ZAPISKI NAUCHN. SEMIN. POMI 222 (1995), 293–306, 311 (RUSSIAN);
TRANSL. IN J. SOVIET MATHEMATICS, TO APPEAR
TRANSLATED BY THE AUTHOR

LOCAL SPECTRAL MULTIPLICITY OF A LINEAR OPERATOR WITH RESPECT TO A MEASURE

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Abstract. Let T be a bounded linear operator in a separable Banach space X and let μ be a nonnegative measure in $\mathbb{C}$ with compact support. A function $m_{T,\mu}$ is considered that is defined μ-a.e. and has nonnegative integers or $+\infty$ as values. This function is called the local multiplicity of T with respect to the measure μ. This function has some natural properties, it is invariant under similarity and quasisimilarity; the local spectral multiplicity of a direct sum of operators equals the sum of local multiplicities, and so on. The definition is given in terms of the maximal diagonalization of the operator T. It is shown that this diagonalization is unique in the natural sense. A notion of a system of generalized eigenvectors, dual to the notion of diagonalization, is discussed. Some examples of evaluation of the local spectral multiplicity function are given. Bibliography: 10 titles.

The spectral multiplicity of a linear operator, which is studied in many papers (let us point out [4, 6, 9]), is one of its invariants. The notion of the local spectral multiplicity has been known only for normal operators. Here we discuss a definition of the local spectral multiplicity of a linear operator in the general case. It is based on the notion of a “maximal diagonalization” and seems to be natural enough. The local multiplicity of a linear operator with respect to a measure μ on the plane is introduced as a measurable nonnegative function defined μ-a.e. It possesses a series of natural properties. In particular, the spectral multiplicity is greater or equal than the essential supremum of the local multiplicity, and the local spectral multiplicity of a direct sum of linear operators equals to the sum of local multiplicities. A connection between the notions of generalized eigenvectors, of the local spectral multiplicity, and of diagonalizations is also explained. Generalized eigenvectors in various forms are actively exploited in the operator theory, see, for example, [1].

The author expresses his gratitude to A. A. Borichev and E. M. Dyn'kin for valuable consultations.

1. Reducing subspaces. Let N be a normal operator on a separable Hilbert space H. A (closed) subspace K in H is called reducing if it is invariant with respect to N and N^*. Assume now that N in its spectral representation has the form of multiplication by the independent variable on the direct integral of Hilbert spaces,

$$H = \oplus \int H(z) d\mu(z). \tag{1}$$

It is easy to see that a subspace K is reducing for N if and only if the equality

$$K = \oplus \int K(z) d\mu(z) \tag{2}$$

holds, where $K(z) \subset H(z)$ for μ-a.e. z.

The function
$$n_K(z) = \dim K(z),$$
defined μ – almost everywhere in $\mathbb{C}$, will be called *the local multiplicity* of the subspace K. If $L \subset H$, then the minimal reducing subspace for N that contains L will be denoted by $\mathrm{Red}(L)$. The symbol span means a closed linear span.

The following statement is easily proved.

Proposition 1. *Let L be a Banach space with respect to its own norm, imbedded continuously into H. Choose a countable subset $\{f_j\}$ dense in L, and put $K(\lambda) = \operatorname{span}_j\{f_j(\lambda)\}$ (the spaces $K(\lambda)$ are defined μ – a.e.). Then $\mathrm{Red}(L) = K$, where K is given by* (2).

2. The space of integer-valued functions and upper bounds. Let μ be a fixed finite measure on the plane with compact support. Consider the set of measurable functions on $\mathbb{C}$ whose values are nonnegative integers or $+\infty$. Two such functions will be called equivalent if they coincide μ – a.e. Let $\mathfrak{N}$ be the set of equivalence classes. Introduce the natural order on $\mathfrak{N}$, putting $f \le g$ if $f(z) \le g(z)$ for μ-a.e. z.

A function $g \in \mathfrak{N}$ is called an upper bound of a family $\mathfrak{A}$ of functions in $\mathfrak{N}$ if $f \le g$ for all f in $\mathfrak{A}$. It is easy to show that every family $\mathfrak{A}$ possesses the least upper bound, that is, an upper bound g_0 such that $g_0 \le g$ for any other upper bound g of $\mathfrak{A}$. The scheme of the proof: let $n \in \mathbb{Z}$, $n \ge 0$, or $n = \infty$. Among the sets B such that $f \ge n$ a.e. on B for all $f \in \mathfrak{A}$, one can find a set of the largest measure, which will be denoted by $B(n)$. It suffices to put $g_0 = n$ on $B(n) \setminus B(n+1)$ for $n \in \mathbb{Z}$, $n \ge 0$, and $g_0 = \infty$ on $B(\infty)$.

In what follows, we fix a linear operator T on a separable Banach space X and a finite Borel measure μ on $\mathbb{C}$ with compact support.

3. Definition of the local multiplicity. Let H be a separable Hilbert space. The vector Lebesgue L^2-space of H-valued functions that corresponds to the measure μ, is denoted by $L^2(\mu; H)$. Let us call a bounded operator

$$J : \mathfrak{X} \longrightarrow L^2(\mu; H)$$

a diagonalization of T if
$$JT = M_z J,$$

where $(M_z f)(z) = zf(z)$ is the operator of multiplication by the independent variable. One can associate with each such operator a collection of spaces $\{H_J(z)\}$ defined by

$$\mathrm{Red}(J\mathfrak{X}) = \oplus \int H_J(z) d\mu(z),$$

and a function n_J in $\mathfrak{N}$,

$$n_J(z) \stackrel{\mathrm{def}}{=} \dim H_J(z),$$

which is called the multiplicity function of the diagonalization J.

Definition. *The least upper bound of functions n_J over all diagonalizations J of T is called the local spectral multiplicity function for the operator T and is denoted by $m_{T,\mu}(z)$.*

So $m_{T,\mu}$ is an element of the space $\mathfrak{N}$.

4. Existence of a universal diagonalization.

Definition. *Let J, J' be two diagonalizations of T. We write $J' \prec J$ (J' is subordinate to J) if there exists a family of possibly unbounded operators $\alpha(z) : \mathcal{D}(\alpha(z)) \to H$, $\mathcal{D}(\alpha(z)) \subset H$, such that*

$$(J'x)(z) = \alpha(z)(Jx)(z), \quad \mu - \text{a.e. } z, \tag{3}$$

for all $x \in \mathfrak{X}$.

Lemma 1. *Assume that J, J' are diagonalizations, $J' \prec J$, and*

$$\mathrm{Red}(J\mathfrak{X}) = \int \oplus N(z) d\mu(z). \tag{4}$$

Then $\alpha(z)$ from (3) can be considered as a closed densely defined operator.

Proof. Choose an extending family of finite-dimensional subspaces R_n, $\dim R_n = n$, such that $\cup R_n$ is dense in $\mathfrak{X}$. Let

$$\mathrm{Red}(JR_n) = \oplus \int L_n(z) d\mu(z). \tag{5}$$

By Proposition 1, $\dim L_n(z) \le n$ μ-a.e. Put $L_0(z) = 0$,

$$T_n(z) = L_n(z) \ominus L_{n-1}(z), \quad T_n(z) = \ominus \int T_n(z) d\mu(z). \tag{6}$$

It is clear that $\overset{\infty}{\underset{n=1}{\oplus}} T_n = \mathrm{Red}(J\mathfrak{X})$, $\overset{\infty}{\underset{n=1}{\oplus}} T_n(x) = N(z)$. Define an operator-valued function $B(z)$: $N(z) \to N(z)$ by

$$B(z) \mid T_n(z) = \big(1 + \|\alpha(z) \mid T_n(z)\|\big)^{-1} I \tag{7}$$

(it is easy to see that $\alpha(z)$ is defined on $T_n(z)$ for a.e. Z). Then

$$\|B(z)\alpha(z) \mid T_n(z)\| \le 1 \tag{8}$$

for all n, hence the operator $B(z)\alpha(z)$ on $\cup L_n(z)$ extends to a continuous operator $C(z) : N(z) \to H_0$ (all this holds for μ-a.e. z). Now define an unbounded operator $\widetilde{\alpha}(z)$, putting $\widetilde{\alpha}(z)u = v$ if $C(z)u = B(z)v$. Since $\mathrm{Ker}B(z) = 0$, this definition is correct and, obviously, $\widetilde{\alpha}(z)$ is a closed operator for a.e. z. This operator is densely defined since it is defined on $\underset{n}{\cup} L_n(z)$.

Multiply (3) by $B(z)$ to get

$$B(z)(J'x)(z) = C(z)(Jx)(z)$$

for μ-a.e. z and for all $x \in \underset{n}{\cup} R_n$. By continuity, this equality is valid for all $x \in \mathfrak{X}$, hence $(J'x)(z) = \widetilde{\alpha}(z)(Jx)(z)$ for all $x \in \mathfrak{X}$. ■

Definition. *A diagonalization J of an operator T is called universal if any other diagonalization of T is subordinate to it.*

Theorem 1. *For any operator T on a separable Banach space $\mathfrak{X}$ and for any finite Borel measure μ in $\mathbb{C}$ with compact support, a universal diagonalization exists.*

To prove this theorem, a few preparations are necessary. For a finite-dimensional subspace R in $\mathfrak{X}$ and for a diagonalization J of T, denote by $f_{R,J}(z)$ the local multiplicity of JR (see Sec. 1). By Proposition 1, $f_{R,J} \le \dim R$ μ-a.e.

Lemma 2. *For a fixed finite-dimensional subspace R, there exists a diagonalization J such that $f_{R,J}$ is the largest possible.*

Proof. If J^1 and J^2 are two diagonalizations, then, obviously, $f_{R,J^1\oplus J^2} \ge \max\big(f_{R,J^1}, f_{R,J^2}\big)$. Put

$$A = \max_J \int f_{R,J} d\mu.$$

Take diagonalizations J_n such that $\int f_{R,J_n} d\mu \ge A - \frac{1}{n}$, then $J = \underset{n}{\oplus} J_n$ is the desired diagonalization. Indeed, it is clear that $\int f_{R,J} d\mu = A$. If there exists a diagonalization J' such that $f_{R,J'} > f_{R,J}$ on a set of positive measure, then the consideration of $\widetilde{J} = J \oplus J'$ leads to a contradiction with the definition of A. ■

Proof of Theorem 1. Choose R_n as in the proof of Lemma 1. For each n, let $J^{(n)}$ be a diagonalization with the largest possible $f_{R_n,J^{(n)}}$. We will show that

$$J = \underset{n}{\oplus} J^{(n)}$$

is a universal diagonalization.

Let J' be another diagonalization of T; we have to show that $J' \prec J$. One can fix bases in the spaces $JR_n \ominus JR_{n-1}$ to consider elements of the linear set $\underset{n}{\cup} R_n$ as functions and not as equivalence classes of functions. Let us use notation (4), (5) from the proof of Lemma 1. Define operators $\alpha(z)$ by

$$\alpha(z)(Jr)(z) = (J'r)(z), \quad r \in \underset{n}{\cup} R_n. \tag{9}$$

For a.e. z, the vector $(Jr)(z)$, $r \in \bigcup_n R_n$, runs over the linear set $\bigcup_n L_n(z)$, and $\alpha(z)$ is correctly defined on this set. Indeed, if $Jr_0(z) = 0$, $J'r_0(z) \neq 0$ for some n, z, $r_0 \in R_n$, then

$$\dim\bigl\{(J^{(n)} \oplus J')r(z) : r \in R_n\bigr\} > \dim\bigl\{J^{(n)}r(z) : r \in R_n\bigr\}.$$

By the choice of $J^{(n)}$, this can happen only on a set of points z of zero measure.

Now let us apply the same argument as in the proof of Lemma 1. Define $B(z)$ by (6), (7); then (8) holds. Now define operators $C(z)$ and $\widetilde{\alpha}(z)$ (see the proof of Lemma 1). It follows from (9) that

$$B(z)(J'x)(z) = C(z)(Jx)(z)$$

μ-a.e. for all $x \in \cup R_n$. By continuity, the same equality holds for all $x \in \mathfrak{X}$, hence $(J'x)(z) = \widetilde{\alpha}(z)(Jx)(z)$ for all $x \in \mathfrak{X}$. ■

If J_1, J_2 are two universal diagonalizations of T, then $(J_1x)(z) = \alpha(z)(J_2x)(z)$, $(J_2x)(z) = \beta(z)(J_1x)(z)$ for certain operator-valued functions α, β, that is, J_1, J_2 are expressed one through another. In this sense, the universal diagonalization is unique.

5. Properties of the local multiplicity. The proofs of most of the properties listed below are omitted because they are simple.

(1) If measures μ_1, μ_2 are mutually absolutely continuous, then $m_{T,\mu} \equiv m_{T,\mu_2}$.

(2) Assume that μ_1, μ_2 are mutually singular, i.e., there are Borel sets A_1 and A_2 such that $A_1 \cap A_2 = \varnothing$ and $\mu_j(A_j) = \mu_j(\mathbb{C})$. Then $m_{T,\mu_1+\mu_2}$ can be calculated by the formula

$$m_{T,\mu_1+\mu_2}(z) = m_{T,\mu_j}(z), \quad z \in A_j. \quad ■$$

(3) Let μ be the Dirac δ-measure, concentrated at a point z_0. Then

$$m_{T,\mu}(z_0) = \dim \operatorname{Ker}\bigl(T^* - z_0 I\bigr). \quad ■$$

An operator $Y : \mathfrak{X}_1 \to \mathfrak{X}_2$ is called a splitting operator for operators T_1, T_2 that act in Banach spaces $\mathfrak{X}_1$, $\mathfrak{X}_2$, respectively, if $YT_1 = T_2Y$. We write $T_1 \overset{d}{\prec} T_2$ if there exists a splitting operator Y whose image is dense in $\mathfrak{X}_2$. If, in addition, $\operatorname{Ker}Y = 0$, then we write $T_1 \prec T_2$. Operators T_1, T_2 are called d-quasisimilar ($T_1 \overset{d}{\sim} T_2$) if $T_1 \overset{d}{\prec} T_2$ and $T_2 \overset{d}{\prec} T_1$; and quasisimilar if $T_1 \prec T_2$ and $T_2 \prec T_1$.

(4) If ($T_1 \overset{d}{\prec} T_2$), then $n_{T_{1,\mu}} \geq n_{T_{2,\mu}}$. Indeed, to each diagonalization J of T_2 there corresponds a diagonalization JY of T_1 with $n_J = n_{JY}$. ■

Thus, the local spectral multiplicity is invariant with respect to d-quasisimilarity, quasisimilarity, and in particular, similarity.

We shall use the following fact.

Theorem (on intersection of ranges, [7])**.** *Assume that N is a normal operator on a Hilbert space H, $x \in H$, G is an open subset of $\mathbb{C}$, and $E(\,\cdot\,)$ is the spectral measure of N. If $x \in (N - \lambda)H$ for all $\lambda \in G$, then $E(G)x = 0$.*

(5) $m_{T,\mu} = 0$ μ-a.e. outside the spectrum of T.

We have to show that $(Jx)(z) \equiv 0$ outside $\sigma(T)$ for any diagonalization J of T and any $x \in \mathfrak{X}$. To do this, notice that

$$Jx(z) = (z - \lambda)\big(J(T - \lambda)^{-1}x\big)(z)$$

for $\lambda \notin \sigma(T)$. Hence, one can apply the theorem on intersection of ranges to the operator M_z on $L^2(\mu, H)$ and to the set $G = \mathbb{C} \setminus \sigma(T)$. ■

(6) $m_{T_1 \oplus T_2, \mu} = m_{T_1,\mu} + m_{T_2,\mu}$.

This follows from the formula

$$\operatorname{Red}\big(J(\mathfrak{X}_1 \oplus \mathfrak{X}_2)\big) = \operatorname{Red}\big(J_1\mathfrak{X}_1\big) \oplus \operatorname{Red}\big(J_2\mathfrak{X}_2\big),$$

valid for any diagonalization $J = J_1 \oplus J_2$ of the operator $T_1 \oplus T_2$. ■

Recall that a subspace K in $\mathfrak{X}$ is called cyclic for T if the space

$$C(K) \stackrel{\text{def}}{=} \operatorname{span}\{T^n K : n \geq 0\}$$

coincides with $\mathfrak{X}$. The least possible dimension of a cyclic subspace is called the (global) spectral multiplicity and will be denoted by ρ_T.

By the Bram theorem [2], the spectral multiplicity of a normal operator coincides with the essential supremum of its function of local multiplicity.

(7) For any T and any μ, $\rho_T \geq \operatorname{sup\,ess} m_{T,\mu}$.

Indeed, it is easy to see that for any subspace K in $\mathfrak{X}$, the local multiplicity of the space $JC(K)$ does not exceed $\dim K$. ■

6. Examples of calculation of the spectral multiplicity function. As a rule, this problem reduces to the description of all diagonalizations for a given operator; then the universal diagonalization can be singled out from all them in a natural way.

(a) Normal operators. Let N be a normal operator given in its spectral representation, $N = M_z$ on the space

$$F = \int \oplus F(z) d\vartheta(z),$$

and let μ be a measure on $\mathbb{C}$ with compact support. It is required to calculate the spectral multiplicity function of N with respect to μ. Let $\vartheta = \vartheta_a + \vartheta_s$ be the decomposition of ϑ into its absolutely continuous and singular parts with respect to μ and let $\mu = \mu_a + \mu_s$ be the corresponding decomposition of μ with respect to ϑ. There exists a partition of $\mathbb{C}$ into disjoint Borel sets A, S_μ, S_ϑ such that ϑ_a, μ_a are concentrated on A, μ_s is concentrated on S_μ and ϑ_s is concentrated on S_ϑ. The Fuglede–Putnam theorem (see [8]) implies that every diagonalization $J : F \to L^2(\mu; H)$ of N has the form

$$(Jf)(z) = \begin{cases} B(z)f(z), & z \in A, \\ 0, & z \in S_\mu, \end{cases}$$

where $\{B(z)\}$ is an operator family defined a.e., $B(z) : F(z) \to H$. It follows that

$$m_{N,\mu}(z) = \begin{cases} \dim F(z), & z \in A, \\ 0, & z \in S_\mu. \end{cases}$$

J is universal if and only if $B(z)$ is invertible ϑ_a – a.e. In particular, the local multiplicity function of N with respect to its own scalar spectral measure ϑ coincides with the multiplicity function of N defined in a usual way, $m_{N,\vartheta}(z) = \dim F(z)$.

(b) Local multiplicity outside the essential spectrum. Let T be an operator on a separable Banach space $\mathfrak{X}$ and let Ω be a domain that does not intersect the essential spectrum $\sigma_{\text{ess}}(T)$ of T. Assume for simplicity that the dimension of the eigenspaces

$$L(z) = \operatorname{Ker}\bigl(T^* - zI\bigr), \quad z \in \Omega,$$

is constant and denote it by n, then $n \in \mathbb{N} \cup \{0\}$. For example, one can put $T = S^n \oplus S^{*m}$ for any finite $m \geq 0$, where $Sf = zf$ is the shift operator on the Hardy space H^2, and $\Omega = \{|z| < 1\}$.

The spaces $L(z)$ form an antianalytic family. By the Grauert theorem [3], one can find an antianalytic family $l_1(z), \ldots, l_n(z)$ of bases in $L(z)$. Denote by $\operatorname{Hol}(\Omega.\mathbb{C}^n)$ the space of all vector-valued analytic functions from Ω to $\mathbb{C}^n$. Define an operator $J_0 : \mathfrak{X} \to \operatorname{Hol}(\Omega, \mathbb{C}^n)$,

$$(J_0 x)(z) = \bigl\{\langle x, l_j(z)\rangle\bigr\}_{j=1}^{n}.$$

Theorem 2. *Assume that T and Ω have the above properties and that a measure μ is such that $\mu(\mathbb{C} \setminus \Omega) = 0$. Then each diagonalization*

$$J : \mathfrak{X} \to L^2(\mu; H)$$

of T has the form

$$(Jx)(z) = \rho(z)(J_0 x)(z), \tag{10}$$

where $\rho(z) : \mathbb{C}^h \to H$ is a measurable family of operators.

Corollary. *In the assumptious of the theorem, the local spectral multiplicity of T with respect to μ is identically equal to n.*

Indeed, it suffices to construct a family of left invertible operators $\rho(\lambda)$ such that (10) defines a bounded operator J.

Proof of Theorem 2. Take an arbitrary point λ_0 in Ω. There exist vectors $x_1, \ldots, x_n$ in $\mathfrak{X}$ and a neighborhood G of λ_0 such that $\overline{G} \subset \Omega$, and the matrix composed of columns $(J_0 x_1)(z) \ldots, (J_0 x_n)(z)$ is invertible for $z \in \overline{G}$. The "splittability" property (2) of the local multiplicity shows that it suffices to consider the case when μ is concentrated at G.

Let K be the linear span of $x_1, \ldots, x_n$. Define a measurable operator-valued function ρ so that (10) takes place for all $x \in K$. It is clear that $\rho(\cdot)(J_0 x)(\cdot)$ is in $L^2(\mu : H)$ for all $x \in \mathfrak{X}$. Let us prove that (10) holds for all $x \in \mathfrak{X}$.

Consider the operator

$$(J_1 x)(z) = (Jx)(z) - \rho(z)(J_0 x)(z), \quad x \in \mathfrak{X},$$

that diagonalizes T. Take any $x \in \mathfrak{X}$ and let $y = J_1 x$. Fix any $\lambda \in \overline{G}$. There exists $k \in K$ such that $J_0(x - k)(\lambda) = 0$. Since $\lambda \notin \sigma_{\text{ess}}(T)$, this implies that $x - k = (T - \lambda)u$ for a certain $u \in \mathfrak{X}$, so that

$$y = J_1(x - k) = J_1(T - \lambda)u = (z - \lambda)J_1 u.$$

Since $y \in L^2(\mu; H)$, the theorem on intersection of ranges yields $y \equiv 0$, which proves (10). ∎

(c) Local multiplicity of the operator of multiplication by the independent variable on spaces of smooth functions. According to our definition, we consider only the separable case.

Let Ω be a bounded domain in $\mathbb{C}$ whose boundary splits into a finite union of disjoint simple C^1-smooth curves. Define spaces $c^\alpha(\overline{\Omega})$, $0 < \alpha < \infty$, in the following way: $c^0(\overline{\Omega}) = C(\overline{\Omega})$; for $0 < \alpha < 1$ set

$$c^\alpha(\overline{\Omega}) = \left\{ f \in C(\overline{\Omega}) : \lim_{\|x-y\| \to 0} \frac{f(x) - f(y)}{\|x - y\|^\alpha} = 0 \right\},$$

and for $[\alpha] = n$, $n \in \mathbb{N}$, set

$$c^\alpha(\overline{\Omega}) = \left\{ f : D^j f \in c^{\alpha - n}(\overline{\Omega}) \quad \forall\, j, \quad |j| = n \right\}$$

(here $[\alpha]$ is the entire part of α, and $j = (j_1, j_2)$ is a multiindex). These are Banach spaces with respect to the natural norms. Put $\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{\partial}{cdy})$ and $\overline{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{cdy})$.

Let $T = T_\alpha$ be the operator of multiplication by the independent variable on $c^\alpha(\overline{\Omega})$. We prove the following statement.

Theorem 3. *Assume that Ω is a domain of the above described type, μ is a measure whose compact support is contained in Ω, and H is a Hilbert space. Let $0 < \alpha < \infty$ and $n = [\alpha]$. An operator*

$$J : c^\alpha(\overline{\Omega}) \to L^2(\mu; H)$$

diagonalizes T_α if and only if J has the form

$$Jx = \sum_{k=0}^{n} \gamma_k \bar{\partial}^k x \tag{11}$$

for certain $\gamma_k \in L^2(\mu; H)$.

Corollary. *The local multiplicity of T_α with respect to any measure concenrated in Ω equals identically $n = [\alpha]$.*

Indeed, if $\gamma_k(z)$ are linearly independent μ-a.e., then it is easy to see that the local multiplicity of the linear set $Jc^\alpha(\overline{\Omega})$ is identically equal to n. ∎

Let us introduce the topological vector space $c^\alpha(\Omega)$ that consists of continuous functions on Ω such that $f \mid \Omega' \in c^\alpha(\overline{\Omega}')$ for any domain Ω' with smooth boundary compactly embedded in Ω.

Proposition 2. *For any $\alpha \geq 0$, the polynomials in coordinate variables are complete in $c^\alpha(\overline{\Omega})$.*

Sketch of the proof. It is sufficient to prove the completeness of restrictions of functions from $C^\infty(\mathbb{C})$ in $c^\alpha(\overline{\Omega})$. First assume that $\alpha < 1$ and take $f \in c^\alpha(\overline{\Omega})$. Then f

can be continued up to a function $\widetilde{f}$ in $c^\alpha(\mathbb{C})$ with compact support. If $\{\phi_n\}$ is an approximate identity, $\phi_n \in C^\infty(\mathbb{C})$, then $\widetilde{f} * \phi_n \in C^\infty(\mathbb{C})$ and $\widetilde{f} * \phi_n \to \widetilde{f}$ in $c^\alpha(\mathbb{C})$. The case $\alpha \geq 1$ reduces to the case $\alpha < 1$, since the operators of differentiation and of convolution commute. ■

The proof of the following statement was kindly communicated to the author by E. M. Dyn'kin.

Lemma A. *For a noninteger α, one has the implication $f \in C^\alpha(\overline{\Omega}) \Rightarrow f * -\frac{\pi^{-1}}{z} \in C^{\alpha+1}(\Omega)$.*

Proof. Put $g = f * -\frac{\pi^{-1}}{z}$, then $\frac{\partial}{\partial \bar{z}}\ g = f$. On the other hand, $\frac{\partial}{\partial \bar{z}} = f * \frac{\pi^{-1}}{z^2}$. The convolution with the kernel z^{-2} acts continuously on C^α [5]. Thus, $\frac{\partial g}{\partial z} \in C^\alpha(\Omega)$, $\frac{\partial g}{\partial \bar{z}} \in C^\alpha(\Omega)$, and therefore $g \in C^{\alpha+1}(\Omega)$. ■

By Proposition 2, the closure of $C^\infty(\mathbb{C})$ in $C^\alpha(\mathbb{C})$ coincides with $c^\alpha(\mathbb{C})$. Hence, the implication $f \in c^\alpha(\overline{\Omega}) \Rightarrow f * -\frac{\pi^{-1}}{z} \in c^{\alpha+1}(\Omega)$ holds.

Denote by $A_0^\alpha(\overline{\Omega})$ the class of functions in $c^\alpha(\overline{\Omega})$ that are analytic in Ω; it is a closed subspace in $c^\alpha(\overline{\Omega})$.

Proposition 3. *Rational functions of the variable z are dense in $A_0^\alpha(\overline{\Omega})$.*

Proof. Let Ω' be a domain with smooth boundary that contains $\overline{\Omega}$ and is close to Ω. It is easy to construct a family of univalent mappings $\phi_t : \Omega' \to \mathbb{C}$, smooth with respect to t, such that $\phi_0 = id$ and $\phi_t(\Omega) \supset \overline{\Omega}$ for $t > 0$. Then, for any $f \in A_0^\alpha(\overline{\Omega})$, the functions $f \circ \phi_t^{-1}$ are analytic in a neighbourhood of $\overline{\Omega}$ and approximate f in the metric of $c^\alpha(\overline{\Omega})$ as $t \to 0$. These functions can easily be approximated by rational ones. ■

Proof of Theorem 3. First consider the case $0 \leq \alpha < 1$. Here one can repeat the argument of the Rosenblum proof of the Fuglede–Putnam theorem (see [8]). Namely, assume that J diagonalizes the operator T_α, then $JM_{z^k} = M_{z^k}J$ for any integer $k \geq 0$. Hence,

$$J = M_{\exp(-\bar{\lambda}z)} J M_{exp(\bar{\lambda}z)}, \quad \lambda \in \mathbb{C}. \tag{12}$$

Put

$$\phi(\lambda) = M_{\exp(-\bar{\lambda}z)} J M_{\exp(\bar{\lambda}z)} = M_{\exp(-2\,\mathrm{Re}(\bar{\lambda}z))} J M_{\exp(2\,\mathrm{Re}(-\bar{\lambda}z))}$$

(the last equality follows from (12)). Since

$$\|M_{\exp(-2\,\mathrm{Re}(\bar{\lambda}z))}\|_{L^2(\mu;H)\to L^2(\mu;H)} = 1,$$

one gets

$$\|\phi(\lambda)\| \leq C\|J\|(1 + |\lambda|)^\alpha.$$

By the Liouville theorem, $\phi(\lambda) \equiv J$. The equality $\phi'(0) = 0$ implies that $M_{\bar{z}}J = JM_{\bar{z}}$. Hence, $Jp = pJ1$ for any polynomial p in z and $\bar{z}$. Now Proposition 2 yields that J has form (11), with $n = 0$ and $\gamma_0 = J1$.

(2) The case of noninteger $\alpha > 1$. Let us use induction in $n = [\alpha]$. Let $J : c^\alpha(\overline{\Omega}) \to L^2(\mu; H)$ be a diagonalization of T_α. Set $\eta = J1$. Define a diagonalization $\widetilde{J} : c^{\alpha-1}(\overline{\Omega}) \to L^2(\mu; H)$ of $T_{\alpha-1}$ by

$$\widetilde{J}\bar{\partial} f \overset{\text{def}}{=} Jf - \eta f, \quad f \in C^\alpha(\overline{\Omega}).$$

The operator $f \mapsto Jf - \eta f$, $f \in c^\alpha(\overline{\Omega})$, vanishes on all rational functions with poles off $\overline{\Omega}$ and therefore, by Proposition 3, on the whole $A_0^\alpha(\overline{\Omega})$. Hence, $\widetilde{J}$ is correctly defined. Since $\widetilde{J}g = J\big(g_1 * -\frac{\pi^{-1}}{z}\big) - \rho \cdot \big(g_1 * -\frac{\pi^{-1}}{z}\big)$, where $g \in C^{\alpha-1}(\overline{\Omega})$ and g_1 is any continuation of g to a greater domain, $\widetilde{J}$ is continuous. It diagonalizes $T_{\alpha-1}$ and thus, by the induction hypothesis, has form (11), where n must be replaced with $n-1$. Therefore, J also has form (11).

(3) The case of integer α. Choose any $\beta \in (\alpha, \alpha+1)$. Apply what has been proved in (2) to $J \mid c^\beta(\overline{\Omega})$ and make use of the contituity. ∎

(d) The last example concerns perturbations of normal operators with two-dimensional Lebesgue spectrum. Here we use the results of [10]. Namely, assume that N is a normal operator of this type, that K is its "smooth" perturbation in the sense of [10], and that ψ is the perturbation determinant. Then the local spectral multiplicities of N and $N+K$ with respect to the two-dimensional Lebesgue measure (restricted to a large disk) coincide everywhere outside the spectral singularity set $\psi^{-1}(0)$. This follows from the functional model of $N+K$, or, to be more exact, from Lemmas 6, 7 from [10].

Similar statements on the invariance of the local multiplicity under perturbations can be formulated for other classes of operators, for instance, for self-adjoint and unitary ones. This deserves a separate study.

7. A dual notion: generalized eigenvectors. Let $\mathfrak{X}$ be a separable Banach space with separable adjoint, let T be a bounded operator on $\mathfrak{X}$, let μ be a finite measure on $\mathbb{C}$ with compact support, and let H be a separable Hilbert space.

Definition. *An operator*

$$E : L^2(\mu; H) \to \mathfrak{X}$$

is called a system of generalized eigenvectors (s.g.e.) *of T with respect to the measure μ if it is bounded and*

$$TE = EM_z. \tag{13}$$

Define a measure μ^* on $\mathbb{C}$, $\mu^*(A) = \mu\big(\{\bar{z} : z \in A\}\big)$, and consider a duality between $L^2(\mu; H)$ and $L^2(\mu^*; H)$ by the equality $\langle f, g\rangle = \int \langle f(z), g(\bar{z})\rangle d\mu$. Then the adjoint to the operator M_z on $L^2(\mu; H)$ is M_z on $L^2(\mu^*; H)$, and (13) is equivalent to the equality $E^*T^* = M_z E^*$. Therefore, E is a system of generalized eigenvectors of T if and only if $E^* : \mathfrak{X}^* \to L^2(\mu^*; H)$ is a diagonalization of T^*.

If $f \in L^2(\mu; H)$, then the vector $Ef \in \mathfrak{X}$ has the sense of a continuous linear combination of generalized eigenvectors of the system E with weight f and can be denoted symbolically by $\int f(z) dE(z)$. In this connection, we call the closure of the image of E, the closed linear span of the system E and denote it by span (E).

Let $K = \int \oplus K(z) d\mu(z)$ be the largest reducing subspace contained in $\operatorname{Ker} E$ and let $L = \int \oplus L(z) d\mu(z) = K^\perp$, where $L(z) = K(z)^\perp$. Then we can treat E as defined on L and continued to K as the zero operator. Let us call the function $\phi_E(z) = \dim L(z)$, the multiplicity of eigenvectors of the system E. Since $\mathrm{Ker}E = (\mathrm{Range}E^*)^\perp$,

$$\phi_E(z) = n_{E^*}(\bar{z}),$$

where n_{E^*} is the multiplicity function of the diagonalization E^* defined in Sec. 3.

All the notions related to diagonalizations transfer also to systems of generalized eigenvectors. For instance, if E and E' are s.g.e. with respect to the same measure μ, then we say that E embraces E' if there exists a family of closed operators $\{\alpha(z)\}$ such that

$$E'f = E(\alpha f) \tag{14}$$

for a set of vectors f, dense in $L^2(\mu; H)$. It is clear that in this case $\mathrm{span}(E) \supset \mathrm{span}(E')$. Equality (14) is equivalent to the equality $E^{'*}x = \widehat{\alpha}E^*x$ for all $x \in \mathfrak{X}$, where $\widehat{\alpha}(z) = \alpha^*(\bar{z})$. Therefore, E embraces E' if and only if the diagonalization $E^{'*1}$ of T^* is subject to E^*. From here and from Theorem 1 we obtain the following result.

Theorem 1′. *For any operator T and any measure μ with the above properties, there exists a maximal s.g.e. E which embraces E' for any other s.g.e. E'.*

Similarly to the situation with universal diagonalizations, maximal system of generalized eigenvectors with respect to a given measure is unique in a natural sense. It is easy to see that, in particular, the closed linear span of a maximal system of generalized eigenvectors does not depend on its choice. One can say that the generalized eigenvectors corresponding to a measure μ are complete if a (any) maximal s.g.e. is complete.

Properties (1)–(7) (from Sec. 5) of the local spectral multiplicity also are translated easily to the language of systems of generalized eigenvectors. Let us mention specially the following analog of property (7).

Proposition 4. *If E is a complete s.g.e. of T (that is, span$(E) = \mathcal{X}$), then the global spectral multiplicity ρ_T does not exceed* $\sup\operatorname{ess}\phi_E$.

Indeed, let L, $L(z)$ be defined as above and let $\phi_E(z) = \dim L(z)$. Then $\rho_{M_z|L} = \sup\operatorname{ess}\phi_E$. If R is a cyclic subspace of $M_z|L$ of dimension $\sup\operatorname{ess}\phi_E$, then ER is a cyclic subspace of T. ■

This research was supported in part by the International Science Foundation, grant N W8000.

Translated by D. V. Yakubovich.

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