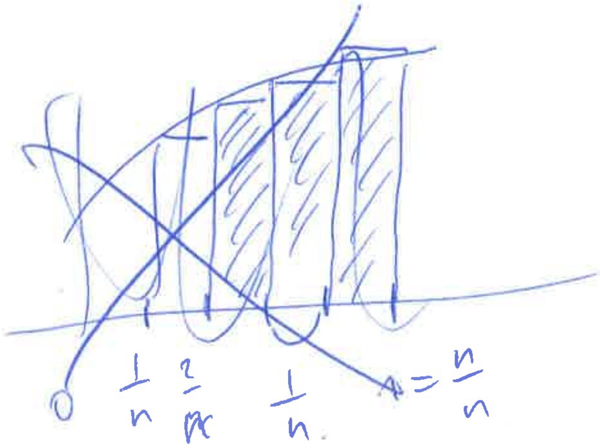
21/12/2021. 1

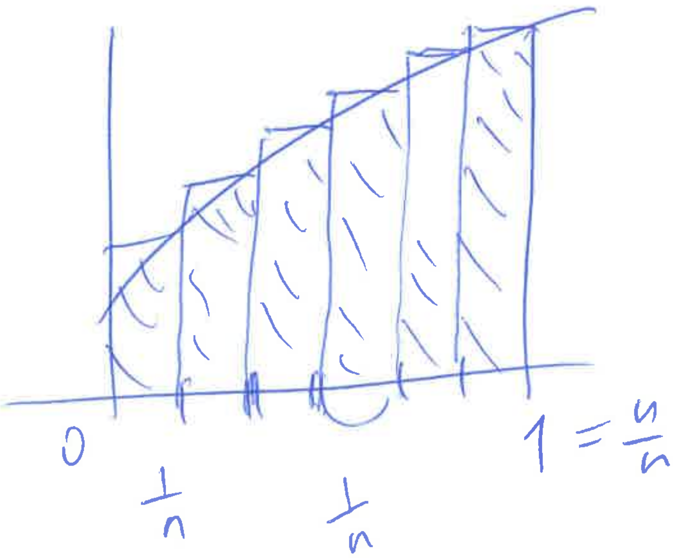
H7, 3B

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k(n-k)}{n^3}.$$

Intentaremos presentarlo en la forma

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} = \int_0^1 f(x)\,dx.$$





$$\sum_{k=1}^{n} \frac{k(n-k)}{n^3} = \frac{1}{n}\sum_{k=1}^{n} \frac{k(n-k)}{n^2} =$$

$$= \frac{1}{n}\sum_{k=1}^{n} \frac{k}{n}\cdot\left(1-\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \xrightarrow[n\to\infty]{} \int_0^1 x(1-x)\,dx.$$

Si $f(x) = x(1-x)$

Integrales impropias

$$\int_a^{\to b} f(x)\,dx \stackrel{df}{=} \lim_{t\to b^-} \underbrace{\int_a^t f(x)\,dx}_{\text{integrales de Riemann.}}$$

(impropia en b)

$b = +\infty$. o f no acotada en un entorno de b.

La integral $\int_a^{\to b}$ converge $\iff$ el límite existe hoja 7, ejercicio ~~1~~ 13 c).

(3)

$$\int_1^{+\infty} \frac{x}{1+x^4}\,dx = \lim_{t\to+\infty} \int_1^{t} \frac{x}{1+x^4}\,dx =$$

$$= \lim_{t\to+\infty} \frac{1}{2} \int_1^{t} \frac{dx^2}{1+x^4} \cancel{dt}$$

$$x^2 = y$$

$$\cancel{x^2 = y}$$

$$\frac{1}{2}\int_1^{t} \frac{dx^2}{1+x^4} = \frac{1}{2}\int_1^{t^2} \frac{dy}{1+y^2} =$$

$$= \frac{1}{2} \operatorname{arctg} y \Big|_{y=1}^{y=t^2} =$$

$$= \frac{1}{2}\left[\operatorname{arctg} t^2 - \frac{\pi}{4}\right].$$

$$\int_1^{+\infty} \frac{x\,dx}{1+x^4} = \lim_{t\to+\infty} \frac{1}{2} \int_1^{t} \frac{dx^2}{1+x^4} =$$

$$= \lim_{t\to\infty} \frac{1}{2}\left[\operatorname{arctg} t^2 - \frac{\pi}{4}\right] = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) =$$

$$= \frac{\pi}{8}.$$ Converge.

Hoja, ejc 13, f).

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$e^{-x^2} \le \begin{cases} 1, & x\in[-1,1] \text{ obvio} \\ e^{-|x|}, & |x|>1 \end{cases}$$

$|x|^2 = x^2 > |x|$ si $|x|>1$

Luego $-x^2 < -|x|$ si $|x|>1$

Luego $e^{-x^2} < e^{-|x|}$.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx \underset{\text{es par}}{=} 2\int_0^{+\infty} e^{-x^2}\,dx \le 2\int_0^1 1\,dx$$

$$+2\int_1^{+\infty} e^{-|x|}\,dx = 2 \cancel{-} 2e^{-x}\Big|_{x=1}^{x=+\infty} =$$

$$= 2 - \left[0 - 2e^{-1}\right] = 2 + 2e^{-1} < \infty.$$

~~Y~~ e^{-x^2} ~~dx~~ ≥ 0.

Luego $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converge.

Se conoce: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

Solución:

$$\int_0^{+\infty} e^{-x^2}\,dx \int_0^{\infty} e^{-y^2}\,dy \to$$

pasar a polares $\quad dx\,dy = r\,dr\,d\theta$

$$= \dots = \frac{\pi}{4} \Rightarrow \int_0^{+\infty} e^{-x^2}\,dx = \sqrt{\frac{\pi}{4}}.$$

Hoja 7, ejc 10

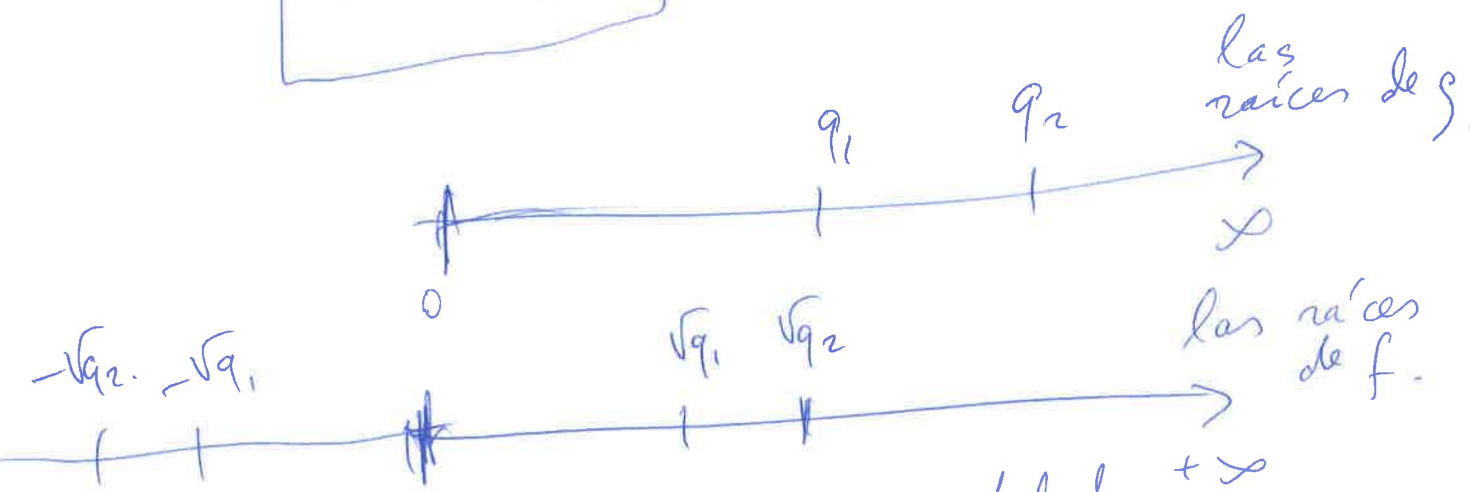
$$f(x) = -1 + \int_0^{x^2} \frac{e^{t^2}}{1+t^2}\,dt.$$

Si ponemos $g(x) = -1 + \int_0^{x} \frac{e^{t^2}}{1+t^2}\,dt$,

entonces $f(x) = g(x^2)$.

Se ve que $f(x) = f(-x)$, $f(0) = -1$.

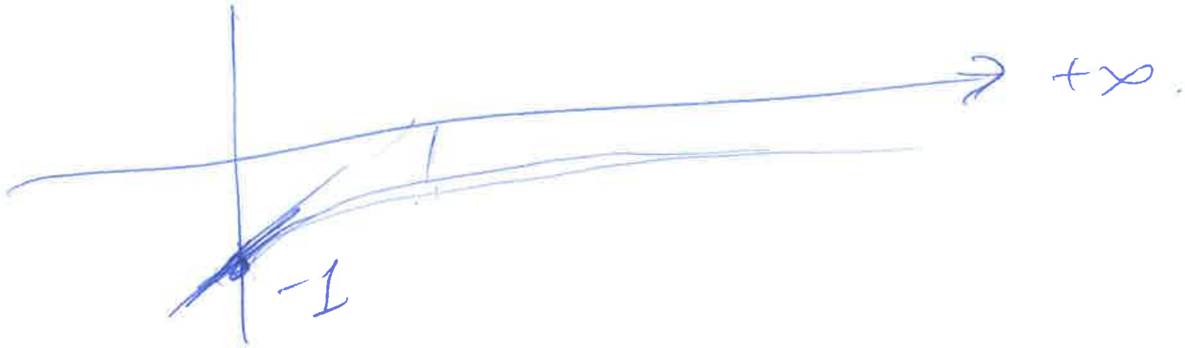
$$\boxed{\begin{cases} g(x) = 0 \\ x \geq 0 \end{cases}} \iff f(\sqrt{x}) = 0$$



la misma cantidad.

Por tanto, basta buscar las raíces no negativas de g.

$$g'(x) = \frac{e^{x^2}}{1+x^2} > 0.$$



$$\lim_{x \to +\infty} g'(x) = +\infty.$$

$\exists\, \varepsilon > 0$ tal que $g'(x) \geq \varepsilon \quad \forall x \geq 0$.

$\exists\, N$: $g'(x) \geq 1$ si $x \geq N$

$\exists \min\limits_{[0,N]} g'(x) = \tau > 0.$

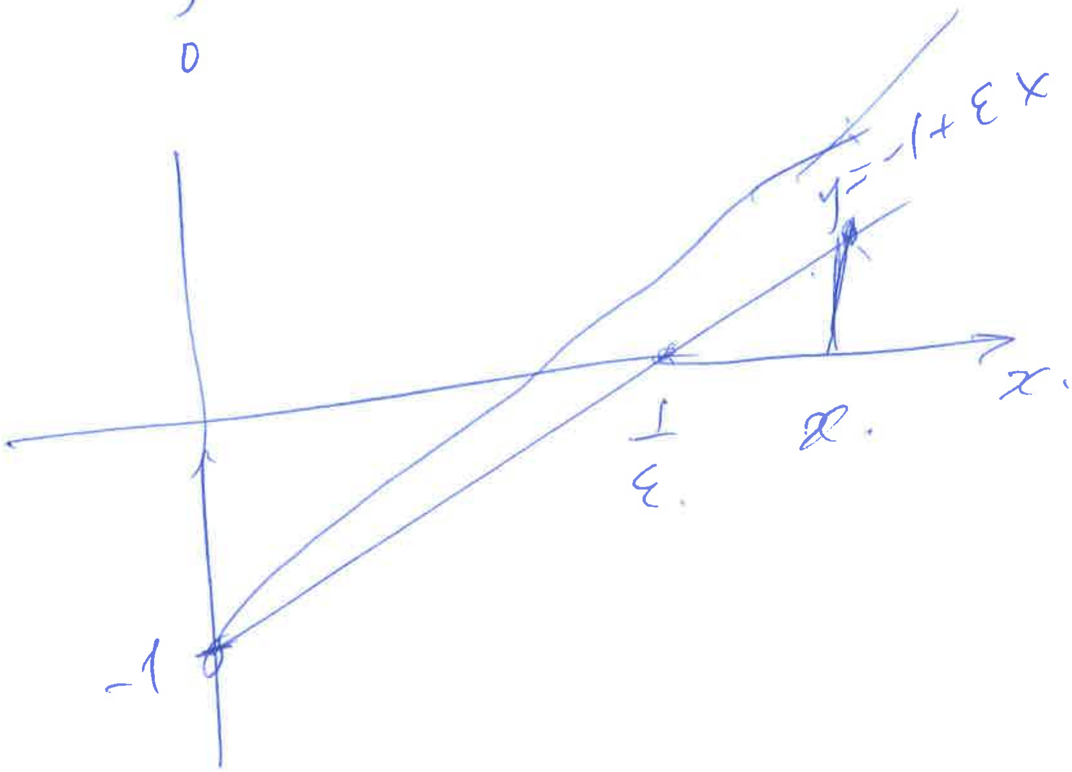
$\uparrow$ continua.

Luego $g'(x) \geq \varepsilon = \min(1, \tau)$,

$\forall x \geq 0$.

$$g(x) = -1 + \int_0^x g'(t)\, dt \geq$$

$$\geq -1 + \int_0^x \varepsilon\, dt \geq -1 + \varepsilon x.$$



Luego $x > \frac{1}{\varepsilon} \Rightarrow$

$$g(x) > -1 + \varepsilon \cdot \frac{1}{\varepsilon} = 0.$$

Luego g tiene al menos una raíz en $[0, +\infty)$. Es sólo una, porque g es estrictamente creciente

9.

Sea $q \in (0,+\infty)$ la ~~[illegible]~~ única raíz de g. Entonces las raíces de f son $\pm\sqrt{q}$.