbf{T}}} \tag{4.2}$$

with respect to this duality. We see that, in the case where Ω_{int} has several connected components $\Omega^1_{\text{int}}, \dots, \Omega^s_{\text{int}}$, the principal model operator $M_z^{\mathbf{T}}$ in the space $\mathcal{H}(\delta) = \mathcal{H}(\delta, \Omega_{\text{int}})$ is similar to the direct sum of the operators $M_z^{\mathbf{T}}$ in the spaces $\mathcal{H}(\delta, \Omega^j_{\text{int}})$.

By (4.2), any $C_{0\cdot}$-operator A in Ω_{int} satisfies the von Neumann inequality

$$\|r(A)\| \le C \sup_{\Omega_{\text{int}}} |r|$$

for any rational function r with poles off $\operatorname{clos} \Omega_{\text{int}}$. As is well known, the question as to whether the von Neumann inequality for $\Omega_{\text{int}} = \mathbb{D}$ implies similarity to a contraction was recently solved in the negative by Pisier [64]. In a recent paper [45], a negative answer was obtained to a certain question of V. Peller about power-bounded operators. Positive results about similarity to a contraction are contained in the book [20], in the papers of Paulsen on completely bounded operators [62], etc. We mention also the papers [18], [4], [6], [8], [9], [10], [12], where various criteria were obtained for similarity to unitary and selfadjoint operators.

Proof of Proposition 2.3. To check 2), it suffices to show that $\operatorname{spec} \delta^{\mathbf{T}} \subset \sigma(M_{z,\delta^{\mathbf{T}}})$. The inclusion $\operatorname{spec} \delta^{\mathbf{T}} \cap \overline{\Omega}_{\text{int}} \subset \sigma(M_{z,\delta^{\mathbf{T}}}) \cap \overline{\Omega}_{\text{int}}$ can be verified directly. Now we prove that $\operatorname{spec} \delta^{\mathbf{T}} \cap \overline{\Gamma} \subset \sigma(M_{z,\delta^{\mathbf{T}}}) \cap \overline{\Gamma}$. Let $\lambda_0 \in \overline{\Gamma}$ and $\lambda_0 \notin \sigma(M_{z,\delta^{\mathbf{T}}})$. Let $\mathcal{W}$ be a neighborhood of the point λ_0 such that $\operatorname{clos}\mathcal{W} \cap \sigma(M_{z,\delta^{\mathbf{T}}}) = \varnothing$. We choose an arbitrary function $y \in E^2(\overline{\Omega}_{\mathrm{int}}, R)$. Then the equation

$$(z-\lambda)x_\lambda = y + \delta^{\mathbf{T}} s_\lambda \tag{4.3}$$

has an analytic family of solutions $\{x_\lambda\} \subset E^2(\overline{\Omega}_{\mathrm{int}}, R)$, $\{s_\lambda\} \subset E^2(\overline{\Omega}_{\mathrm{int}}, R_*)$ defined for $\lambda \in \mathcal{W}$. It is easily seen that $\|s_\lambda\| \le C\|y\|$. We denote by α_λ the norm of the functional $\varphi \to \varphi(\lambda)$, $\varphi \in E^2(\overline{\Omega}_{\mathrm{int}})$. Substituting $z = \lambda$ in (4.3), we obtain $s_\lambda(\lambda) = -\delta^{\mathbf{T}-1}(\lambda) y(\lambda)$, $\lambda \in \mathcal{W} \cap \overline{\Omega}_{\mathrm{int}}$ (observe that the function $\delta^{\mathbf{T}}$ is invertible in $\mathcal{W} \cap \overline{\Omega}_{\mathrm{int}}$). Now we set $y = y_{r,\mu}$ in (4.3), where $y_{r,\mu}$ satisfies the relations $y_{r,\mu}(\lambda) = r$, $\|r\| = \alpha_\mu \|y_{r,\mu}\|$; let $\{x_{r,\mu,\lambda}\}$, $\{s_{r,\mu,\lambda}\}$ be the corresponding families of solutions of (4.3). Taking $\lambda = \mu$, we obtain

$$\|\delta^{\mathbf{T}-1}(\lambda) r\| = \|s_{r,\lambda,\lambda}(\lambda)\| \le \alpha_\lambda \|s_{r,\lambda,\lambda}\| \le C\alpha_\lambda \|y_{r,\lambda}\| = C\|r\|, \quad \lambda \in \mathcal{W} \cap \overline{\Omega}_{\mathrm{int}},$$

which means that $\|\delta^{\mathbf{T}-1}(\lambda)\| \le C$ for $\lambda \in \mathcal{W} \cap \overline{\Omega}_{\mathrm{int}}$. In particular, $\lambda_0 \notin \operatorname{spec} \delta^{\mathbf{T}}$. Thus, we have checked statement 2). Statement 1) is a consequence of it. Statement 3) follows from the fact that the eigenvectors of the operator $M_z^{\mathbf{T}} \lceil \mathcal{H}(\delta)$ are exactly the vectors of the form $(z-\lambda)^{-1}c$, where $\lambda \in \Omega_{\mathrm{int}}$, $\delta(\lambda)c = 0$. □

We consider the case of a two-sided admissible function δ. Let

$$V : \mathcal{H}(\delta) \to \widehat{E}^2(\Omega_{\mathrm{int}}, R_*)/\delta \widehat{E}^2(\Omega_{\mathrm{int}}, R)$$

be the linear operator acting by the formula

$$Vg = \delta g | \Gamma + \delta \widehat{E}^2(\Omega_{\mathrm{int}}, R), \quad g \in \mathcal{H}(\delta).$$

Proposition 4.1. *If the function δ is two-sided admissible, then*
1) *the map V is an isomorphism*;
2) *V intertwines the operator $M_z^{\mathbf{T}} \lceil \mathcal{H}(\delta)$ with $M_{z,\delta}$.*

Proof. Let $y \in \widehat{E}^2(\Omega_{\mathrm{int}}, R)$. The equation $Vx = y + \delta \widehat{E}^2(\Omega_{\mathrm{int}}, R)$ for $x \in \mathcal{H}(\delta)$ has a unique solution $x = P_{\mathrm{ext}}\big(\delta^{-1} y | \Gamma\big)$, whence 1) follows directly.

By Proposition 2.3, $\mathcal{H}(\delta)$ is an admissible space of functions on $\mathbb{C} \setminus \operatorname{spec}\delta$. Hence, $(M_z^{\mathbf{T}} - \lambda)^{-1} g$ can be calculated by formula (1.4). This implies 2) immediately. □

Proposition 4.2. *If the function δ is two-sided admissible, then the spaces $\mathcal{H}(\delta)$ and $\mathcal{H}(\delta^{\mathbf{T}})$ are dual to each other with respect to the duality pairing*

$$\langle f, g\rangle_\delta = \frac{1}{2\pi i} \int_\Gamma \langle \delta(z) f_e(z), g_e(\bar z)\rangle \, dz, \quad f \in \mathcal{H}(\delta), \ \ g \in \mathcal{H}(\delta^{\mathbf{T}}).$$

Proof. By (2.6), with respect to the Cauchy duality we have the identification

$$\big(\widehat{E}^2(\Omega_{\mathrm{int}}, R_*)/\delta \widehat{E}^2(\Omega_{\mathrm{int}}, R)\big)^* = \big(\delta \widehat{E}^2(\Omega_{\mathrm{int}}, R)\big)^\perp = \mathcal{H}(\delta^{\mathbf{T}}).$$

If $f \in \mathcal{H}(\delta)$ and $g \in \mathcal{H}(\delta^{\mathbf{T}})$, then

$$\langle f, g\rangle_\delta = \langle Vf, g\rangle = \frac{1}{2\pi i} \int_\Gamma \langle \widetilde{f}(z), g(\bar z)\rangle \, dz,$$

where $\widetilde{f}$ is any representative of the coset Vf. The assertion follows from Proposition 2.5, 2). □

If δ is a two-sided admissible function, then $\dim R = \dim R_*$, and in many cases it is convenient to assume that $R = R_*$.

In general, it is impossible to find a two-sided inner function δ_1 such that $\mathcal{H}(\delta) = \mathcal{H}(\delta_1)$ and $\mathcal{H}(\delta^{\mathbf{T}}) = \mathcal{H}(\delta_1^{\mathbf{T}})$. So, the analysis of the model spaces $\mathcal{H}(\delta)$ and $\mathcal{H}(\delta^{\mathbf{T}})$ cannot be reduced to the case of two-sided inner functions.

Observe that the map $f + \delta\widehat{E}^2(\Omega_{\rm int}, R) \mapsto P_K f$ is an isomorphism of the quotient space $\widehat{E}^2(\Omega_{\rm int}, R_*)\big/\delta\widehat{E}^2(\Omega_{\rm int}, R)$ onto the standard model space $K = \widehat{E}^2(\Omega_{\rm int}, R_*) \ominus \delta\widehat{E}^2(\Omega_{\rm int}, R)$ (here P_K is the orthogonal projection onto K). This isomorphism transforms the quotient operator $M_{z,\delta}$ into the restricted shift operator $\varphi \mapsto P_K(z\varphi)$ in the space K (for an unbounded domain $\Omega_{\rm int}$, it is easier to talk about the resolvents of these operators). This gives a simple relationship between our model for the case of a two-sided admissible δ and a restricted shift operator. For a disk or a half-plane, the restricted shift operator is a special case of the model operator of Sz.-Nagy–Foiaş. There are many papers devoted to restricted shift operators. We mention, in particular, [14], [18], [30], [31], [47], [48], [60]. The book [13] by Nikolskiĭ is especially devoted to the spectral properties of these operators and applications. The restricted shift operators for multiply connected domains and other similar situations have also been studied intensively. We mention [15], [27], [58], and also the theory of operator vessels (see [25] and the book [55]).

§5. Relationship with the Szökefalvi-Nagy–Foiaş model

Here we discuss the general case of a $*$-admissible function δ.

Definitions [20, §V.2]**.** Suppose that all components of $\Omega_{\rm int}$ are simply connected. The function δ is said to be

1) *inner* if the operators $\delta(\lambda)$ are isometric for almost all $\lambda \in \Gamma$;
2) $*$-*inner* if the operators $\delta^*(\lambda)$ are isometric for almost all $\lambda \in \Gamma$;
3) *two-sided inner* if the operators $\delta(\lambda)$ are unitary for almost all $\lambda \in \Gamma$.

Clearly, any inner function is admissible, any $*$-inner function is $*$-admissible, and any two-sided inner function is two-sided admissible. Moreover, a function δ in the space $H^\infty(\Omega_{\rm int}, \mathcal{L}(R, R_*))$ is $*$-admissible if and only if it has a factorization of the form

$$\delta(z) = \delta^{(e)}(z)\delta^{(i)}(z), \quad z \in \Omega_{\rm int},$$

where $\delta^{(i)} \in H^\infty(\Omega_{\rm int}, \mathcal{L}(R, R_*))$ is $*$-inner and $\delta^{(e)}, \delta^{(e)\,-1} \in H^\infty(\Omega_{\rm int}, \mathcal{L}(R_*))$. (This follows from the results of [20, §V.4]. The above factorization is a special case of a $*$-canonical one and is unique up to the substitution $\delta^{(e)} \mapsto \delta^{(e)}u^{-1}$, $\delta^{(i)} \mapsto u\delta^{(i)}$, where u is a unitary constant.) Similar statements are true for factorizations of the inner and the two-sided inner functions.

Also, we recall that a contractive function $\delta \in H^\infty(\Omega_{\rm int}, \mathcal{L}(R, R_*))$ is said to be *pure* if $\|\delta(0)r\| < \|r\|$ for all $r \in R$, $r \neq 0$. Any contractive function $\delta \in H^\infty(\Omega_{\rm int}, \mathcal{L}(R, R_*))$ can uniquely be decomposed in a direct sum $\delta = \delta^{(0)} \oplus w$ (with respect to some orthogonal decompositions of the spaces R and R_*), where $\delta^{(0)}$ is a pure contractive function and w is a unitary constant; then the function $\delta^{(0)}$ is called the *pure part* of δ; see [20, §V.2].

Proposition 5.1. 1) *Let $\delta_j \in H^\infty(\Omega_{\rm int}, \mathcal{L}(R, R_j))$ be $*$-admissible functions, $j = 1, 2$. Then $\mathcal{H}(\delta_1) = \mathcal{H}(\delta_2)$ if and only if $\delta_2 = \psi \cdot \delta_1$, where the function ψ belongs to the space $H^\infty(\Omega_{\rm int}, \mathcal{L}(R_1, R_2))$ and satisfies $\|\psi^{-1}(z)\| \le C$, $z \in \Omega_{\rm int}$.*

2) *Suppose that, moreover, all components of $\Omega_{\rm int}$ are simply connected. Let $\delta_1 = \delta_1^{(e)}\delta_1^{(i)}$, $\delta_2 = \delta_2^{(e)}\delta_2^{(i)}$ be the $*$-canonical factorizations of the functions δ_1 and δ_2. Then $\mathcal{H}(\delta_1) = \mathcal{H}(\delta_2)$ if and only if $\delta_2^{(i)}(z) \equiv u\delta_1^{(i)}(z)$, $z \in \Omega_{\rm int}$, where u is a unitary constant.*

Proof. Statement 1) easily follows from Proposition 2.5, and 2) follows from 1) and the uniqueness of a $*$-canonical factorization. □

Suppose that the components of the domain $\Omega_{\rm int}$ are simply connected and are Smirnov domains [36], [16]. Let $\delta \in H^\infty(\Omega_{\rm int}, \mathcal{L}(R, R_*))$ be an arbitrary $*$-admissible function. We show how an analog of the operator V can be defined.

We assume (without loss of generality) that δ is a $*$-inner function, and set $\Delta = (I-\delta^*\delta)^{1/2} = I-\delta^*\delta$; then Δ is a projection-valued function on Γ of class $L^\infty(\Gamma,\mathcal{L}(R))$.

Let

$$\mathcal{H}_{SNF}(\delta,\Omega_{\rm int}) = \mathcal{H}_{SNF}(\Omega_{\rm int}) = \begin{pmatrix} \widehat{E}^2(\Omega_{\rm int},R_*) \\ \Delta L^2(\Gamma,R) \end{pmatrix} \Big/ \left(\begin{smallmatrix} \delta \\ \Delta \end{smallmatrix}\right) \widehat{E}^2(\Omega_{\rm int},R)$$

be a Sz.-Nagy–Foiaş model space. The coset of the function $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \widehat{E}^2(\Omega_{\rm int},R_*) \oplus \Delta L^2(\Gamma,R)$ will be denoted by $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\widehat{\ }$. The Smirnov class $\mathcal{N}(\Omega_{\rm int})$ is defined as the set of functions that are analytic in $\Omega_{\rm int}$ and have the form f/g, where $f,g\in H^\infty(\Omega_{\rm int})$ and g is outer.

If $\psi\in\mathcal{N}(\Omega_{\rm int})$, then we define the quotient multiplication operator $M_\psi\lceil\mathcal{H}_{SNF}(\Omega_{\rm int})$ and its domain by formulas similar to (4.1), so that

$$M_\psi\left(\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)\widehat{\ }\right) = \left(\left(\begin{smallmatrix} \psi f \\ \psi g \end{smallmatrix}\right)\widehat{\ }\right), \quad \text{where } \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right), \left(\begin{smallmatrix} \psi f \\ \psi g \end{smallmatrix}\right) \in \begin{pmatrix} \widehat{E}^2(\Omega_{\rm int},R_*) \\ \Delta L^2(\Gamma,R) \end{pmatrix}.$$

This operator is densely defined.

Now, we introduce an operator

$$\widetilde{V} \,:\, \mathcal{H}(\delta) \to \mathcal{H}_{SNF}(\Omega_{\rm int})$$

by the formula

$$\widetilde{V}f \stackrel{\rm def}{=} \left(\begin{smallmatrix} \delta \\ \Delta \end{smallmatrix}\right) f|\Gamma + \left(\begin{smallmatrix} \delta \\ \Delta \end{smallmatrix}\right) \widehat{E}^2(\Omega_{\rm int},R), \quad f\in\mathcal{H}(\delta)\subset\widehat{E}^2(\Omega_{\rm ext},R).$$

It is easy to check that $\widetilde{V}$ is an isomorphism, and for $\widetilde{V}^{-1}$ we have the relation

$$\widetilde{V}^{-1}\Big(\left(\begin{smallmatrix} g \\ u \end{smallmatrix}\right) + \left(\begin{smallmatrix} \delta \\ \Delta \end{smallmatrix}\right) \widehat{E}^2(\Omega_{\rm int},R)\Big) = P_{\rm ext}(\delta^* g + u).$$

The operator $\widetilde{V}$ intertwines the operator $M_z^{\mathbf{T}}$ on $\mathcal{H}(\delta)$ with the quotient operator M_z on $\mathcal{H}_{SNF}(\Omega_{\rm int})$.

The construction of the Sz.-Nagy–Foiaş model space in the domain $\Omega_{\rm int}$ can easily be generalized to the case of a multiply connected domain. Then the $*$-inner function δ becomes multivalued and character-automorphic. Generalizations like this were considered in the papers of Ball [29], McCullough [58], Pavlov [15], Fedorov [21], etc. It is easy to define the isomorphism $\widetilde{V}$ in this case.

It is easily seen that the quotient operator M_z on $\mathcal{H}_{SNF}(\Omega_{\rm int})$ is densely defined. Therefore, under the assumptions of Theorem 3.1, the operators A and $M_z^{\mathbf{T}}\lceil\mathcal{H}(\delta)$ are densely defined.

We remind the reader that an operator T on a Hilbert space H is called a *contraction* if $\|T\|\le 1$, and a $C_{0\cdot}$*-contraction* if, moreover, $\lim_n\|T^n x\| = 0$ for all $x\in H$.

The relationship between the model of §§2, 3 and the Sz.-Nagy–Foiaş model is described in the following.

Proposition 5.2. *An operator A is a $C_{0\cdot}$-operator in the disk $\mathbb{D}=\{|z|<1\}$ in the sense of §3 if and only if it is similar to a $C_{0\cdot}$-contraction.*

Proposition 5.3. *Suppose that a domain $\Omega_{\rm int}$ is connected and simply connected.*

1) *Let $\varphi:\mathbb{D}\to\Omega_{\rm int}$ be a conformal mapping. An operator A is a $C_{0\cdot}$-operator in the domain $\Omega_{\rm int}$ if and only if it is similar to the operator $\varphi(T)$ for some $C_{0\cdot}$-contraction T* (*the operator $\varphi(T)$* (*possibly, unbounded*) *is defined in the theory of Sz.-Nagy–Foiaş*).

2) *Let A be a $C_{0\cdot}$-operator in $\Omega_{\rm int}$, let $\delta\in H^\infty(\Omega_{\rm int},\mathcal{L}(R,R_*))$ be its generalized characteristic function, and $\delta^{(i0)}$ the pure part of its $*$-inner factor. Then the contraction T can be chosen so that $\delta^{(i0)}\circ\varphi$ will be the characteristic function of T in the sense of Sz.-Nagy–Foiaş.*

Proof of Propositions 5.2 *and* 5.3. If $\Omega_{\text{int}} = \mathbb{D}$, the quotient operator M_z on $\mathcal{H}_{SNF}(\mathbb{D})$ basically coincides with the model operator of Sz.-Nagy–Foiaş. Since by assumption the function δ is $*$-inner, $M_z\lceil\mathcal{H}_{SNF}(\mathbb{D})$ is a C_0.-contraction, and its characteristic function is the pure part of δ (see [20, Chapter VI]).

Now let Ω_{int} be an arbitrary simply connected domain. Let δ be a $*$-admissible function in Ω_{int} and $\delta^{(i)}$ its $*$-inner factor. Then the operator

$$W\begin{pmatrix} f \\ g \end{pmatrix}^{\frown} = \left(\begin{pmatrix} f\circ\varphi \\ g\circ\varphi \end{pmatrix} \cdot \varphi'^{\frac{1}{2}}\right)^{\frown}$$

is an isomorphism of $\mathcal{H}_{SNF}(\delta^{(i)}, \Omega_{\text{int}})$ onto $\mathcal{H}_{SNF}(\delta^{(i)} \circ \varphi, \mathbb{D})$. Set

$$T = M_z\lceil\mathcal{H}_{SNF}(\delta^{(i)} \circ \varphi, \mathbb{D}).$$

Then T is a C_0.-contraction, and W intertwines $M_z\lceil\mathcal{H}_{SNF}(\delta^{(i)}, \Omega_{\text{int}})$ with the operator $\varphi(T)$. This implies Proposition 5.3. $\square$

It is easy to formulate an assertion similar to Proposition 5.3 for the case where Ω_{int} consists of several simply connected components.

Now the relationship between Theorem 3.1 and the dilation theory of Sz-Nagy–Foiaş is clear: under the assumptions of the theorem the operator A admits a normal dilation up to similarity; the spectrum of this dilation is included in Γ.

Similarity to a Sz.-Nagy–Foiaş model operator established in Theorem 3.1 allows us to obtain an overwhelming majority of the known consequences of the model: a description of the commutant, invariant and hyperinvariant subspaces, and criteria of similarity to a normal operator. In connection with this, we note that positive, as well as negative, results on the similarity of a C_{00}-operator in a domain to a normal operator immediately follow from the results of Benamara–Nikolski [31], Vasyunin–Kupin [3], Kupin [47], and Kupin–Treil [48].

If an operator A has a two-sided admissible generalized characteristic function in Ω_{int}, then every generalized characteristic function of it is two-sided admissible. In the simply connected case this follows from Proposition 5.3, and in the general case, a few additional arguments will be necessary.

§6. Examples

6.1. Contractions. Let T be a C_0.-contraction in a Hilbert space H. The defect operator of T is the operator $D_T = (I - T^*T)^{1/2} \geq 0$. Set $\Omega_{\text{int}} = \mathbb{D} = \{\lambda : |\lambda| < 1\}$, $J = D_T$, $R = \mathcal{D}_T \overset{\text{def}}{=} \operatorname{clos} D_T H$. The following proposition is well known.

Proposition 6.1. *The system (T, J) is exact for the disk $\mathbb{D}$; moreover, $\|U_{T,J}x\| = \|x\|$ for any $x \in H$.*

The proof can be found, for instance, in the introductory lecture to the book [13] (theorem on the model). For the choice of Ω_{int} and J as above, the model obtained from Theorem 3.1 with the help of the isomorphism $\widetilde{V}$ coincides with the Sz.-Nagy–Foiaş model.

The next proposition is an easy consequence of Theorem 3.1 and Proposition 5.2.

Proposition 6.2. 1) *A 2-system (A, J) is a C_0.-system for the disk $\mathbb{D}$ if and only if it is similar to a system (T, D_T) for some C_0.-contraction T.*

2) *Let (A, J) be a C_0.-system in the disk $\mathbb{D}$, δ a generalized characteristic function of it, and $\delta_{(i0)}$ the pure part of the $*$-inner part of the function δ. Then the 2-system (A, J) is similar to a 2-system (T, D_T), where T is a C_0.-contraction with characteristic function $\delta_{(i0)}$ (in the sense of Sz.-Nagy–Foiaş).*

6.2. Dissipative operators. Let A be a maximal dissipative operator. In accordance with [20], with A we can associate the Cayley transform $T=(A-iI)(A+iI)^{-1}$, which is a contraction, and the c_0-semigroup $\{T(s)\}_{s\geq 0}$ of contractions the generator of which is the operator iA. We assume that $T\in C_{0\cdot}$; this condition is equivalent to the fact that $\text{s-lim}_{t\to\infty} T(t)=0$ (see [20, §3, Proposition 9.1]). For simplicity we assume that $A=A_r+iA_i$, where the operators A_r and A_i are selfadjoint, $A_i\geq 0$, and $\mathcal{D}(A_r)\subset\mathcal{D}(A_i)$. We put $J=(2\operatorname{Im}A)^{1/2}=(2A_i)^{1/2}$, $R=\operatorname{clos}\operatorname{Range}J$, and $\Omega_{\text{int}}=\mathbb{C}_+=\{\lambda:\operatorname{Im}\lambda>0\}$.

Proposition 6.3. *The system* (A,J) *is exact for the half-plane* $\mathbb{C}_+$; *moreover*, $\|U_{A,J}\,x\|=\|x\|$ *for all* $x\in H$.

Proof. Applying the results of [20, Chapter 1, §10] and the fact that the Laplace transform of the semigroup $\{T(t)\}$ is the resolvent $(\lambda-iA)^{-1}$, $\operatorname{Re}\lambda>0$, we see that

$$\begin{aligned}\|x\|^2&=\int_0^\infty\|JT(t)x\|^2\,dt+\lim_{t\to+\infty}\|T(t)x\|^2\\&=\int_0^\infty\|JT(t)x\|^2\,dt=\|J(\cdot-iA)^{-1}x\|^2_{E^2(\{\operatorname{Re}\lambda>0\},R)}=\|\,U_{A,J}\,x\|^2_{E^2(\mathbb{C}_+,R)}\end{aligned}$$

for all $x\in\mathcal{D}(A)$ (observe that $E^2(\mathbb{C}_+,R)=H^2(\mathbb{C}_+,R)$). So, $U_{A,J}$ extends up to an isometry from H to $H^2(\mathbb{C}_+,R)$ that acts by the same formula. □

Apparently, there is a relationship between our results applied to continuous linear systems (see the commentary at the beginning of §3, and also §§8, 9 and Subsection 10.2) and the results of [67].

6.3. Differentiation on an interval with nondissipative boundary conditions. Let $H=L^2\big([0,w]\to\mathbb{C}^n\big)\overset{\text{def}}{=}L^2$. Suppose that ρ is a complex matrix of size $n\times n$ and β is an $(n\times n)$-matrix-valued complex measure on $[0,w]$ such that $\beta(\{0\})=\beta(\{w\})=0$. Consider the operator

$$A\varphi=-\varphi',\tag{6.1}$$

$$\mathcal{D}(A)\overset{\text{def}}{=}\big\{\varphi\in W_2^1\big([0,w]\to\mathbb{C}^n\big):L\varphi=0\big\},\tag{6.2}$$

where

$$L\varphi=\varphi(0)-\int d\beta(x)\varphi(x)-\rho\cdot\varphi(w).$$

We show that Theorem 3.1 allows us to construct a C_{00}-model of the operator A^*. This means also (see §9) that a C_{00}-model of the operator A exists; in §10 it will be computed explicitly.

In the scalar case ($n=1$), consider an operator in $L^2([0,l])$ of the form $B\psi=\alpha\psi'+\gamma\psi$, where α and γ are continuous, $\alpha<0$, with the boundary condition as in (6.2). The isomorphism of $L^2([0,l])$ onto $L^2([0,w])$ defined by $\psi\mapsto f\cdot(\psi\circ\tau)$, where f and τ are appropriate functions, allows us to reduce the study of the operator B to that of A.

We put $B(x)=\beta([0,x])$; then the function B is continuous at the points 0 and w, and $B(0)=0$. The operator A^* is given by the relations

$$\begin{aligned}\mathcal{D}(A^*)=\big\{\psi\in L^2:\exists c\in\mathbb{C}^n:\psi(x)+B^{\mathbf{T}}(x)c\in W_2^1([0,w],\mathbb{C}^n),\\ c=\psi(0),\ \psi(w)=\rho^{\mathbf{T}}\cdot\psi(0)\big\},\end{aligned}$$

$$\big(A^*\psi\big)(x)\overset{\text{def}}{=}\big(\psi(x)+B^{\mathbf{T}}(x)\psi(0)\big)'.$$

Set $\Delta(\lambda)c = Le^{-\lambda x}c$, $c \in \mathbb{C}^n$. Then

$$\Delta(\lambda) = I - \int_{[0,w]} e^{-\lambda x}\, d\beta(x) - e^{-\lambda w}\rho. \tag{6.3}$$

Consider the operator

$$(Fr)(\lambda) = \int_{[0,w]} d\beta(x) \int_0^x e^{\lambda(t-x)} r(t)\, dt + \rho \int_0^w e^{\lambda(t-w)} r(t)\, dt, \quad r \in L^2.$$

Then $\sigma(A) = \big\{\lambda : \det \Delta(\lambda) = 0\big\}$, and

$$\big((\lambda - A)^{-1} r\big)(x) = e^{-\lambda x}\Delta(\lambda)^{-1}(Fr)(\lambda) + \int_0^x e^{\lambda(t-x)} r(t)\, dt, \quad \lambda \notin \sigma(A). \tag{6.4}$$

From this formula we obtain

$$\begin{aligned} &\big((\lambda - A^*)^{-1}\psi\big)(t) \\ &\quad = \int_t^w e^{\lambda(t-x)}\psi(x)\, dx \\ &\qquad + \left(\int_t^w e^{\lambda(t-y)}\, d\beta^*(y)dy + e^{\lambda(t-w)}\rho^*\right)\Delta^{\mathbf{T}}(\lambda)^{-1}\int_0^w e^{-\lambda x}\psi(x)\, dx. \end{aligned} \tag{6.5}$$

We set

$$J_*\psi = \psi(0), \quad \psi \in \mathcal{D}(A^*), \tag{6.6}$$

and consider the function

$$\delta(\lambda) = e^{\lambda w}\Delta(\lambda) = e^{\lambda w} I - \int e^{\lambda(w-x)}\, d\beta(x) - \rho, \quad \lambda \in \mathbb{C}. \tag{6.7}$$

By (6.3) and (6.5), the operator U_{A^*,J_*} has a simple form:

$$U_{A^*,J_*}\psi(\lambda) = \delta^{\mathbf{T}}(\lambda)^{-1}\int_0^w e^{\lambda(w-s)}\psi(s)\, ds. \tag{6.8}$$

We take an arbitrary number a such that $\det \Delta(z) \neq 0$ and $\|\Delta(z)^{-1}\| \le C$ for $\operatorname{Re} z \ge a$. Next, we put

$$\Omega_{\mathrm{int}} = \overline{\Omega}_{\mathrm{int}} = \{\operatorname{Re} z < a\}, \quad \Omega_{\mathrm{ext}} = \overline{\Omega}_{\mathrm{ext}} = \{\operatorname{Re} z > a\}.$$

It is easily seen that the matrix-valued functions δ and $\delta^{\mathbf{T}}$ are (two-sided) admissible for the domain Ω_{int}. This leads to the following statement.

Proposition 6.4. *(A^*, J^*) is a C_{00}-system in the domain Ω_{int}.*

Proof. From the Parseval identity, it follows that $\|\widetilde{\psi}\|_{E^2(\Omega_{\mathrm{ext}},\mathbb{C}^n)} \asymp \|\psi\|_{L^2}$, where $\widetilde{\psi}(\lambda) = \int_0^w e^{-\lambda s}\psi(s)\, ds$. Since $U_{A^*,J_*}\psi(\lambda) = \Delta^{\mathbf{T}}(\lambda)^{-1}\widetilde{\psi}(\lambda)$ and the functions $\Delta^{\mathbf{T}}$, $\Delta^{\mathbf{T}-1}$ belong to $H^\infty(\Omega_{\mathrm{ext}}, \mathbb{C}^{n\times n})$, we get

$$\|U_{A^*,J_*}\psi\|_{E^2(\Omega_{\mathrm{ext}},\mathbb{C}^n)} \asymp \|\psi\|_{L^2}.$$

$\square$

Since A^* has discrete spectrum, Theorem 3.1 implies that U_{A^*,J_*} is an isomorphism of L^2 onto the space $\mathcal{H}(\delta_1)$ for some function $\delta_1 \in H^\infty(\Omega_{\mathrm{int}}, \mathbb{C}^{n\times n})$. In §10 we shall show that we can take $\delta_1 = \delta^{\mathbf{T}}$.

6.4. Generators of systems with delay. For $h > 0$, we put

$$H = \mathbb{C}^n \oplus L^2([-h, 0] \to \mathbb{C}^n)$$

and assume that the operators $M : C([-h, 0], \mathbb{C}^n) \to \mathbb{C}^n$, $L : W_2^1([-h, 0], \mathbb{C}^n) \to \mathbb{C}^n$ can be represented as follows:

$$M\varphi = \int_0^h d\mu(\theta)\varphi(-\theta); \quad L\varphi = \int_0^h \zeta(\theta)\varphi(-\theta)\,d\theta + \int_0^h \eta(\theta)\dot{\varphi}(-\theta)\,d\theta.$$

Consider the unbounded operator A on H defined by

$$\mathcal{D}(A) = \big\{(c, \varphi) \in H : \varphi \in W_2^1([-h, 0], \mathbb{C}^n),\ c = M\varphi\big\}, \quad A(c, \varphi) = (L\varphi, \dot{\varphi}).$$

We assume that $\zeta, \eta \in L^2([0, h] \to \mathbb{C}^{n\times n})$, and that $d\mu$ is an $(n \times n)$-matrix-valued measure with $\mu(\{0\}) = I$. In this case, A is the generator of a c_0-semigroup associated with a neutral linear system with delay,

$$\begin{gathered}\frac{d}{dt}(\mu * x)(t) = (\zeta * x)(t) + (\eta * \dot{x})(t),\\ (\mu * x)(0) = c, \quad x(\theta) = \varphi(\theta), \ -h \le \theta \le 0.\end{gathered}$$

In [56] a function model of the operator A and the corresponding semigroup of operators were constructed. Now we show that the result of [56] can easily be deduced from the construction of the present paper.

As is well known, the function

$$\begin{aligned}\Delta(z) &= zMe^{z\cdot} - Le^{z\cdot}\\ &\overset{\text{def}}{=} z\bigg(\int_0^h e^{-z\theta}\,d\mu(\theta) - \int_0^h e^{-z\theta}\eta(\theta)\,d\theta\bigg) - \int_0^h e^{-z\theta}\zeta(\theta)\,d\theta\end{aligned}$$

plays the principal role in the spectral analysis of the operator A. In particular, $\sigma(A) = \big\{z : \det\Delta(z) = 0\big\}$. There exists a number $a \in \mathbb{R}$ such that $\|\Delta(z)^{-1}\| \le C(1 + |z|)^{-1}$ for $\operatorname{Re} z > a$. We fix this a and an arbitrary number b with $\operatorname{Re} b > a$. Let $\Omega_{\text{int}} = \{\operatorname{Re} z < a\}$, $\Omega_{\text{ext}} = \{\operatorname{Re} z > a\}$. The function

$$\delta(z) = (z - b)^{-1}e^{hz}\Delta(z)$$

is admissible for Ω_{int}. Consider the operator

$$J_0 : H \to \mathbb{C}^n, \quad J_0(d, \psi) = d$$

and put

$$\widehat{V}_1 y(z) = J_0(z - A^*)^{-1}y, \quad y \in H.$$

It is easily seen that

$$\widehat{V}_1(d, \psi)(z) = \Delta^{\mathbf{T}}(z)^{-1}\bigg[m^*(\bar{z})d + \int_{-h}^0 e^{zs}\psi(s)\,ds\bigg], \tag{6.9}$$

where

$$m(z) = \int_0^h e^{-z\theta}\,d\mu(\theta)$$

(see [56]). Since $\Delta^{\mathbf{T}}(z)^{-1} \to 0$ as $z \to \infty$, $z \in \partial\Omega_{\text{int}}$, the relation $\|\widehat{V}_1 y\|_{E^2(\Omega_{\text{ext}}, \mathbb{C}^n)} \asymp \|y\|_H$ is impossible. We define

$$J_* = J_0(A^* - \bar{b}), \quad J_* : \mathcal{D}(A^*) \to \mathbb{C}^n;$$

then

$$U_{A^*, J_*}y(z) = (z - \bar{b})\widehat{V}_1 y(z) - J_0 y. \tag{6.10}$$

Proposition 6.5. *The system (A^*, J_*) is exact.*

Proof. Let $y = (d, \psi) \in H$. Suppose that $U_{A^*,J_*}y(z) \equiv 0$ for all $z \in \rho(A^*)$. Putting $z = \bar{b}$ in (6.10), we obtain $d = J_0 y = 0$. By (6.9) and (6.10), the Laplace transform of the function ψ is identically equal to zero, whence $\psi \equiv 0$, $y = 0$. Thus, $\operatorname{Ker} U_{A^*,J_*} = 0$.

From (6.9) and (6.10) it follows that

$$U_{A^*,J_*}(d,\psi)(z) = \delta^{\mathbf{T}}(z)^{-1}\Big[\big(e^{hz}m^{\mathbf{T}}(z) - \delta^{\mathbf{T}}(z)\big)d + \int_{-h}^{0} e^{z(h+s)}\psi(s)\,ds\Big]$$

(cf. formula (1.15) in [56]). Now it is clear that $U_{A^*,J_*} : H \to E^2(\Omega_{\text{ext}}, \mathbb{C}^n)$ is a bounded operator. Obviously, $\|U_{A^*,J_*}(0,\psi)\|_{E^2(\Omega_{\text{ext}},\mathbb{C}^n)} \asymp \|\psi\|$. Since the vectors of the form $(0,\psi)$ constitute a subspace of finite codimension in H, we conclude that the system (A^*, J_*) is exact. $\square$

In §10 we shall show that $\delta^{\mathbf{T}}$ is a generalized characteristic function of the system (A^*, J_*).

§7. Relationship with control systems and the exact controllability

The relationship of function models with control theory is well known; see [28], [30], [39], [61]. In this section we briefly elucidate the relationship between $C_{0\cdot}$-operators and the exact controllability. For more details about the exact controllability, we refer the reader to the papers [59], [78], the books [28], [34], [46], [54], [61], etc. In the books [54], [46], [78], in particular, a method of proving the exact controllability was presented for specific systems given by partial differential equations (the so-called HUM, the method of Lions). In [59, 61] the Sz.-Nagy–Foiaş function model was applied systematically to the study of the possible "size" of control operators under the conditions of exact controllability, zero-controllability, etc. Note that an apparent contradiction of some results of [34], [59], [61] to Theorem 7.1 is related to the fact that in those books only bounded control and observation operators were considered (see, e.g., [61, Vol. 2, Theorem 3.8.2]).

Let (A, J) be a 2-system with a densely defined operator A. We fix a point $\lambda_0 \in \rho(A)$ and set $A_0 = A - \lambda_0 I$. We formally introduce the vectors $A_0^n h$ for $n \in \mathbb{Z}$ and $h \in H$ and the vector spaces $A_0^n H = \{A_0^n h : h \in H\}$. Suppose that $A_0^n h_1 \neq A_0^n h_2$ if $h_1 \neq h_2$; then $A_0^n H$ is a Hilbert space with the range norm, and it is isomorphic to H. For $n < m$ we assume that $A_0^n h_1 = A_0^m h_2$ if $A_0^{n-m} h_1 = h_2$. Under this assumption, we get a scale of spaces

$$\cdots \subset A_0^{-2}H \subset A_0^{-1}H \subset H \subset A_0 H \subset A_0^2 H \subset \cdots;$$

clearly, $A_0^{-k}H = \mathcal{D}(A_0^k)$, $k > 0$. The operators A_0, A, and A_0^{-1} are well defined on the linear set $\bigcup_{n\in\mathbb{Z}} A_0^n H$, so that A_0 and A_0^{-1} are mutually inverse. In a similar way, we can define the scale of spaces $A_0^{*n}H$, $n \in \mathbb{Z}$.

We shall use the following realization of the dual space: $(A_0^n H)^* = A_0^{*\,-n}H$, where we set $\langle A_0^n h_1, A_0^{*\,-n}h_2\rangle_n \overset{\text{def}}{=} \langle h_1, h_2\rangle$, $n \in \mathbb{Z}$. These definitions are compatible: $\langle h_1, h_2\rangle_n = \langle h_1, h_2\rangle_m$ if both quantities make sense. We shall write $\langle h_1, h_2\rangle$ instead of $\langle h_1, h_2\rangle_n$.

Suppose that $\sigma(A) \subset \{\operatorname{Re} z \le \gamma\}$ for some $\gamma \in \mathbb{R}$ and that

$$\|(A - zI)^{-1}\| \le C(\operatorname{Re} z - \gamma)^{-1}, \quad \operatorname{Re} z > \gamma$$

(we do not assume *a priori* that A is the generator of a semigroup). Set

$$\Gamma = \big\{\operatorname{Re} z = \gamma\big\}, \quad \Omega_{\text{int}} = \big\{\operatorname{Re} z < \gamma\big\}, \quad \Omega_{\text{ext}} = \big\{\operatorname{Re} z > \gamma\big\}. \tag{7.1}$$

With the system (A, J) we associate the (dual) linear system

$$\dot{x}(t) = A^* x(t) + J^* u(t), \quad t \le 0. \tag{7.2}$$

Here $x(t)$ is the state vector, u is the control, $u(t) \in R$, $J^* : R \to \mathcal{D}(A)^* = A_0^* H$. Equality in (7.2) should be understood as equality in the Hilbert space $A_0^* H$. We look for a solution $x(t)$ that tends to zero at $-\infty$ (in some sense).

Consider the Hilbert space

$$L^2_\gamma(\mathbb{R}_-, R) = \bigl\{\varphi : \varphi(t) = e^{\gamma t}\psi(t),\ \psi \in L^2(\mathbb{R}_-, R)\bigr\}.$$

Observe that the Fourier–Laplace transformation $\mathcal{L}u(p) = \int e^{-pt}u(t)\,dt$ is a unitary isomorphism of $L^2_\gamma(\mathbb{R}_-, R)$ onto $L^2(\Gamma, R, \frac{|dz|}{2\pi})$. We define a map

$$W : u \in L^2_\gamma(\mathbb{R}_-, R) \mapsto Wu \stackrel{\text{def}}{=} x(0)$$

as follows. First, suppose that $u \in C^3(\mathbb{R}_-, R)$ and u has compact support contained in $\mathbb{R}_-$, i.e., $u \equiv 0$ on $(-\infty, -S)$ and on $(-\epsilon, 0)$, with $\epsilon, S > 0$. Applying the Laplace transformation on $[-S, +\infty)$, we see that (7.2) has a solution

$$x(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{pt}(p - A^*)^{-1}J^*\tilde{u}(p)\,dp, \quad t \le 0, \tag{7.3}$$

where $\tilde{u}(p) = \mathcal{L}u$, and $\alpha > \gamma$ is arbitrary. It is easily seen that $\bigl(p - (\gamma+1)\bigr)^3\tilde{u} \in H^2(\Omega_{\text{int}}, R)$ and $e^{-Sp}\bigl(p - (\gamma-1)\bigr)^3\tilde{u} \in H^2(\Omega_{\text{ext}}, R)$, which implies that the element $e^{-Sp}\bigl(p - (\gamma-1)\bigr)^3(p - A^*)^{-1}J^*\tilde{u}$ lies in $H^2(\{\operatorname{Re} z > \alpha\}, R)$ for any $\alpha > \gamma$. Therefore, $x(t) \equiv 0$ for $t < -S$. Obviously, $x \in C^1(\mathbb{R}_-, R)$. Taking the Laplace transform on $(-\infty, 0)$, it is easy to show that (7.3) is a unique compactly supported solution of (7.2) on $\mathbb{R}_-$. For functions $u \in C^3(\mathbb{R}_-, R)$ of the above form, we set $Wu \stackrel{\text{def}}{=} x(0)$, where x is the solution of (7.2) defined by formula (7.3).

Definition. A system (7.2) is said to be *exactly γ-controllable* if the operator W can be extended continuously to the entire space $u \in L^2_\gamma(\mathbb{R}_-, R)$, and the range of this extension coincides with all of the space H.

In other words, a system (7.2) is exactly γ-controllable if, with the help of a control $u \in L^2_\gamma(\mathbb{R}_-, R)$, any state $x(0)$ can be reached.

Theorem 7.1. *A system* (7.2) *is exactly γ-controllable if and only if the system* (A, J) *is exact for the choice* (7.1) *of the domains.*

Proof. Note that $\Omega_{\text{int}} = \overline{\Omega}_{\text{int}}$, $\Omega_{\text{ext}} = \overline{\Omega}_{\text{ext}}$. We choose numbers γ_0, γ_1 so that $\gamma < \gamma_0 < \gamma_1$, and define the operator $\widetilde{W} : \frac{1}{z-\gamma_1}E^2(\{\operatorname{Re} z < \gamma_0\}, R) \to H$ by

$$\widetilde{W}v = \frac{1}{2\pi i}\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty}(p - A^*)^{-1}J^*v(p)\,dp, \quad v \in \frac{1}{z - \gamma_1}E^2(\{\operatorname{Re} z < \gamma_0\}, R). \tag{7.4}$$

By (7.3), $\widetilde{W}\mathcal{L}u = Wu$ for functions u in $C^3(\mathbb{R}_-, R)$ with compact support. Since $\mathcal{L}$ is a unitary isomorphism of $L^2_\gamma(\mathbb{R}_-, R)$ onto $\widehat{E}^2(\Omega_{\text{int}}, R)$, the system (7.2) is exactly γ-controllable if and only if $\widetilde{W}$ can be extended by continuity to $\widehat{E}^2(\Omega_{\text{int}}, R)$ and the range of this extension coincides with H. From (7.4) and (2.3) it is easily seen that $\langle U_{A,J}\, x, v\rangle = \langle x, \widetilde{W}v\rangle$ for $v \in \frac{1}{z-\gamma_1}E^2(\{\operatorname{Re} z < \gamma_0, R\}) \subset \widehat{E}^2(\Omega_{\text{int}}, R)$ and $x \in H$. Hence, if one of the operators $\widetilde{W} : \widehat{E}^2(\Omega_{\text{int}}, R) \to H$ and $U_{A,J} : H \to \widehat{E}^2(\Omega_{\text{int}}, R)$ is bounded, then the other is also bounded and they are mutually adjoint. In this case, by the Banach theorem, the relation $\widetilde{W}\widehat{E}^2(\Omega_{\text{int}}, R) = H$ for the extension $\widetilde{W}$ is equivalent to the condition $\exists \epsilon > 0 : \|\, U_{A,J}\, x\| \ge \epsilon\|x\|$, $x \in H$. □

From Theorem 3.1 it follows that, under the assumptions of Theorem 7.1, A is the generator of a c_0-semigroup.

A system (7.2) is said to be *exactly controllable on* $[-\tau, 0]$ if any state $x(0)$ can be reached by application of a control $u \in L^2\big([-\tau, 0], R\big)$ (we then assume that $u \equiv 0$ on $(-\infty, -\tau)$; as before, we understand the solutions of the system (7.2) in the sense of extending W by continuity). Clearly, exact controllability on $[-\tau, 0]$ implies exact γ-controllability for any γ such that the operator $W : L^2_\gamma(\mathbb{R}_-, R) \to H$ is continuous.

§8. Modeling of generators of groups

Let $T = \{T(t)\}_{t\in\mathbb{R}}$ be a c_0-group of linear operators. Then the limits

$$\alpha_0(T) = \lim_{t\to-\infty} \frac{\ln \|T(t)\|}{t}, \quad \beta_0(T) = \lim_{t\to+\infty} \frac{\ln \|T(t)\|}{t},$$

exist and $\alpha_0(T) \le \beta_0(T)$.

The next theorem shows that for modeling a generator of a group we can take $J = I$.

Theorem 8.1. 1) *Let A be the generator of a c_0-group $T = \{T(t)\}$, $\beta > \beta_0(T)$. Then*

$$\| U_{A,I}\, x\|_{E^2(\{\operatorname{Re} z>\beta\},H)} \asymp \|x\|, \quad x \in H. \tag{8.1}$$

Similarly, if $\alpha < \alpha_0(T)$, then $\| U_{A,I}\, x\|_{E^2(\{\operatorname{Re} z<\alpha\},H)} \asymp \|x\|$.

In particular, A is a C_{00}-operator in any strip of the form $\alpha < \operatorname{Re} z < \beta$, where $\alpha < \alpha_0(T) \le \beta_0(T) < \beta$.

2) *Conversely, let A be a closed operator with $\sigma(A) \subset \{\alpha \le \operatorname{Re} z \le \beta\}$. If*

$$\| U_{A,I}\, x\|_{E^2(\{\operatorname{Re} z<\alpha,H\})} + \| U_{A,I}\, x\|_{E^2(\{\operatorname{Re} z>\beta,H\})} \asymp \|x\|, \quad x \in H,$$

then A is the generator of a c_0-group.

Proof. 1) As is well known,

$$-(A - zI)^{-1} = \int_0^\infty e^{-zt} T(t)\, dt, \quad \operatorname{Re} z > \beta.$$

Therefore, by the Paley–Wiener theorem we have

$$\frac{1}{2\pi} \int_{\beta'-i\infty}^{\beta'+i\infty} \|(A - zI)^{-1}x\|^2\, |dz| = \int_0^\infty e^{-2\beta' t} \|T(t)x\|^2\, dt \tag{8.2}$$

for $\beta' > \beta$. Let $\|T(t)x\| \ge \epsilon\|x\|$ for $t \in [0, 1]$. From (8.2) it follows that

$$\|U_{A,I}x\|^2_{E^2(\{\operatorname{Re} z>\beta,H\})} = \int_0^\infty e^{-2\beta t}\|T(t)x\|^2\, dt \le \left(\int_0^\infty e^{-2\beta t}\|T(t)\|^2\, dt\right)\|x\|^2$$

for $x \in H$, where the last integral converges because $\beta > \beta_0(T)$. On the other hand,

$$\|U_{A,I}x\|^2_{E^2(\{\operatorname{Re} z>\beta,H\})} \ge \int_0^1 e^{-2\beta t}\|T(t)x\|^2\, dt \ge \left(\epsilon \int_0^1 e^{-2\beta t}\, dt\right)\|x\|^2.$$

The second statement in 1) can be proved in a similar way.

2) We put $\Omega_{\text{int}} = \big\{\alpha < \operatorname{Re} z < \beta\big\}$, $\Omega_{\text{ext}} = \big\{\alpha < \operatorname{Re} z\big\} \cup \big\{\operatorname{Re} z < \beta\big\}$. By Theorem 3.1, A is similar to the operator $M_z^{\mathbf{T}}$ on the space $\mathcal{H}(\delta, \Omega_{\text{int}})$ for some $*$-inner function δ on Ω_{int}. Using the Sz.-Nagy–Foiaş representation of a model operator (see §5), we can conclude that A is the generator of a c_0-group $\big\{M_{e^{zt}} \lceil \mathcal{H}_{SNF}(\delta, \Omega_{\text{int}})\big\}_{t\in\mathbb{R}}$ of linear operators. □

In particular, Theorem 8.1 implies the existence of an H^∞-calculus for the generators of groups in strips of the form $\{\alpha < \operatorname{Re} z < \beta\}$, where $\alpha < \alpha_0(T) \le \beta_0(T) < \beta$. The McIntosh techniques of constructing an H^∞-calculus with the help of quadratic estimates [57] together with the techniques of completely bounded operators [62] were applied systematically by Le Merdy in the papers [49]–[53] in connection with modeling semigroups of linear operators, commuting families of semigroups, and operators with spectrum in the unit disk. The example presented in [52] shows that there exists a compact c_0-semigroup such that its generator is not a C_{00}-operator in any half-plane $\Omega_{\rm int} = \{\operatorname{Re} z < a\}$.

We note that Theorem 1.1 in [53] is very close to our Theorem 8.1, but was proved by totally different (and less explicit) method.

It would be of interest to elaborate a function model for the generators of groups in a strip $\alpha_0(T) < \operatorname{Re} z < \beta_0(T)$.

From the theorem of Datko [35] it follows that the relation $\|U_{A,I}x\|_{E^2(\{\operatorname{Re} z>\beta, H\})} \ge C\|x\|$, $x \in H$, implies the strict inequality $\beta > \beta_0(T)$.

§9. Reproducing kernels. Calculation of the generalized characteristic function via duality

Let $\Omega_{\rm int}$, $\Omega_{\rm ext}$, Γ, R, R_* be given and have the same meaning as in §2. Consider the space

$$\begin{aligned}\widehat{E}^2\big(\Omega_{\rm int}, \mathcal{L}(R_*, R)\big) &\overset{\rm def}{=} \mathcal{L}\big(R_*, \widehat{E}^2(\Omega_{\rm int}, R)\big)\\ &= \big\{f \in \operatorname{Hol}(\Omega_{\rm int}, \mathcal{L}(R_*, R)) : f(\cdot)r \in \widehat{E}^2(\Omega_{\rm int}, R), \text{ for } r \in R_*\big\}.\end{aligned}$$

This is a Banach space with the norm $\|f\| = \sup_{r\in R_*, \|r\|=1} \|f(\cdot)r\|_{\widehat{E}^2(\Omega_{\rm int}, R)}$. In a similar way, we introduce $\widehat{E}^2\big(\Omega_{\rm ext}, \mathcal{L}(R_*, R)\big)$ and $L^2\big(\Gamma, \mathcal{L}(R_*, R)\big)$. If f is a function in $\widehat{E}^2\big(\Omega_{\rm int}, \mathcal{L}(R_*, R)\big)$ (or in $\widehat{E}^2\big(\Omega_{\rm ext}, \mathcal{L}(R_*, R)\big)$), then the map $r \mapsto fr|\Gamma$ is continuous from R_* to $L^2(\Gamma, R)$, which means that it is an element of the space $L^2\big(\Gamma, \mathcal{L}(R_*, R)\big) \overset{\rm def}{=} \mathcal{L}\big(R_*, L^2(\Gamma, R)\big)$. The images of this map will be called the *boundary values of the function f*.

The classes introduced above are direct analogs of the classes of strong H^2-functions in terms of the book [13]. Note that the boundary values of strong H^2-functions cannot be regarded in the sense of nontangential strong operator convergence.

The definition of the projections $P_{\rm int}$, $P_{\rm ext}$ can easily be transferred to the $\mathcal{L}(R)$-valued functions; in this sense, these projections are also bounded.

Definition. Let δ be a two-sided admissible function of class $H^\infty(\Omega_{\rm int}, \mathcal{L}(R, R_*))$. We shall say that an $\mathcal{L}(R_*, R)$-valued function $\varphi(z)$ holomorphic in $\Omega_{\rm ext}$ *corresponds to the function $\delta(z)$* if for some (and then for all) $\lambda_{\rm int} \in \Omega_{\rm int}$, $\lambda_{\rm ext} \in \Omega_{\rm ext}$ and for some function $\tau \in (z - \lambda_{\rm ext})\widehat{E}^2\big(\Omega_{\rm int}, \mathcal{L}(R_*, R)\big)$ we have

$$\begin{aligned}&\Phi|\Omega_{\rm ext} \in (z - \lambda_{\rm int})\widehat{E}^2\big(\Omega_{\rm ext}, \mathcal{L}(R_*, R)\big);\\ &\Phi_e = (\delta^{-1} + \tau)_i \quad \text{a.e. on } \Gamma\end{aligned} \tag{9.1}$$

(the latter means that $(\Phi r)_e = \delta^{-1}r + (\tau r)_i$ a.e. on Γ for all $r \in R_*$).

Proposition 9.1. *Conditions* (9.1) *determine* Φ *up to the replacement* $\Phi \mapsto \Phi + \Phi_0$ *with* $\Phi_0 = \text{const} \in \mathcal{L}(R_*, R)$. *One of the functions* Φ *can be calculated by the rule*

$$\Phi(z) = (z - \lambda_{\rm ext})\left(P_{\rm ext}\frac{\delta^{-1}}{z - \lambda_{\rm ext}}\right)(z), \tag{9.2}$$

and then

$$\tau(z) = -(z - \lambda_{\mathrm{ext}})\Big(P_{\mathrm{int}}\frac{\delta^{-1}}{z - \lambda_{\mathrm{ext}}}\Big)(z), \quad z \in \Omega_{\mathrm{int}}. \tag{9.3}$$

Here λ_{ext} *is an arbitrary fixed point in* Ω_{ext}.

Proof. Suppose that Φ_1, Φ_2 satisfy (9.1), $\Phi_0(z) = \Phi_1(z) - \Phi_2(z)$, and

$$\Phi_{00}(z) = (z - \lambda_{\mathrm{ext}})^{-1}\big(\Phi_0(z) - \Phi_0(\lambda_{\mathrm{ext}})\big).$$

Then $(\Phi_{00})_i = (\Phi_{00})_e$ a.e. on Γ, $\Phi_{00}|\Omega_{\mathrm{int}} \in \widehat{E}^2\big(\Omega_{\mathrm{int}}, \mathcal{L}(R_*, R)\big)$, and $\Phi_{00}|\Omega_{\mathrm{ext}} \in \widehat{E}^2\big(\Omega_{\mathrm{ext}}, \mathcal{L}(R_*, R)\big)$. Consequently, $\Phi_{00} \equiv 0$ and $\Phi_0 = \mathrm{const}$. The possibility of calculation of Φ and τ by formulas (9.2), (9.3) is obvious. □

If Φ corresponds to δ, then we agree to extend Φ to $\Omega_{\mathrm{int}} \setminus \operatorname{spec}\delta$ by the formula

$$\Phi(\lambda) = \delta^{-1}(\lambda) + \tau(\lambda), \quad \lambda \in \Omega_{\mathrm{int}} \setminus \operatorname{spec}\delta; \tag{9.4}$$

clearly, $\Phi_e = \Phi_i$ a.e. on $\Gamma \setminus \operatorname{spec}\delta$. By this definition, Φ becomes a holomorphic function on $\mathbb{C} \setminus \operatorname{spec}\delta$. Soon, we shall see that this extension is natural.

With δ as above we also associate the following family of functions:

$$\tilde{g}_{\mu,m}(z) = -\frac{\Phi(z) - \Phi(\mu)}{z - \mu}m, \quad \tilde{h}_{\nu,l}(z) = -\frac{\Phi^{\mathbf{T}}(z) - \Phi^{\mathbf{T}}(\nu)}{z - \nu}l; \tag{9.5}$$

here $\mu \in \mathbb{C} \setminus \operatorname{spec}\delta$, $\nu \in \mathbb{C} \setminus \operatorname{spec}\delta^{\mathbf{T}}$, $m \in R_*$, and $l \in R$. These functions are uniquely determined by δ.

Proposition 9.2. *The functions* $\tilde{g}_{\mu,m}$ *lie in* $\mathcal{H}(\delta)$. *The functions* $\tilde{h}_{\nu,l}$ *lie in* $\mathcal{H}(\delta^{\mathbf{T}})$. *The following reproducing formulas are true*:

$$\langle \varphi, \tilde{h}_{\bar{\nu},l}\rangle_\delta = \langle \varphi(\nu), l\rangle, \quad \varphi \in \mathcal{H}(\delta), \tag{9.6}$$

$$\langle \psi, \tilde{g}_{\bar{\mu},m}(z)\rangle_{\delta^T} = \langle \psi(\nu), m\rangle, \quad \psi \in \mathcal{H}(\delta^{\mathbf{T}}). \tag{9.7}$$

Proof. From (9.1) it easily follows that $\tilde{g}_{\mu,m} \in \widehat{E}^2(\Omega_{\mathrm{ext}}, R)$, $\delta\tilde{g}_{\mu,m} \in \widehat{E}^2(\Omega_{\mathrm{int}}, R_*)$, whence we conclude that $\tilde{g}_{\mu,m}|\Gamma \in \mathcal{H}(\delta)$. Similarly, $\tilde{h}_{\nu,l} \in \mathcal{H}(\delta^{\mathbf{T}})$. Let $\varphi \in \mathcal{H}(\delta)$, and let $\nu \in \mathbb{C} \setminus (\operatorname{spec}\delta \cup \Gamma)$. Then

$$\begin{aligned}\langle \varphi, \tilde{h}_{\bar{\nu},l}\rangle_\delta &= \int_\Gamma \Big\langle \delta(z)\varphi_e(z), \frac{\Phi^*(\nu)l}{\bar{z} - \bar{\nu}}\Big\rangle \frac{dz}{2\pi i} - \int_\Gamma \Big\langle \delta(z)\varphi_e(z), \frac{\Phi^*(z)l}{\bar{z} - \bar{\nu}}\Big\rangle \frac{dz}{2\pi i} \\ &\stackrel{\mathrm{def}}{=} I_1 - I_2.\end{aligned}$$

Using the Cauchy theorem for Ω_{int}, Ω_{ext}, the conditions $\delta \cdot \varphi_e|\Gamma \in \widehat{E}^2(\Omega_{\mathrm{int}}, R_*)$, $\varphi_e \in \widehat{E}^2(\Omega_{\mathrm{ext}}, R)$, and formulas (9.4), (2.10), we obtain

$$\begin{aligned}I_1 &= \int_\Gamma \Big\langle \frac{\Phi(\nu)\delta(z)\varphi_e(z)}{z - \nu}, l\Big\rangle \frac{dz}{2\pi i} \\ &= \begin{cases} \langle \big(\delta^{-1}(\nu) + \tau(\nu)\big)\delta(\nu)\varphi(\nu), l\rangle & \text{if } \nu \in \Omega_{\mathrm{int}} \setminus \operatorname{spec}\delta, \\ 0 & \text{if } \nu \in \Omega_{\mathrm{ext}}; \end{cases} \\ I_2 &= \int_\Gamma \Big\langle \frac{1 + \tau\delta}{z - \nu}\varphi_e, l\Big\rangle \frac{dz}{2\pi i} = \int_\Gamma \Big\langle \frac{\varphi_e}{z - \nu}, l\Big\rangle \frac{dz}{2\pi i} + \int_\Gamma \Big\langle \frac{\tau\delta\varphi_i}{z - \nu}, l\Big\rangle \frac{dz}{2\pi i} \\ &= \begin{cases} \langle \tau(\nu)\delta(\nu)\varphi(\nu), l\rangle & \text{if } \nu \in \Omega_{\mathrm{int}} \setminus \operatorname{spec}\delta, \\ -\langle \varphi(\nu), l\rangle & \text{if } \nu \in \Omega_{\mathrm{ext}}. \end{cases}\end{aligned}$$

Thus, formula (9.6) is valid for $\nu \notin \Gamma$. For $\nu \in \Gamma \setminus \operatorname{spec}\delta$, formula (9.6) follows by continuity. Formula (9.7) can be proved in a similar way. □

We call the functions $\tilde{h}_{\bar{\nu},l}$ *reproducing kernels of the space* $\mathcal{H}(\delta)$, and the functions $\tilde{g}_{\bar{\mu},m}$ *reproducing kernels of the space* $\mathcal{H}(\delta^{\mathbf{T}})$. The reproducing formulas imply that the functions $\tilde{h}_{\bar{\nu},l}$ are complete in $\mathcal{H}(\delta^{\mathbf{T}})$, and the functions $\tilde{g}_{\bar{\mu},m}$ are complete in $\mathcal{H}(\delta)$.

It is obvious that, for instance, the functions $\tilde{h}_{\bar{\nu},l}$ depend not only on the space $\mathcal{H}(\delta)$ but also on the specific choice of the function δ. Let $j : \mathcal{D}(M_z^{\mathbf{T}}\lceil\mathcal{H}(\delta)) \to R$ be an operator defined by formula (1.2), and $j_* : \mathcal{D}(M_z^{\mathbf{T}}\lceil\mathcal{H}(\delta^{\mathbf{T}})) \to R_*$ a similar operator corresponding to the space $\mathcal{H} = \mathcal{H}(\delta^{\mathbf{T}})$. Clearly,

$$\tilde{h}_{\nu,l} = \big(\nu - M_z^{\mathbf{T}}\lceil\mathcal{H}(\delta^{\mathbf{T}})\big)^{-1} j^* l, \quad \tilde{g}_{\mu,m} = \big(\mu - M_z^{\mathbf{T}}\lceil\mathcal{H}(\delta)\big)^{-1} j_*^* m \tag{9.8}$$

(here $j^* : R \to (M_z^{\mathbf{T}} - \bar{\lambda}_{\mathrm{ext}})\mathcal{H}(\delta^{\mathbf{T}})$, $j_*^* : R_* \to (M_z^{\mathbf{T}} - \lambda_{\mathrm{ext}})\mathcal{H}(\delta)$, where $\lambda_{\mathrm{ext}} \in \Omega_{\mathrm{ext}}$; see §7). Indeed, for any function φ in $\mathcal{H}(\delta)$, by Proposition 1.1, we have

$$\big\langle \varphi, \big(\nu - M_z^{\mathbf{T}}\lceil\mathcal{H}(\delta^{\mathbf{T}})\big)^{-1} j^* l\big\rangle = \big\langle j\big(\bar{\nu} - M_z^{\mathbf{T}}\lceil\mathcal{H}(\delta)\big)^{-1}\varphi, l\big\rangle = \langle \varphi(\bar{\nu}), l\rangle,$$

whence the first formula follows. The second formula can be proved similarly.

We shall use the notion of a 3-system that appeared in the Introduction. Observe that if (A, J, J_*) is a 3-system, then the operator A^* is densely defined.

Definition. Suppose (A, J, J_*) is a 3-system. Then the operators $J : \mathcal{D}(A) \to R$ and $J_* : \mathcal{D}(A^*) \to R_*$ are bounded in the graph norm. Hence, $J_*^* : R_* \to A_0 H$ is a bounded operator. By the Hilbert identity, relation (0.2) determines a well-defined $\mathcal{L}(R_*, R)$-valued analytic function Φ on $\rho(A)$, which is called the *transfer function* of the system (A, J, J_*).

The only nonuniqueness in the choice of Φ is related to the replacement $\Phi \to \Phi + \Phi_0$, $\Phi_0 = \mathrm{const}$. In the case of a bounded operator A, we can get rid of this indeterminacy by the requirement $\Phi(\infty) = 0$, in other words, by putting $\Phi(\lambda) = J(A - \lambda)^{-1} J_*^*$.

We set

$$h_{\nu,l} = (\nu I - A^*)^{-1} J^* l, \quad g_{\mu,m} = (\mu I - A)^{-1} J_*^* m,$$

where $l \in R$, $m \in R_*$, and $\mu, \bar{\nu} \in \rho(A)$. Since $J^* : R \to A_0^* H$ and $J_*^* : R_* \to A_0 H$ are bounded operators, we have $h_{\nu,l}$, $g_{\mu,m} \in H$ for all l, m, μ, ν. Clearly,

$$\begin{aligned} \langle \big(U_{A,J}\, x\big)(\lambda), l\rangle &= \langle x, h_{\bar{\lambda},l}\rangle, \quad \lambda \in \rho(A), \\ \langle \big(U_{A^*,J_*} x\big)(\lambda), m\rangle &= \langle x, g_{\bar{\lambda},m}\rangle, \quad \lambda \in \rho(A^*), \end{aligned} \tag{9.9}$$

for all $x \in H$, $l \in R$, $m \in R_*$. The Hilbert formula implies the identities

$$U_{A,J}\, g_{\mu,m} = \tilde{g}_{\mu,m}, \quad U_{A^*,J_*} h_{\nu,l} = \tilde{h}_{\nu,l}. \tag{9.10}$$

Now suppose that $\sigma(A) \subset \operatorname{clos}\Omega_{\mathrm{int}}$, where Ω_{int} and Ω_{ext} satisfy the conditions of §2. We shall use the notion of duality of 2-systems (A, J) and (A^*, J_*) with respect to a two-sided admissible function, as defined in the Introduction (see (0.1)). Note that if the systems (A, J) and (A^*, J_*) are dual with respect to a function δ, then δ is a generalized characteristic function of the system (A, J), and $\delta^{\mathbf{T}}$ is a generalized characteristic function of the system (A^*, J_*).

Proposition 9.3. *Let (A, J) be a C_{00}-system in a domain Ω_{int}. Fix its generalized characteristic function δ. Then the operators A and A^* are densely defined, and there exists a unique operator $J_* : \mathcal{D}(A^*) \to R$ such that the system (A^*, J_*) is dual to (A, J) with respect to δ.*

Proof. Since (A, J) is a C_0.-system, $\mathcal{D}(A)$ is dense in H. Therefore, A^* is well defined and $\mathcal{D}(A^*)$ is dense in H. By assumption, $U_{A,J}$ is an isomorphism of H onto $\mathcal{H}(\delta)$. Consider the isomorphism $W = \big(U_{A,J}{}^*\big)^{-1} : H \to \mathcal{H}(\delta^{\mathbf{T}})$. It intertwines A^* with the operator M_z^T on $\mathcal{H}(\delta^{\mathbf{T}})$. It is easily seen that the system (A^*, J_*) is dual to (A, J) with respect to the function δ if and only if $U_{A^*,J_*} = W$. Now the existence and uniqueness of J_* follow from Theorem 3.1. $\square$

Theorem 9.4. *Let (A, J, J_*) be a 3-system and Φ its transfer function. Let Ω_{int} be an admissible domain, and let $\sigma(A) \subset \operatorname{clos}\Omega_{\mathrm{int}}$, $\delta \in H^\infty(\Omega_{\mathrm{int}}, \mathcal{L}(R))$. If the systems (A, J) and (A^*, J_*) are dual with respect to the function δ, then $\Phi|\Omega_{\mathrm{ext}}$ corresponds to δ, and $\Phi|\Omega_{\mathrm{int}}$ is an extension of $\Phi|\Omega_{\mathrm{ext}}$ by formula* (9.4).

Proof. Starting with δ, we construct a function $\widetilde{\Phi}$ holomorphic in Ω_{ext} and extend it to $\Omega_{\mathrm{int}} \setminus \operatorname{spec}\delta$ by the rule (9.4). By (9.5) and (9.6), we have the relation

$$\langle \tilde{g}_{\mu,m}, \tilde{h}_{\nu,l}\rangle_\delta = \langle \tilde{g}_{\mu,m}(\nu), l\rangle = -\Big\langle \frac{\widetilde{\Phi}(\nu) - \widetilde{\Phi}(\mu)}{\nu - \mu} m, l\Big\rangle.$$

From the definition of the transfer function it follows that

$$\langle g_{\mu,m}, h_{\nu,l}\rangle = -\Big\langle \frac{\Phi(\nu) - \Phi(\mu)}{\nu - \mu} m, l\Big\rangle.$$

Since the systems (A, J) and (A^*, J_*) are dual with respect to δ, we have

$$\langle g_{\mu,m}, h_{\nu,l}\rangle = \langle U_{A,J}\, g_{\mu,m}, U_{A^*,J_*} h_{\nu,l}\rangle_\delta = \langle \tilde{g}_{\mu,m}, \tilde{h}_{\nu,l}\rangle_\delta$$

for all $\mu, \bar{\nu} \in \rho(A) \setminus \operatorname{spec}\delta$, $m \in R_*$, $l \in R$. We conclude that $\Phi - \widetilde{\Phi} = \mathrm{const}$. $\square$

Theorem 9.5. *Let (A, J, J_*) be a 3-system, and Φ its transfer function. Suppose that Ω_{int} is an admissible domain and $\sigma(A) \subset \operatorname{clos}\Omega_{\mathrm{int}}$. Let δ be a two-sided admissible function in $H^\infty(\Omega_{\mathrm{int}}, \mathcal{L}(R))$. The systems (A, J) and (A^*, J_*) are dual with respect to the function δ if and only if the following conditions are fulfilled*:

1) *$U_{A,J} : H \to \widehat{E}^2(\Omega_{\mathrm{ext}}, R)$ and $U_{A^*,J_*} : H \to \widehat{E}^2(\overline{\Omega}_{\mathrm{ext}}, R)$ are bounded operators*;
2) *the operators $U_{A,J}$ and U_{A^*,J_*} are injective*;
3) *Φ corresponds to δ.*

Proof. If the systems (A, J) and (A^*, J_*) are dual with respect to δ, then, by definition, conditions 1) and 2) are fulfilled. The relation $U_{A,J}\, g_{\mu,m} = \tilde{g}_{\mu,m}$ (see (9.10)) implies that Φ corresponds to δ.

Conversely, suppose that conditions 1)–3) are fulfilled. Since the operator $U_{A,J}$ is injective, by (9.9) the vectors $h_{\nu,l}$ are dense in H. By 3), identities (9.10) are fulfilled. Since $\mathcal{H}(\delta^{\mathbf{T}})$ is a closed subspace of $E^2(\overline{\Omega}_{\mathrm{ext}}, R)$ and the vectors $U_{A^*,J_*} h_{\nu,l} = \tilde{h}_{\nu,l}$ generate $\mathcal{H}(\delta^{\mathbf{T}})$, we have $U_{A^*,J_*} H \subset \mathcal{H}(\delta^{\mathbf{T}})$ by 1). Similarly, $U_{A,J}\, H \subset \mathcal{H}(\delta)$. The same arguments show that $U_{A^*,J_*} H$ is dense in $\mathcal{H}(\delta^{\mathbf{T}})$ and $U_{A,J}\, H$ is dense in $\mathcal{H}(\delta)$. We agree that $U_{A,J}$ acts from H to $\mathcal{H}(\delta)$, and U_{A^*,J_*} acts from H to $\mathcal{H}(\delta^{\mathbf{T}})$. Since

$$\langle U_{A,J}\, x, U_{A^*,J_*} h_{\nu,l}\rangle_\delta = \langle U_{A,J}\, x, \tilde{h}_{\nu,l}\rangle_\delta = \langle (U_{A,J}\, x)(\bar{\nu}), l\rangle = \langle x, h_{\nu,l}\rangle \tag{9.11}$$

for any $\nu \in \rho(A^*)$, $l \in R$, and $x \in H$, we obtain $U^*_{A^*,J_*} U_{A,J} = I$. Since the range of the operator $U_{A,J}$ is dense in $\mathcal{H}(\delta)$, the operator $U^*_{A^*,J_*}$ is inverse to $U_{A,J}$. Consequently, the system (A^*, J_*) is dual to (A, J) with respect to δ. $\square$

In particular, under the assumptions of the theorem, the operator A is similar to $M_z^{\mathbf{T}}$ on $\mathcal{H}(\delta)$.

The following version of Theorem 9.5 is useful.

Theorem 9.6. *Let (A, J, J_*) be a 3-system, and Φ its transfer function. Suppose that Ω_{int} is an admissible domain and $\sigma(A) \subset \operatorname{clos}\Omega_{\mathrm{int}}$. Let δ be an admissible function in $H^\infty(\Omega_{\mathrm{int}}, \mathcal{L}(R))$. Suppose also that the following conditions are fulfilled*:

1′) *$U_{A,J}$ is a bounded operator from H to $\mathcal{H}(\delta)$, and U_{A^*,J_*} is a bounded operator from H to $E^2(\overline{\Omega}_{\mathrm{ext}}, R_*)$*;

2′) *the operator* $U_{A,J}$ *is injective*;
3′) Φ *corresponds to* δ.
Then the systems (A, J) *and* (A^*, J_*) *are dual with respect to* δ.

Proof. From 2′) and (9.9) we deduce that the vectors $h_{\nu,l}$ are dense in H. By (3′), condition (9.10) is fulfilled. In the same way as in the proof of Theorem 9.5, we see that $U_{A^*,J_*}H \subset \mathcal{H}(\delta^{\mathbf{T}})$ and $U_{A^*,J_*}H$ is dense in $\mathcal{H}(\delta^{\mathbf{T}})$. The computations (9.11) yield ${U_{A,J}}^* U_{A^*,J_*} = I$, and now the density of the range U_{A^*,J_*} implies that the system (A^*, J_*) is dual to (A, J) with respect to δ. □

It would be of interest to combine our approach with constructions from the theory of optimal control by infinite-dimensional systems (see [24], [7], [65], [34], [66], [37], etc.).

§10. Calculation of generalized characteristic functions in examples

We consider the examples of §6, and also the generators of c_0-groups.

10.1. Suppose that T is a C_{00}-contraction on H, i.e., $\|T\| \le 1$, and $\lim_n T^n x = \lim_n T^{*n} x = 0$ for all $x \in H$. Set $D_{T^*} = (I - TT^*)^{1/2}$, $\mathcal{D}_{T^*} = \operatorname{clos} D_{T^*}H$. As earlier, we put $J = D_T$, $R = \mathcal{D}_T$ and $J_* = D_{T^*}$, $R_* = \mathcal{D}_{T^*}$. Then the transfer function of the 3-system (A, J, J_*) has the form

$$\Phi(\lambda) = D_T(\lambda - T)^{-1} D_{T^*}|\mathcal{D}_{T^*} \,:\, R_* \to R, \quad \lambda \in \rho(T).$$

Proposition 10.1. *Let* T *be a* C_{00}*-contraction. Then the systems* (T, D_T) *and* (T^*, D_{T^*}) *are dual with respect to the Sz.-Nagy–Foiaş characteristic function*

$$\delta(\lambda) = -T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T|\mathcal{D}_T, \quad \lambda \in \mathbb{D}.$$

Proof. By [20], the condition $T \in C_{00}$ implies that $\delta \in H^\infty\big(\mathbb{D}, \mathcal{L}(R, R_*)\big)$ and the values of δ are unitary operators a.e. on $\mathbb{T}$. We set $\tau(\lambda) \equiv T^*|R_*$; it is easily seen that $\tau \in E^2(\mathbb{D}, \mathcal{L}(R_*, R))$. We have

$$\delta(\lambda)^* + T^*|R_* = \Phi(\bar\lambda^{-1}), \quad \lambda \in \mathbb{D}.$$

Thus, $\Phi \in H^\infty(\widehat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}, \mathcal{L}(R_*, R))$ and, in particular, $\Phi \in zH^2(\widehat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}, \mathcal{L}(R_*, R))$; we also obtain $\delta_i^{-1} + T^*|R_* = \Phi_e$ a.e. on $\partial\mathbb{D}$. Therefore, $\Phi|\Omega_{\mathrm{ext}}$ corresponds to δ. The duality of the systems (T, D_T) and (T^*, D_{T^*}) with respect to δ follows from Theorem 9.5. □

10.2. Similarly, let A be a maximal dissipative operator satisfying the conditions of Subsection 6.2. Suppose that, moreover, $A \in C_{00}$. Put $\Omega_{\mathrm{int}} = \{\operatorname{Re} z > 0\}$, $J = (2\operatorname{Im} A)^{1/2}$, $J_* = -iJ$, and $R = R_* = \operatorname{clos}\operatorname{Range} J$. The values of the Sz.-Nagy–Foiaş characteristic function

$$\delta_A(z) = I - J_*(A^* - zI)^{-1} J^*|R$$

(initially defined on the set $\{r \in R : Jr \in R\}$) are unitary operators a.e. on $\mathbb{R}$. In the same way as in Subsection 10.1, from the relation

$$\delta_A^*(z) = I + J(\bar z I - A)^{-1} J_*^* = I + \Phi(\bar z), \quad \operatorname{Re} z > 0,$$

we deduce that the transfer function $\Phi(z) = J(\bar z I - A)^{-1} J_*^*$ is well defined on Ω_{ext} and that the function $\Phi|\Omega_{\mathrm{ext}}$ corresponds to δ. Proposition 6.3 and Theorem 9.5 permit us to conclude that the systems (A, J) and (A^*, J_*) are dual with respect to the characteristic function δ.

The techniques of the papers [19], [17], [26] make it possible to get rid of the assumption $\mathcal{D}(A) = \mathcal{D}(A^*)$.

10.3. Let A be the operator as in Example 6.3, and let δ and J_* be determined by formulas (6.7), (6.6). We shall show that the generalized characteristic function of the operator A coincides with δ. We must prove that $U_{A^*,J_*}H = \mathcal{H}(\delta^{\mathbf{T}})$ and find an operator $J : \mathcal{D}(A) \to \mathbb{C}^n$ such that the system (A, J) is dual to the system (A^*, J_*) with respect to δ.

Observe that $\delta^{-1}(\lambda) = e^{-\lambda w}\Delta^{-1}(\lambda) \in H^\infty(\Omega_{\text{ext}}, \mathbb{C}^{n\times n})$; this implies that the function Φ constructed via δ coincides with δ^{-1} (see (9.1)). We want to find an operator J such that the transfer function of the system (A, J, J_*) coincides with δ^{-1}.

Each $c \in \mathbb{C}^n$ gives rise to a δ-function $\psi_w c$ defined by the relation $\langle \varphi, \psi_w c\rangle = \langle \varphi(w), c\rangle$. Assuming that $\varphi \in \mathcal{D}(A)$, we obtain $\psi_w c \in \mathcal{D}(A)^* = A^*H$. Identity (6.7) formally yields

$$J_*(\lambda - A^*)^{-1}\psi_w c = \delta^{\mathbf{T}}(\lambda)^{-1}c = \Phi^{\mathbf{T}}(\lambda)^{-1}c, \quad \lambda \in \rho(A). \tag{10.1}$$

Note that the elements $\big((\lambda - A^*)^{-1}\psi_w c\big)(x)$ are continuous at $x = 0$ but do not belong to $\mathcal{D}(A^*)$. Formulas (10.1) and (0.1) lead to the conjecture that the required operator J can be found by the formula $J^*c = \psi_w c$, $c \in \mathbb{C}^n$, which is equivalent to the relation

$$J\varphi = \varphi(w), \quad \varphi \in \mathcal{D}(A). \tag{10.2}$$

Proposition 10.2. *Let A be the operator determined by formulas* (6.1) *and* (6.2)*, and let J, J^*, and δ be defined by* (10.2), (6.6), *and* (6.7)*. Then the systems (A, J) and (A^*, J_*) are dual with respect to the function δ, and the function δ^{-1} is the transfer function of the* 3*-system (A, J, J_*).*

Proof. Let Φ_1 denote the transfer function of the system (A, J, J_*). Then $\Phi_1^{\mathbf{T}}$ is the transfer function of the system (A^*, J_*, J). From (10.1) it follows that

$$J_*\big[(\lambda - A^*)^{-1} - (\mu - A^*)^{-1}\big]J^* = \delta^{\mathbf{T}}(\lambda)^{-1} - \delta^{\mathbf{T}}(\mu)^{-1}.$$

Therefore, $\Phi_1^{\mathbf{T}}(\lambda) = \delta^{\mathbf{T}}(\lambda)^{-1} = \Phi^{\mathbf{T}}(\lambda)$. We apply Theorem 9.6 to the system (A^*, J_*, J). From (6.4) we obtain

$$\begin{aligned} U_{A,J}\, x(\lambda) = \delta(\lambda)^{-1}\bigg[\int d\beta(x)\int_0^x e^{\lambda(t-x)}x(t)\,dt + \rho\int_0^w e^{\lambda(t-w)}x(t)\,dt\bigg] \\ + \int_0^w e^{\lambda(t-w)}x(t)\,dt. \end{aligned}$$

Now we see that $U_{A,J}$ is a bounded operator from H to $\widehat{E}^2(\Omega_{\text{ext}}, R)$. Formula (6.8) and Proposition 6.4 imply that U_{A^*,J_*} is a bounded and injective operator from H to $\mathcal{H}(\delta^{\mathbf{T}})$. Thus, the assumptions of Theorem 9.6 are fulfilled, and we obtain all the statements of the proposition. □

10.4. Let A be the generator of a neutral semigroup with delay as in Example 6.4. We put

$$J(c, \varphi) = \varphi(-h), \quad (c, \varphi) \in \mathcal{D}(A). \tag{10.3}$$

Proposition 10.3. *In the notation of Subsection* 6.4 *and* (10.3)*, the systems (A, J) and (A^*, J_*) are dual with respect to the function δ, and the function δ^{-1} is the transfer function of the system (A, J, J_*).*

Proof. Consider the following operator D_- on $L^2([-h, 0] \to \mathbb{C}^n)$:

$$\mathcal{D}(D_-) = \{\varphi \in W_1^2([-h, 0] \to \mathbb{C}^n) : \varphi(-h) = 0\}, \quad D_-\varphi = \varphi'.$$

Then the operator $(z - D_-)^{-1}$ is bounded for all $z \in \mathbb{C}$, and

$$\big((z - D_-)^{-1}\varphi\big)(x) = -\int_{-h}^x e^{z(x-t)}\varphi(t)\,dt.$$

It is easily seen that

$$U_{A,J}(c,\varphi)(z) = J(z-A)^{-1}(c,\varphi) = e^{-hz}\Delta(z)^{-1}\big(c-(zM-L)(z-D_-)^{-1}\varphi\big).$$

Hence,

$$J(z-A)^{-1}J_0^* = e^{-hz}\Delta(z)^{-1} = (z-b)\delta(z)^{-1}.$$

Observe that

$$\begin{aligned} J\big[(\lambda-A)^{-1}-(\mu-A)^{-1}\big]J_*^* &= J\big[(\lambda-A)^{-1}-(\mu-A)^{-1}\big](A-b)J_0^* \\ &= J\big[(\lambda-b)(\lambda-A)^{-1}-(\lambda-b)(\mu-A)^{-1}\big]J_0^* \\ &= \delta^{-1}(\lambda)-\delta^{-1}(\mu). \end{aligned}$$

Thus, the transfer function Φ of the system (A,J,J_*) coincides with δ^{-1}.

Direct inspection shows that all the assumptions of Theorem 9.6 are fulfilled, and the proposition follows. $\square$

Note that in [44] a general "algebraic" definition of the characteristic function was presented. It can also be applied to operators like those considered in Examples 6.3 and 6.4. It would be of interest to elucidate the relationship between the construction of [44] and that of ours in a maximally general setting.

10.5. Let A be the generator of a c_0-group, and let $\alpha < \alpha_0(A) \le \beta_0(A) < \beta$ (see §8). We put $J = I$, $\Pi = \{\alpha < \operatorname{Re} z < \beta\}$. By Theorem 3.1 and Remark 2 to it, $U_{A,J}$ is an isomorphism of H onto the space $\mathcal{H}(\delta)$ for some two-sided admissible function $\delta \in H^\infty(\Pi,\mathcal{L}(H))$. The following statement gives a formula for δ among other things.

Proposition 10.4. *Let A be the generator of a c_0-group. Let s be a complex number with $|\operatorname{Re} s| > \beta-\alpha$, and let $R = R_* = H$, $J_* = \bar{s}I$, and $\delta(z) = (z-A)(z+s-A)^{-1}$. Then $\delta \in H^\infty(\Pi,\mathcal{L}(H))$, δ is a two-sided admissible function in the strip* Π, *and the systems (A,J) and (A^*,J_*) are dual with respect to the function δ.*

Proof. Clearly, δ is a two-sided admissible function in Π. Putting $\tau(z) \equiv I$, we see that the function $\Phi(z) = s(z-A)^{-1}$ corresponds to δ; also we can see that Φ is the transfer function of the system (A,J,J_*). Thus, conditions 1)–3) of Theorem 9.5 are fulfilled. $\square$

Note that the condition $\delta(z) - I \in S_p$ for $z \in \Pi$ is equivalent to $(z-A)^{-1} \in S_p$ for $z \in \rho(A)$; here S_p stands for the Schatten–von Neumann class. We also note that for $\operatorname{Re} s > \beta-\alpha$ the same function δ is a generalized characteristic function of a 3-system in the half-plane $\alpha < \operatorname{Re} z$, and if $\operatorname{Re} s < -(\beta-\alpha)$, then this is true for the half-plane $\operatorname{Re} z < \beta$.

§11. Nonuniqueness of the generalized characteristic function

In Examples 3, 4, and 5 of the preceding section, explicit formulas were given for generalized characteristic functions, while, most likely, the inner factors of these functions cannot be calculated explicitly. The nature of nonuniqueness of a generalized characteristic function depends on whether we deal with an operator A, a 2-system (A,J), or a 3-system (A,J,J_*). In this section we always assume that A is a C_{00}-operator in a domain Ω_{int}. To simplify the notation, we also assume (without loss of generality) that $R = R_*$.

Theorem 11.1. *Suppose that all components of a domain Ω_{int} are simply connected. Let $\delta \in H^\infty\big(\Omega_{\text{int}},\mathcal{L}(R)\big)$ be a generalized two-sided admissible characteristic function of an operator A in the domain Ω_{int}, let L be a Hilbert space, and let ζ be a two-sided admissible function in $H^\infty\big(\Omega_{\text{int}},\mathcal{L}(L)\big)$. Then ζ is also a generalized characteristic function of A if and only if there exist functions $\Sigma_1,\ \Sigma_2 \in H^\infty\big(\Omega_{\text{int}},\mathcal{L}(R,L)\big)$,*

$\Psi_1,\ \Psi_2 \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(L,R)\big)$, $\Pi \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(R)\big)$, *and* $\Lambda \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(L)\big)$ *such that* $\Sigma_1\delta = \zeta\Sigma_2,\ \Psi_1\zeta = \delta\Psi_2,\ \Psi_1\Sigma_1 = I + \delta\Pi,\ \Sigma_1\Psi_1 = I + \zeta\Lambda.$

Proof. We note that ζ is a generalized characteristic function of A if and only if there exists an isomorphism

$$\varphi\,:\,\widehat{E}^2(\Omega_{\rm int},R)/\delta\widehat{E}^2(\Omega_{\rm int},R) \to \widehat{E}^2(\Omega_{\rm int},L)/\zeta\widehat{E}^2(\Omega_{\rm int},L)$$

such that $\varphi M_z = M_z\varphi$ (here M_z means a quotient operator). By the commutant lifting theorem (see [38, Theorem VII.1.2]), there exists a function $\Sigma_1 \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(R,L)\big)$ such that $\varphi\big(f + \delta\widehat{E}^2(\Omega_{\rm int},R)\big) = \Sigma_1 f + \zeta E^2(\Omega_{\rm int},L)$, $f \in \widehat{E}^2(\Omega_{\rm int},R)$. Since φ is well defined, we arrive at the necessity of the identity $\Sigma_1\cdot\delta = \zeta\cdot\Sigma_2$. Applying the same arguments to the isomorphism $\psi = \varphi^{-1}$ and using the identities $\varphi\psi = I$ and $\psi\varphi = I$, we conclude that all four relations are necessary. Their sufficiency follows by direct inspection. □

In the case where the space R is finite-dimensional, the commutant lifting theorem can be generalized to the multiply connected case because of the results of [29]; partial generalizations for infinite-dimensional R and multiply connected domains $\Omega_{\rm int}$ can be found in [58].

The following fact is a consequence of Proposition 5.1.

Proposition 11.2. *Let* (A, J) *be a* 2*-system, and let* $\delta \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(R)\big)$ *be its generalized two-sided admissible characteristic function. If* $\delta_1 \in H^\infty(\Omega_{\rm int}, \mathcal{L}(R))$, *then the following statements are equivalent*:

1) δ_1 *is also a generalized characteristic function of the system* (A, J);
2) $\mathcal{H}(\delta) = \mathcal{H}(\delta_1)$;
3) *there exists a function* ψ *such that* $\psi, \psi^{-1} \in H^\infty\big(\Omega_{\rm int} \to \mathcal{L}(R)\big)$ *and* $\delta_1 = \psi\delta$.

Proposition 11.3. *Let* (A, J, J_*) *be a* 3*-system such that* (A^*, J_*) *is dual to the system* (A, J) *with respect to a function* δ. *If* $\delta_1 \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(R)\big)$ *is a two-sided admissible function, then the following statements are equivalent*:

1) *the system* (A^*, J_*) *is dual to the system* (A, J) *with respect to the function* δ_1;
2) $\mathcal{H}(\delta) = \mathcal{H}(\delta_1)$, $\mathcal{H}(\delta^{\mathbf{T}}) = \mathcal{H}(\delta_1^{\mathbf{T}})$, *and the forms* $\langle\cdot,\cdot\rangle_\delta$ *and* $\langle\cdot,\cdot\rangle_{\delta_1}$ *coincide*;
3) *there exists a function* $\tau \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(R)\big)$ *such that* $\delta^{-1} - \delta_1^{-1} = \tau$;
4) *there exists a function* τ *such that* $\tau, (I - \delta\tau)^{-1} \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(R)\big)$ *and* $\delta_1 = (I - \delta\tau)^{-1}\delta$;
5) *there exists a function* ρ *such that* $\rho, (I + \delta\rho)^{-1} \in H^\infty\big(\Omega_{\rm int}, \mathcal{L}(R)\big)$ *and* $\delta_1 = (I + \delta\rho)\delta$.

Proof. The equivalence of 1) and 2) is obvious. We check the equivalence of 3), 4), and 5). If 3) is fulfilled, then $\delta = (I - \delta\tau)\delta_1$ and $\delta_1 = (I + \delta_1\tau)\delta$, whence 4) follows. Putting $\rho = \tau(I - \delta\tau)^{-1}$ so that $(I + \delta\rho)^{-1} = I - \delta\tau$, we see that 4) implies 5). Finally, taking $\tau = \rho(I + \delta\rho)^{-1}$, we see that 5) $\Longrightarrow$ 3).

Now we check that 2) $\Longrightarrow$ 3). If 2) is fulfilled, then the reproducing kernels of the spaces $\mathcal{H}(\delta)$ and $\mathcal{H}(\delta_1)$ coincide, which implies that the functions Φ corresponding to δ and δ_1 coincide. Thus, there exist $\tau_0, \tau_1 \in (z - \lambda_{\rm ext})\widehat{E}^2\big(\Omega_{\rm int}, \mathcal{L}(R_*, R)\big)$ such that $\Phi = \delta^{-1} + \tau_0 = \delta_1^{-1} + \tau_1$. Taking $\tau = \tau_1 - \tau_0 = \delta^{-1} - \delta_1^{-1}$, we see that $\tau \in H^\infty\big(\Omega \to \mathcal{L}(R)\big)$.

It remains to check that 5) $\Longrightarrow$ 2). Suppose 5) is fulfilled. Then $\mathcal{H}(\delta) = \mathcal{H}(\delta_1)$. Next, the function $\eta = I + \rho\delta$ is also invertible in H^∞, and $(I + \rho\delta)^{-1} = I - \rho(I + \delta\rho)^{-1}\delta$. Since $\delta_1 = \delta\eta$, $\delta_1^{\mathbf{T}} = \eta^{\mathbf{T}}\delta^{\mathbf{T}}$, we have $\mathcal{H}(\delta^{\mathbf{T}}) = \mathcal{H}(\delta_1^{\mathbf{T}})$. The forms $\langle\cdot,\cdot\rangle_\delta$ and $\langle\cdot,\cdot\rangle_{\delta_1}$ coincide by their definitions and by 5). □

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