Stable solutions to some elliptic problems: minimal cones, the Allen-Cahn equation, and blow-up solutions

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Abstract:

We will present several results on the classification of stable solutions to some nonlinear elliptic equations. These results are a crucial step within the regularity theory of minimizers to such problems. We will mainly center our attention to three different (but connected) equations. The techniques and ideas in the three settings are quite similar.

The first one is the celebrated result of Simons on the flatness of minimal cones in low dimensions, that we will describe in some detail. Its semilinear analogue is a conjecture on the Allen-Cahn equation posed by E. De Giorgi in 1978. This is our second problem, for which we will discuss some proofs, as well as an open problem (for high dimensions) on the saddle-shaped solution vanishing on the Simons cone.

The third problem concerns the boundedness of stable solutions to reaction-diffusion equations in bounded domains. We will present proofs on their regularity in low dimensions and discuss the still main open problem. Finally, we will briefly comment on related results for harmonic maps and for nonlocal minimal cones.

Stable solns to some elliptic pbs: minimal cones, the Allen-Cahn eq'n, and blow-up solns

• 5 hours course at COLUMBIA UNIV. May 2016

Contents:

1. Minimal cones
   - The Simons cone. Minimality.
   - Simon's lemma on minimal cones.
   - Comments on:
     - Harmonic maps
     - Free bdry pbs
     - Nonlocal minimal surfaces

2. Allen-Cahn equation
   - Minimality of monotone sol'ns
   - Conjecture of De Giorgi ($n \leq 3$).
   - The Saddle-shaped solution & the Simons cone
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3. Blow-up & extremal solns: semilinear eqns
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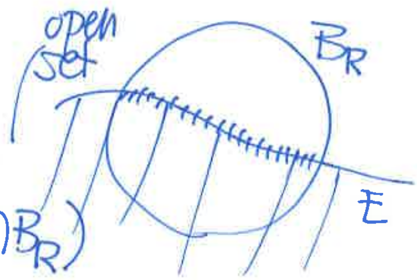
   - Regularity for $n=4$.

stable solns to some elliptic pbs: minimal cones, the Allen-Cahn equ & blow-up solns

5 hours course at COLUMBIA UNIV. May 2016 & 6h course CIME-CETRARO 2017

- **1. MINIMAL CONES**



$E\subset\mathbb{R}^n$ regular enough

$P(E;B_R)=\mathcal{H}^{(n-1)}(\partial E\cap B_R)$

Def'n $E\subset\mathbb{R}^n$ is a minimal set (or set of minimal perimeter)

iff $\forall F\subset\mathbb{R}^n$, $\forall R$, $B_R=B_R(0)$,

$$E\cap(\mathbb{R}^n\setminus B_R)=F\cap(\mathbb{R}^n\setminus B_R)\ \Rightarrow\ P(E;B_R)\le P(F;B_R)$$

[Giusti]: $\begin{cases}\varphi_t:\mathbb{R}^n\to\mathbb{R}^n,\ \varphi_0=Id\\ \varphi_t-Id \text{ compact support in } B_R\end{cases}$

$\to F=E_t:=\varphi_t(E)$

$\varphi_t=Id+t\,\xi\,\nu$ with $\operatorname{supp}\xi\subset B_R$ & ν unit normal to ∂E.

Then:

$$\boxed{(1)}\quad \frac{d}{dt}\Big|_{t=0}P(E_t;B_R)=\int_{\partial E}H\xi$$

$$\boxed{(2)}\quad \frac{d^2}{dt^2}\Big|_{t=0}P(E_t;B_R)=\int_{\partial E}\left\{|\delta\xi|^2-(c^2-H^2)\,\xi^2\right\},\quad \text{where}$$

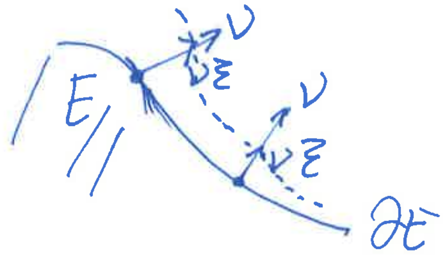
enough to have ν & ξ defined on ∂E.

$\delta\xi = \nabla_T \xi$ tangential (to ∂E) gradient

H = mean curv = $\kappa_1 + \dots + \kappa_{n-1}$

$c^2 = \kappa_1^2 + \dots + \kappa_{n-1}^2 = |A|^2$ (squared of second fund. form)

κ_i = principal curv. of ∂E



If $u : \mathbb{R}^n \to \mathbb{R}$ &

$E = \{u < 0\}$ then

$$H = H_E = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)\Big|_E$$

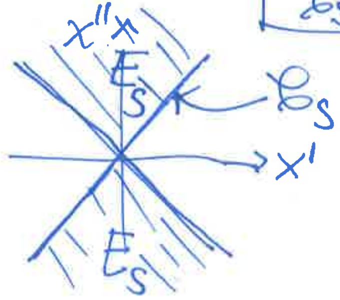
$\mathcal{C}_S := \{|x'|^2 = |x''|^2\} \subset \mathbb{R}^{2m} = \{x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^m\}$

$= \{x_1^2 + \dots + x_m^2 = x_{m+1}^2 + \dots + x_{2m}^2\}$

has zero mean curvature $\forall m \geq 1$: $\boxed{H_{\mathcal{C}_S} \equiv 0.}$ Exer. 1

Ex. 1 -see -3bis-

Simons cone



Compute $\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ for

$u = |x'|^2 - |x''|^2$

$E_S = \{u < 0\} = \{|x'|^2 < |x''|^2\} \longrightarrow \partial E_S = \mathcal{C}_S$

- $\boxed{m=2} \longrightarrow \mathcal{C}_S$ is not a minimizer.
- Thm 1 [B-DG-G, 1969] If $2m \geq 8$, E_S is minimal

($\partial E_S = \mathcal{C}_S$ is a minimizing minimal surface)

$H = 0$

minimizer

Terminology

- Not in $\mathbb{R}^2$!!! (obvious)

→ See a related diff. proof in [Cabré-Capella, 2007] ← Pf of Massari-Miranda

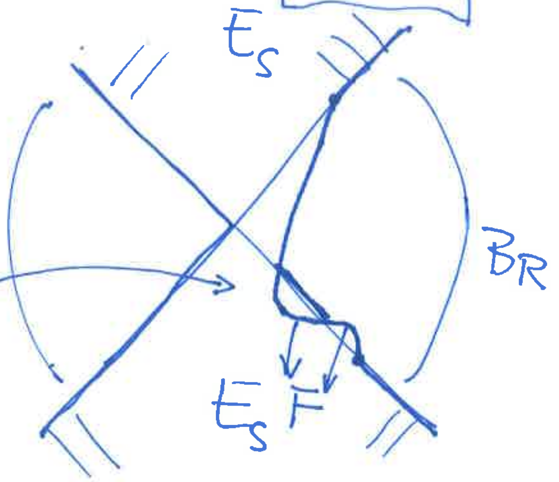
$\boxed{\text{Ex 2}}$ GIVE SOLN

Proof: Computation $\xrightarrow{m \geq 4}$ $\operatorname{div}(X)$ has the same sign as $\tilde{u} = |x'|^4 - |x''|^4$ in $\mathbb{R}^{2m}$

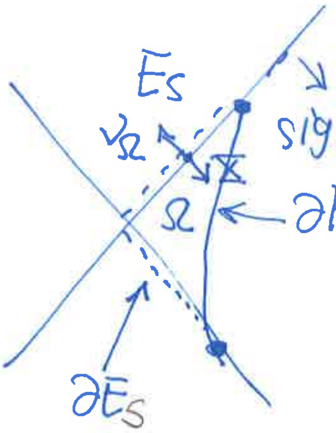
[G. de Philippis - E. Paolini, 2009]

where $X = \frac{\nabla \tilde{u}}{|\nabla \tilde{u}|}$; $\tilde{u} = |x'|^4 - |x''|^4$

see -3 bis-$_1$



• [redacted]: check this cutting method.



$\text{sign}(\tilde{u}) > 0$

$\Downarrow$

$\boxed{\operatorname{div} X \geq 0 \text{ in } E_S^c \setminus B_R}$ ①

$\boxed{\text{Ex 4: Calibration for isop. ineq -3bis-}_2}$

$$0 \leq \int_\Omega \operatorname{div} X = \int_{\partial\Omega} X \cdot \nu_\Omega = \int_{\partial E_S \cap \bar{\Omega}} \underbrace{X \cdot \nu_\Omega}_{-1} + \int_{\partial F \cap \bar{\Omega}} \underbrace{X \cdot \nu_\Omega}_{\leq 1}$$

③ $|X| \leq 1$ in B_R

② on E_S: $X = \nu_{E_S}$ exterior

50 min CIME

$$\mathcal{H}^{n-1}(\partial E_S \cap \bar{\Omega}) \leq \mathcal{H}^{n-1}(\partial F \cap \bar{\Omega}) \qquad \square$$

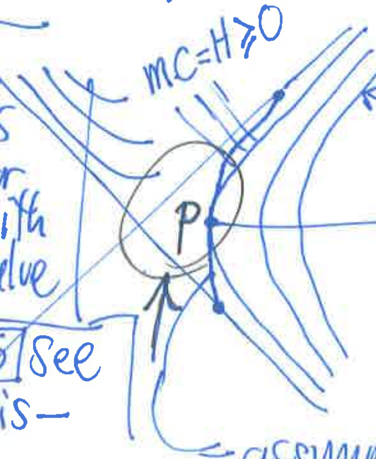
①②③ } CALIBRATION

Ex 3: Similar (simpler) argument to show that a hyperplane is minimizing.

• Another "way" to understand the proof:

FOLIATION by $H \geq 0$

RK: This argument gives UNIQUENESS for Dirichlet pb with $\mathscr{C}_S$ as bdry. value



$\{\tilde{u} = \lambda\}$

touching pt → contradiction. $\square$

$\downarrow r^{n-2} \ll 1$ as $n \uparrow$ near $r \simeq 0$

1st lecture (1h)

$\boxed{\text{Ex 5}}$ See -3 bis-$_3$

assuming $\exists$ minimizer (which does $\exists$).

• RK: other minimizing cones are some LAWSON cones: $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^{n-k}$

[Davini, 2004, ...] ($\perp$ for $n \geq 8$) $\mathscr{C}_{n,k} = \{|x'| = c_{n,k}|x''|\}$. Note: $\exists\, \mathscr{C}_{n,k}$ ... So $k \geq$ ...

Ex 1: $u=|x'|^2-|x''|^2$, $(x',x'')\in\mathbb{R}^m\times\mathbb{R}^m$

$$\nabla u = 2(x',-x'')$$
$$|\nabla u| = 2|x|$$

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \mathrm{div}_{x'}\frac{x'}{|x|} - \mathrm{div}_{x''}\frac{x''}{|x|}$$

$$= \frac{m}{|x|} - \frac{|x'|^2}{|x|^3} + \frac{m}{|x|} + \frac{|x''|^2}{|x|^3} = -\frac{|x'|^2-|x''|^2}{|x|^3} = 0 \Big|_{|x'|=|x''|}.$$

instead

$$\circledast = (s^2+t^2)\{(m-1)(s^4+t^4) - (m+2)s^2t^2\}$$

as in DePhilippis-Paolini

Ex 2

$$\tilde{u} = |x'|^4-|x''|^4$$
$$\nabla\tilde{u} = 4(|x'|^2x', -|x''|^2x'')$$
$$|\nabla\tilde{u}| = 4(|x'|^6+|x''|^6)^{1/2}$$

$$\mathrm{div}\left(\frac{\nabla\tilde{u}}{|\nabla\tilde{u}|}\right) = \mathrm{div}_{x'}\left(\frac{|x'|^2x'}{(|x'|^6+|x''|^6)^{1/2}}\right) - \mathrm{div}_{x''}\left(\frac{|x''|^2x''}{(|x'|^6+|x''|^6)^{1/2}}\right)$$

$$= \frac{m|x'|^2}{(|x'|^6+|x''|^6)^{1/2}} + \frac{2|x'|^2}{(\ \ )^{1/2}} - \frac{1}{2}\frac{|x'|^2x'}{(|x'|^6+|x''|^6)^{3/2}}\cdot 3|x'|^4\cdot 2x'$$

$$-\frac{m|x''|^2}{(\ \ )^{1/2}} - \frac{2|x''|^2}{(\ \ )^{1/2}} + 3\frac{|x''|^8}{(\ \ )^{3/2}}$$

$$= \frac{1}{(|x'|^6+|x''|^6)^{3/2}}\left\{(2+m)(|x'|^2-|x''|^2)(|x'|^6+|x''|^6) - 3(|x'|^8-|x''|^8)\right\}$$

$$= \left|\begin{matrix} s=|x'| \\ t=|x''| \end{matrix}\right| = \frac{s^2-t^2}{(s^6+t^6)^{3/2}}\Big\{\underbrace{(2+m)(s^6+t^6) - 3(s^2+t^2)(s^4+t^4)}_{\circledast}\Big\}$$

$$\circledast \to \boxed{(m-1)(s^6+t^6) - 3s^2t^4 - 3s^4t^2 \geq 0} \quad \text{if } m\geq 4$$

Young:

$$s^2t^4 \leq \frac{s^6}{3} + \frac{t^6}{3/2}; \quad 3s^2t^4 \leq s^6+2t^6$$

□

EX3: Another CALIBRATION giving the optimal ISOPERIMETRIC INEQ:

see [Cabré, Isop... survey, 2017]

$\Omega \subset \mathbb{R}^n$ smooth bdd domain $\Rightarrow$ $\boxed{\dfrac{|\partial\Omega|}{|\Omega|^{\frac{n-1}{n}}} \geq \dfrac{|\partial B_1|}{|B_1|^{\frac{n-1}{n}}}}$

Pf:

$$|\partial\Omega| = \int_{\partial\Omega} X\cdot\nu_\Omega = \sup_{\|X\|_{L^\infty}\le 1} \int_\Omega \operatorname{div} X = \sup_{\|X\|_{L^\infty}\le 1} \int_{\mathbb{R}^n} (\operatorname{div} X)\mathbb{1}_\Omega$$

$$\sup_{\|X\|_{L^\infty}\le 1} \qquad = \sup_{\|X\|_{L^\infty}\le 1} -\int_{\mathbb{R}^n} X\cdot\nabla \mathbb{1}_\Omega = \|\nabla \mathbb{1}_\Omega\|_{L^1(\mathbb{R}^n)}$$

$$= [\mathbb{1}_\Omega]_{W^{1,1}(\mathbb{R}^n)}.$$

(De Giorgi: sets of finite perimeter)

Take $X = \nabla u$ for some u:

$$|\partial\Omega| = \int_{\partial\Omega} \nabla u\cdot\nu_\Omega = \int_{\partial\Omega} \frac{\partial u}{\partial\nu} \quad \text{if} \quad \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega} = 1. \quad \xrightarrow{\text{Solve}}$$

$$\begin{cases} \Delta u = \dfrac{|\partial\Omega|}{|\Omega|} & \text{in } \Omega \quad \to \operatorname{div}(\nabla u) = |\partial\Omega|/|\Omega| \quad (1\text{bis}) \\ \dfrac{\partial u}{\partial\nu} = 1 & \text{on } \partial\Omega \quad \to \nabla u\cdot\nu_\Omega = 1 \text{ on } \partial\Omega \quad (2\text{bis}) \end{cases}$$

(3bis): $\nabla u(\Gamma_u) \supset B_1(0)$ $\quad$ ($\Gamma_u \subset \Omega$: lower contact set: $= \{x\in\Omega : u(y) \geq u(x) + \nabla u(x)\cdot(y-x)\ \forall y\in\overline{\Omega}\}$ points where $u \geq$ tang. plane)

Now: $|B_1| \leq |\nabla u(\Gamma_u)| \leq$ area formula $\leq \ldots$

Note. See [A. Henrot book 2017: Shape optim.... Brasco-De Philippis chapter] to learn Isop. ineq $\Rightarrow$ Schwarz rearrang. decreases Dirichlet energy.

Ex 5. Write the ODE in (s,t)-variables equivalent to $H=0$.

Answers:

[Bombieri-DeG-Giusti]:

$$\begin{cases} s=s(\tau) \\ t=t(\tau) \end{cases} \ \& \ \frac{d}{d\tau}=' \ : \quad \boxed{s''t'-s't''+(m-1)\{(s')^2+(t')^2\}\cdot\left(\frac{s'}{t}-\frac{t'}{s}\right)=0}$$

[Davini, On calibrations for Lawson's cones]

$$\begin{cases} s=e^{z(\theta)}\cos\theta \\ t=e^{z(\theta)}\sin\theta \end{cases} \rightarrow \int_a^b e^{(2m-1)z(\theta)}(\cos\theta)^{m-1}(\sin\theta)^{m-1}\sqrt{1+\dot z^2(\theta)}\,d\theta$$

$$\boxed{\ddot z=(1+\dot z^2)\left\{(2m-1)-\frac{2(m-1)\cos(2\theta)}{\sin(2\theta)}\dot z\right\}}$$

If ∂E is a cone, we take $\xi \in C_c^\infty(\mathbb{R}^n \setminus \{0\})$

If $H=0$ (∂E stationary minimal surface)

$\boxed{(3)}$ then
$$\frac{\partial^2}{\partial t^2}\Big|_{t=0} P(E_t; B_R) = \int_{\partial E} |\delta\xi|^2 - c^2\xi^2 \qquad \text{from (2).}$$

• Suppose $\boxed{E \text{ is a cone}}$ (i.e., $\lambda E = E \ \forall \lambda > 0$)

<u>Thm 2</u> [Simons, J.; 1968] $E \subset \mathbb{R}^n$ <u>stationary cone</u> ($H=0$) & $\partial E \setminus \{0\}$ is smooth

& it is <u>stable</u> (2^{nd} var. of area ≥ 0)

[in particular, both hold if E is a minimal set].

Then, if $n \leq 7$, ∂E is a <u>hyperplane</u>. ($3 \leq n$)

← Comment $n=2$ (minimality)

Δ_{LB}: Laplace-Beltrami opr on $\partial E = S$

$\Delta_{LB} v = \mathrm{div}_T(\nabla_T v) = \mathrm{div}_T(\delta v)$ tangential gradient & divergence. tangential

for $v: \partial E \to \mathbb{R}^n$)

• Replace ξ by $\tilde{c}\eta$, $\xi = \tilde{c}\eta$, in (3) (any $\tilde{c}$; later "$\tilde{c} = |A|$")

$$0 \leq \frac{d^2}{dt^2} P = \int_{\partial E} |\delta\xi|^2 - c^2\xi^2 = \int_{\partial E} \tilde{c}^2|\delta\eta|^2 + \eta^2|\delta\tilde{c}|^2 + \underbrace{\tilde{c}\,\delta\tilde{c}}\cdot\delta(\eta^2) - c^2\tilde{c}^2\eta^2$$

$$= \int_{\partial E} \tilde{c}^2|\delta\eta|^2 - \{\Delta_{LB}\tilde{c} + c^2\tilde{c}\}\tilde{c}\,\eta^2$$

↑ Lemma 10.8 [Giusti] (typo): $\varphi \in C_c^\infty$ $\forall E$
$$\int_{\partial E} (\delta_i\varphi)\psi = -\int_{\partial E} \varphi\,\delta_i\psi - \int_{\partial E} \varphi\psi H\nu^i$$

2½h CIME

$\boxed{(4)}$
$$\int_{\partial E} \underbrace{\{\Delta_{LB}\tilde{c} + c^2\tilde{c}\}}_{\text{Linearized opr at } \partial E}\tilde{c}\,\eta^2 \leq \int_{\partial E} \tilde{c}^2|\delta\eta|^2$$

similar also in semilinear case

$$\boxed{\tilde{c} = c} \to \int_{\partial E} \underbrace{\{c\Delta_{LB}c + c^4\}}_{= \frac{1}{2}\Delta_{LB}c^2 - |\delta c|^2 + c^4}\eta^2 \leq \int_{\partial E} c^2|\delta\eta|^2$$

$$\boxed{(5)}\quad \int_{\partial E} \left\{\tfrac{1}{2}\Delta_{LB}c^2 - |\delta c|^2 + c^4\right\}\eta^2 \leq \int_{\partial E} c^2|\delta\eta|^2$$

& $\partial E\setminus\{0\}$ smooth

<u>Lemma 3</u> [Simons] E cone stationary in $\mathbb{R}^n$, $\forall n$, $\Rightarrow$

$$\tfrac{1}{2}\Delta_{LB}c^2 - |\delta c|^2 + c^4 \geq \frac{2}{|x|^2}c^2 \quad \text{on } \partial E\setminus\{0\}.$$

(To be proved later).

<u>Pf of thm 2 (using Lemma 3)</u> For $n\geq 3$.

$$0 \leq \int_{\partial E} c^2\left\{|\delta\eta|^2 - \frac{2}{|x|^2}\eta^2\right\}.$$

$\underline{|x|=r}$ $\quad \eta(r) = \begin{cases} r^{-\alpha}, & r\leq 1 \\ r^{-\beta}, & r\geq 1\end{cases}$ $\quad$ want $\alpha^2 < 2$, $\beta^2<2$ so that $\left\{|\delta\eta|^2 - \frac{2}{r^2}\eta^2\right\}<0.$

<u>Ex 4</u>: Cut-offs at $r=0$ & $r=\infty$ $\Rightarrow$ must go to zero
integrals

$$\Leftarrow \int_{\partial E} c^2|\delta\eta|^2 < \infty \text{ at } 0 \,\&\, \infty$$

$\Updownarrow \leftarrow$ c^2 homog. of degree -2

∂E: $(n-1)$-dim'l

$\downarrow$

$$\begin{cases}(n-2)-2-2\alpha-2 > 1 \\ (n-2)-2-2\alpha-2<-1\end{cases} \Leftrightarrow \quad \alpha < \frac{n-5}{2}, \quad \frac{n-5}{2}<\beta$$

OK if $-\sqrt{2}<\frac{n-5}{2}$ (α), $\frac{n-5}{2}<\sqrt{2}$ (β)

OK if $n\leq 7$

OK if $n\geq 3$

$$c^2\equiv 0 \Leftarrow$$

$\Downarrow$

∂E union of hyperplanes
$\llcorner$ smooth in $\mathbb{R}^n\setminus\{0\}$ $\Rightarrow$ ∂E hyperpl $\square$

<u>RK</u> $n=2$ & Union of hyperplanes (see page 8)

- ∂E stationary cone. (2^{nd} var^n)

$$\Rightarrow \quad 0 \leq \int_{\partial E} |\delta \xi|^2 - c^2 \xi^2 = \int_{\partial E} (-\Delta_{LB}\xi - c^2 \xi)\xi$$

& $\quad -\Delta_{LB} - c^2 = \underbrace{-\Delta_{LB} - \frac{d(\sigma)}{|x|^2}}_{\text{Hardy type opr}}$ $\qquad x = |x|\sigma = r\sigma$, $d(\sigma) =$ cst dep. on σ

(Same scaling Δ_{LB} & $\frac{d(\sigma)}{|x|^2}$)

$$0 \leq \int_{\partial E} |\delta \xi|^2 - \frac{d(\sigma)}{|x|^2}\xi^2 \quad \text{if } \partial E \text{ stable}$$

<u>Prop^n 4</u> <u>(Hardy inequality)</u> $\boxed{n \geq 3}$ &

$\xi \in C^1_c(\mathbb{R}^n \setminus \{0\}) \Rightarrow \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{\xi^2}{|x|^2} \leq \int_{\mathbb{R}^n} |\nabla \xi|^2$

& $\frac{(n-2)^2}{4}$ is the best constant & the ineq. is <u>not</u> achieved in $\xi \in H^1(\mathbb{R}^n)$. In addition,

if $\quad c > \frac{(n-2)^2}{4} \Rightarrow \inf_{H^1_0(B_1)} \frac{\int_{B_1} |\nabla \xi|^2 - c\,\xi^2/|x|^2}{\int_{B_1} |\xi|^2} = -\infty$

2 hours

<u>Pf</u>: Polar coordinates: $\sigma \in S^{n-1}$ fixed

$$\int_0^{+\infty} \underbrace{r^{n-1} r^{-2}}_{r^{n-3} = \left(\frac{r^{n-2}}{n-2}\right)'} \xi^2(r\sigma)\, dr = -\frac{1}{n-2}\int_0^{+\infty} r^{n-2}\, 2\,\xi\, \xi_r \, dr$$

$r^{\frac{n-3}{2}} r^{-\frac{n-3}{2}}$

$$\leq \frac{2}{n-2}\left(\int r^{n-3}\xi^2 dr\right)^{1/2}\left(\int r^{n-1}\xi_r^2\right)^{1/2}$$

$$\frac{(n-2)^2}{4}\int r^{n-1}\frac{\xi^2}{r^2}(r\sigma)\,dr \leq \int r^{n-1}\xi_r^2\,dr$$

integrate in σ ($\square$ Hardy)

$c > \frac{(n-2)^2}{4} \Rightarrow \bar{u} = r^{-\alpha}_{\searrow 1}$ cutted-off near 0

$u \in H_0^1(B_1)$

Main term: $[r^{-\alpha}\cancel{(-1)}]$

$$\frac{(\alpha^2-c)\int r^{-2\alpha-2}dx}{\int r^{-2\alpha}dx} \to -\infty$$

$$\Uparrow$$

$\frac{(n-2)^2}{4} < \alpha^2 < c \to \alpha^2 - c < 0$

$\&\ \alpha > \frac{n-2}{2} \Rightarrow -2\alpha-2 < -n$

$\alpha\downarrow$

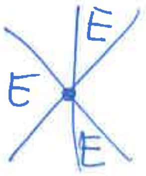
$\Downarrow$

$-2\alpha\uparrow 2-n > -n \Rightarrow \int_{B_1} r^{-2\alpha}\,dx < \infty$ & can be cutted-off remains $<\infty$

but $-2\alpha-2\uparrow -n$ $(<-n)$

$\Rightarrow \int r^{-2\alpha-2} \to +\infty.$ $\square$

RK: One must discuss ($n\geq 3$) the case $\partial E = \bigcup_{\text{finite}}$ hyperplanes & ∂E minimizing.
Done by dimension reduction & using that for $n=2$



is not minimizer.

• Proof of Lemma 3 [Simons lemma]

$$E \text{ stationary cone in } \mathbb{R}^n \ (\forall n) \overset{?}{\Longrightarrow} \tfrac{1}{2}\Delta_B c^2 - |\delta c|^2 + c^4 \geq \frac{2}{r^2} c^2$$

RK: We follow [Giusti]-Chapter 10

- Two typos
  - Lemma 10.8: $\int_{\partial E} \delta_i \varphi = -\int_{\partial E} H \varphi \nu^i$ — missed
  - (10.18): missed label in [line −8, page 122].

→ Alternative proofs using intrinsic Riemannian connection: something more clear.
- original [Simons, 1968] paper
- book by [Colding-Minicozzi]

E stationary cone; $H \equiv 0$!

$E\setminus\{0\}$ regular. $d(x) := \begin{cases} \operatorname{dist}(x,\partial E), & x\in E \\ -\operatorname{dist}(x,\partial E), & x \in E^c \end{cases}$ C^2 in neighb. of ∂E

$\nu = \nabla d = \dfrac{\nabla d}{|\nabla d|}$. Summation convention over repeated indices always. & $u_i = u_{x_i}$; $u_{ij} = u_{x_i x_j}$

$\nu = (\nu^1, \dots, \nu^n) \subset \mathbb{R}^n$
$\quad = (d_1, \dots, d_n)$

$$1=|\nu|^2=\sum_{k=1}^n d_k^2 \;\Rightarrow\; d_{jk}d_k=0 \quad \forall j.$$

$$\delta_i w \to \boxed{\begin{array}{l}\delta_i := \partial_i - \nu^i\nu^k\partial_k \\ \delta_i w = w_i - \nu^i\nu^k w_k\end{array}} \;\to\; \delta_i\nu^j = \delta_i d_j = d_{ij} - d_i d_k d_{kj} = $$

$$= d_{ij} = d_{ji} \;\Rightarrow$$

$$\boxed{\delta_i\nu^j = \delta_j\nu^i}$$

Ex: $|\nabla_T w|^2 = \underline{|\delta w|^2} = \sum_{i=1}^n \underline{|\delta_i w|^2}$ (even $(n-1)$-dim) $\forall w$

$$\boxed{\begin{array}{l}\nu^i\delta_i \equiv 0 \\ \hline \nu^i\delta_j\nu^i \equiv 0 \quad \forall j\end{array}}$$

Use it constantly!!

$$\boxed{\begin{array}{l} H = \delta_i\nu^i \\ c^2 = \delta_i\nu^j\,\delta_j\nu^i = \sum_{i,j=1}^n (\delta_i\nu^j)^2 \\ \hline \Delta_{LB} = \delta_i\delta_i = \sum_{i=1}^n \delta_i\delta_i \end{array}}$$

(6) Lemma 10.7 $\quad \delta_i\delta_j = \delta_j\delta_i + (\nu^i\delta_j\nu^k - \nu^j\delta_i\nu^k)\,\delta_k \qquad \forall i,j$

(7) $\forall$ hypersurf. & ∂E, smooth $\quad \Delta_{LB}\nu^j + c^2\nu^j = \delta_j H \quad$ ($=0$ if ∂E stationary)

[This eq'n reflects that a translation of ∂E also has $H=0$

(8) (10.18) $\quad \Delta_{LB}\delta_k = \delta_k\Delta_{LB} - 2\nu^k(\delta_i\nu^j)\,\delta_i\delta_j - 2(\delta_k\nu^j)(\delta_j\nu^i)\,\delta_i$

$$c^2 = \sum_{i,j}(\delta_i\nu^j)^2 \;\Rightarrow\; \tfrac{1}{2}\Delta_{LB}c^2 = (\delta_i\nu^j)\,\Delta_{LB}\,\delta_i\nu^j + \sum_{i,j,k}(\delta_k\delta_i\nu^j)^2$$

Using (7), (8), and $H\equiv 0 \longrightarrow$

$$\frac{1}{2}\Delta_{LB} c^2 = -(\delta_i \nu^j)\,\delta_i(c^2\nu^j) - 2(\delta_i\nu^j)(\delta_k\nu^l)(\delta_l\nu^j)(\delta_i\nu^k) + \sum_{i,j,k}(\delta_k\delta_i\nu^j)^2$$

$$\underset{\text{by (6)}}{=} -c^4 - 2\nu^i\nu^l(\delta_j\delta_l\nu^k)(\delta_k\delta_i\nu^j) + \sum_{i,j,k}(\delta_k\delta_i\nu^j)^2.$$

$x_0\in\partial E\setminus\{0\} \longrightarrow \nu(x_0)=(0,\dots,0,1)$ at x_0: $\nu^n=1$, $\nu^\alpha=0$, $\alpha=1,\dots,(n-1)$ [always greek indices], $\delta_n=0$, $\delta_\alpha=\partial_\alpha$

By (6) again & $\delta_n\nu^\alpha=\delta_\alpha\nu^n$

$$\frac{1}{2}\Delta_{LB} c^2 = -c^4 + \sum_{\alpha,\beta,\gamma}(\delta_\gamma\delta_\alpha\nu^\beta)^2 + 2\sum_{\alpha,\gamma}(\delta_\gamma\delta_\alpha\nu^n)^2 - 2\sum_{\alpha,\beta}(\delta_\alpha\delta_\beta\nu^n)^2$$

$$= -c^4 + \sum_{\alpha,\beta,\gamma}(\delta_\gamma\delta_\alpha\nu^\beta)^2$$

$$|\delta c|^2 = \frac{1}{c^2}(\delta_\alpha\nu^\beta)(\delta_\gamma\delta_\alpha\nu^\beta)(\delta_\sigma\nu^\tau)(\delta_\gamma\delta_\sigma\nu^\tau)$$

$$\frac{1}{2}\Delta_{LB}c^2 + c^4 - |\delta c|^2 = \frac{1}{2c^2}\sum_{\alpha,\beta,\gamma,\sigma,\tau}\left[(\delta_\sigma\nu^\tau)(\delta_\gamma\delta_\alpha\nu^\beta) - (\delta_\alpha\nu^\beta)(\delta_\gamma\delta_\sigma\nu^\tau)\right]^2$$

Γ E cone with vertex at 0

Coord $x_0\in\langle x_{n-1}\text{ axis}\rangle$; $\delta_i\nu^{n-1}=0$ at x_0

$A,B,S,T: 1\div(n-2)$

$$\frac{1}{2}\Delta_{LB}c^2 + c^4 - |\delta c|^2 = \frac{1}{2c^2}\sum_{A,B,S,T,\gamma}\left[(\delta_S\nu^T)(\delta_\gamma\delta_A\nu^B) - (\delta_A\nu^B)(\delta_\gamma\delta_S\nu^T)\right]^2 \quad (+)$$

$$\oplus\ \sum_{S,T,\gamma,\alpha}\frac{2}{c^2}(\partial_S v^T)^2(\partial_\gamma \partial_{n-1} v^\alpha)^2 \geq 2\sum_{\alpha,\gamma}(\partial_\gamma \partial_{n-1} v^\alpha)^2$$

$$\parallel$$

$$2\,\frac{1}{|x|^2}\sum_{i,\alpha}(\partial_i v^\alpha)^2 = \frac{2c^2}{|x|^2}$$

$\partial_i v^\alpha$ homog. of degree -1

$\square$

Rk: Similar but simpler in harmonic maps & in free bdry pb.

- Harmonic maps:

$u:\Omega\subset\mathbb{R}^n \longrightarrow \overline{S^N_+}=\{y\in\mathbb{R}^{N+1} : |y|=1,\ y_{N+1}\geq 0\}$, $E(u)=\int_\Omega \frac{|\nabla u|^2}{2}$

Eq'n: $-\Delta u = |\nabla u|^2 u$ in Ω.

Thm A $u_*(x)=(x/|x|,0)$, $u_*: B_1\subset\mathbb{R}^n\to \overline{S^n_+}$ is minimizing among $\{u$ st $u|_{\partial B_1}=u_*\}$ if & only if $n\geq 7$. [Calibration for the (if)]

Thm B $u: B_1\subset\mathbb{R}^n\to\overline{S^N_+}$ minimizing harmonic map, homogeneous of degree zero. If $3\leq n\leq 6 \Rightarrow u\equiv$ cst.

[Giaquinta-Souček & Schoen-Uhlenbeck]

Pf Thm B: After Stereographic projection, $E(v)=\int_{B_1}\frac{|\nabla v|^2}{(1+|v|^2)^2}\,dx$ & $|v|\leq 1$. Later $|v|\equiv 1$.

Test fcn in stability: $\xi(x)=v(x)\,|\nabla v(x)|\,\eta(|x|)$

$c=|\nabla v(x)|$

Lemma $\frac{1}{2}\Delta c^2-|\nabla c|^2+c^4 \geq c^2/|x|^2+\frac{c^4}{n-1}$. $\square$

• Free bdry problems [Savin & Jerison, arXiv 2014]

$$\begin{cases} \Delta u = 0 \text{ in } E \subset \mathbb{R}^n, \; E \text{ cone} \\ u = 0, \; |\nabla u| = 1 \text{ on } \partial E \\ u \text{ homogeneous of degree } 1 \end{cases}$$

$$\mathcal{E}_{B_1}(u) = \int_{B_1} |\nabla u|^2 + 1\!\!1_{\{u>0\}}$$

Thm C: u stable & $n \le 4 \Rightarrow u = (x\cdot\nu)^+$ for some ν, $|\nu| = 1$ (1d soln)

($n \le 6$? Conjecture) ($\exists$ minimizer in dim 7: De Silva - Jerison)

Pf: Linearized pb $\begin{cases} \Delta v = 0 \text{ in } E \\ v_\nu + Hv = 0 \text{ on } \partial E \end{cases}$

$$c^2 = \|D^2 u\|^2 = \sum_{i,j=1}^n u_{ij}^2$$

3rd hour columbia

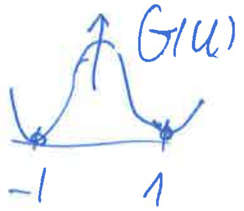
Interior ineq $\longrightarrow \frac{1}{2}\Delta c^2 - |\nabla c|^2 \ge 2\frac{n-2}{n-1}\frac{c^2}{|x|^2} + \frac{2}{n-1}|\nabla c|^2$
+ bdry ineq □

• NONLOCAL MINIMAL SURFACES (all open except cones minimizing are lines in $\mathbb{R}^2$: [Savin - Valdinoci].

[2] The Allen-Cahn eq'n

$$-\Delta u = u - u^3 \text{ in } \mathbb{R}^3 \quad \text{(crystals; Peierls-Nabarro)}$$

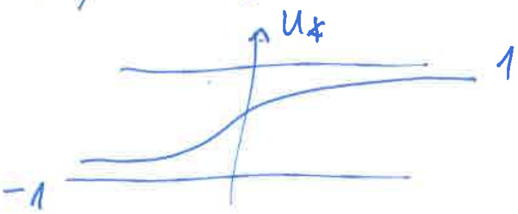
$$\mathcal{E}_\Omega(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \underbrace{\frac{1}{4}(1-u^2)^2}_{G(u)}$$



MP $\Downarrow$ $|u| \le 1$ $\xrightarrow{\text{SMP}}$ $-1 < u < 1$

1-d solns: $u(x) = u_*(x\cdot e)$, $e \in \mathbb{R}^n$, $|e| = 1$

$$u_*(y) = \tanh\left(\frac{y}{\sqrt{2}}\right)$$



Ex: check it is soln.

Def'n [9] (10) }

Thm 5 [Alberti-Ambrosio-Cabré 2001]

include Thm 5bis here

$\forall u, \forall e \in \mathbb{R}^n, |e|=1, \quad u(x)=u_*(x\cdot e)$ is a minimizer of the Allen-Cahn eqn (in $B_R(0)$, $\forall R$, under the Dir B.C. of u)

$\Updownarrow$

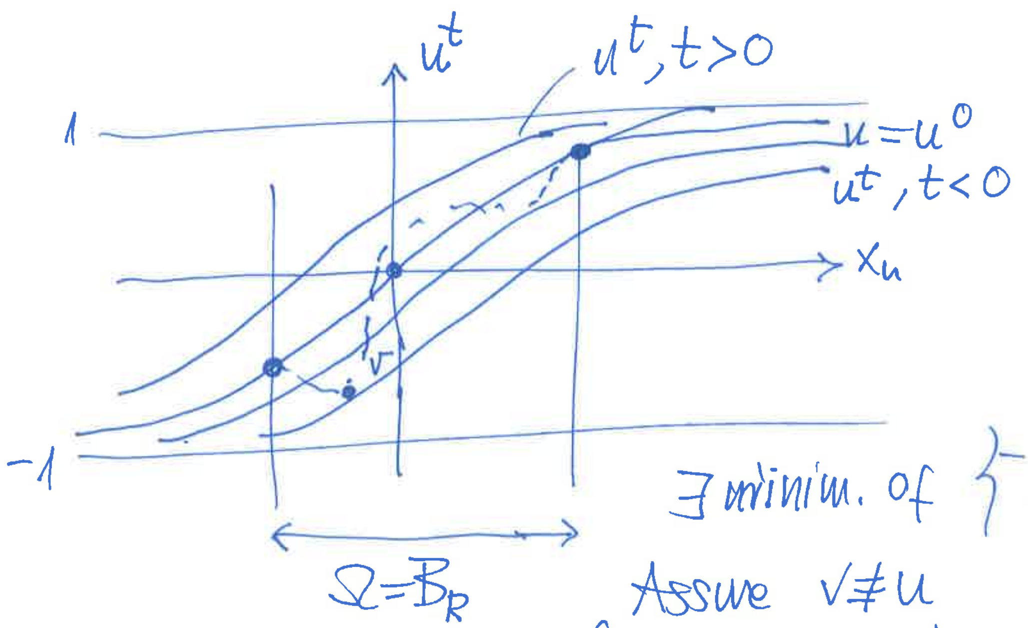
$E_{B_R}(u) \leq E_{B_R}(v)$ for all $v:\overline{B}_R \to \mathbb{R}$ st $v_{|\partial B_R} = u_{\partial B_R}$.

Proof: A foliation: $e=(0,\dots,0,1)$

$t \in \mathbb{R} \longmapsto u^t(x) = u(x', x_n+t)$, $\quad x=(x',x_n)\in \mathbb{R}^{n-1}\times\mathbb{R}$

$u(x)=u_*(x_n)=\tanh\left(\frac{x_n}{\sqrt{2}}\right) \to \begin{cases} u_{x_n}>0 & \boxed{(9)} \\ u(x',x_n) \xrightarrow[x_n\to\pm\infty]{} \pm 1 & \boxed{(10)} \end{cases}$

$t<t' \Rightarrow u^t < u^{t'}$ in $\mathbb{R}^n$



$\exists$ minim. of $\begin{cases} -\Delta v = v - v^3 & \text{in } B_R \\ v = u & \text{on } \partial B_R \end{cases}$

Assume $v \not\equiv u$

$\hookrightarrow$ Foliation + Strong MP $\to$ contradiction.

Thm 5bis [Same proof] $\forall u$ soln $-\Delta u = u - u^3$ in $\mathbb{R}^n$ satisfying (9) & (10), is a minimizer

RK: This proof is simpler than the one in [Alb-Amb-C.] (2001) where the CALIBRATION is built.

↑ vector field Ξ in $\mathbb{R}^n \times (-1,1)$ st...

Corol 6

Thm 7 [Savin, 2009]

$-\Delta u = u - u^3$ in $\mathbb{R}^n$ is a minimizer & $n \le 7$ $\Rightarrow$
$\Rightarrow$ u is a 1-d solution

CONJ. OF DE GIORGI

Corollary 6 (9)+(10) $\Rightarrow$ $E_{B_R}(u) \le C R^{n-1}$ (energy estimates)

- $n \le 3$ [Ambrosio-Cabré, 2000] see later
- Pf of Thm 7 uses: improvement of flatness; minimal cones $n \le 7$ ($\nexists$)

Thm 8 [del Pino-Kowalczyk-Wei, 2011]

$n \ge 9$ $\Rightarrow$ $\exists$ a sol'n satisfying (9)+(10) $\Rightarrow$ $\exists$ minimizer not 1d

Thm 9 [Cabré, JMPA 2012]

$\exists!$ unique soln of $-\Delta u = u - u^3$ in $\mathbb{R}^{2m}$, $m \ge 1$,
st
- $u = u(s,t)$
- $u > 0$ in $\{s > t\}$
- $u(s,t) = -u(t,s)$ in $\mathbb{R}^{2m}$

$\mathcal{C}_S$ Simons cone $\subset \mathbb{R}^{2m} = \{s = t\}$
$|x'| = s$, $|x''| = t$

$\left(\underset{RK}{\Rightarrow} u_{|\mathcal{C}_S} \equiv 0\right)$

This sol'n is called saddle-shaped solution

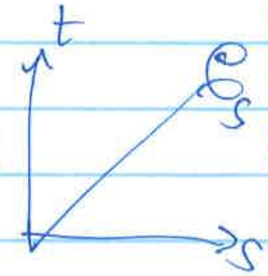
Open pb : Is the saddle soln a minimizer in $\mathbb{R}^8$? Or in $\mathbb{R}^{2m}$ for some $2m \geq 8$? Nothing is known except

Prop'n 10 [Cabré JMPA 2012]

$2m \geq 14 \Rightarrow$ the saddle soln is stable in $\mathbb{R}^{2m}$, i.e.,
$\int f'(u)\xi^2 \leq \int |\nabla \xi|^2 \quad \forall \xi \in C_c^1(\mathbb{R}^{2m})$

Pf: A new PDE:

$$u_{ss} + u_{tt} + (m-1)\left\{ \frac{u_s}{s} + \frac{u_t}{t} \right\} + \underbrace{f(u)}_{= u - u^3} = 0 \quad \underbrace{\{s>0, t>0\}}_{\subset \mathbb{R}^2}$$



$$\varphi := t^{-b} u_s - s^{-b} u_t$$

$2m \geq 14 \Rightarrow \exists b > 0$ st $\begin{cases} \Delta\varphi + f'(u)\varphi \leq 0 & \text{in } \mathbb{R}^{2m} \setminus \{st = 0\} \\ \varphi > 0 & \text{in } \mathbb{R}^{2m} \setminus \{st = 0\} \end{cases}$. Now

RKs: (1) $\exists$ positive supersol $\Rightarrow$ stability

Two ways to prove this:
- MP [Beres-Nir-Var]
- Variational proof ~~[illegible]~~ & do Cauchy-Schwarz
- $f'(u) \leq \frac{-\Delta\varphi}{\varphi}$

(2) Motivation for the proof of Conj of De Giorgi in dim $n \leq 3$

$(-\Delta - f'(u))\, u_n = 0$, $u_n > 0$ $\Rightarrow$ $\frac{u_{x_i}}{u_{x_n}} = \text{ct}$!! □

see: next page

u_n should be the "1st eigenf" in $\mathbb{R}^n$. $\Rightarrow$ Unique (it is simple)

$(-\Delta - f'(u))\, u_{x_i} = 0$

$\nwarrow$ This is not a proof $S_0 = \mathbb{R}^n$ not bounded

RK (1): $\exists \varphi>0$ in $\mathbb{R}^n$ st $-\Delta\varphi - f'(u)\varphi \geq 0$ in $\mathbb{R}^n$

$\Rightarrow$ u stable.

Proof: Given $\xi \in C^1_c(\mathbb{R}^n)$,

$$\int f'(u)\xi^2 \leq \int \frac{-\Delta\varphi}{\varphi}\xi^2 = \int \frac{\nabla\varphi}{\varphi} 2\xi\nabla\xi - \int \frac{|\nabla\varphi|^2}{\varphi^2}\xi^2$$

$$\underset{\uparrow \text{ CS}}{\leq} \int |\nabla\xi|^2 \qquad \square$$

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[3] BLOW-UP & STABLE EXTREMAL SOLUTIONS

$\Omega \subset \mathbb{R}^n$ bdd smooth, $f: \mathbb{R}^+ \to \mathbb{R}$

$$\boxed{(11)} \quad \begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u>0 & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega. \end{cases} \quad \to \quad E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u) \quad (F' = f)$$

Def'n u stable iff $D^2E(u) \geq 0$

iff $\int_\Omega f'(u)\xi^2 \leq \int_\Omega |\nabla\xi|^2 \quad \forall \xi \in H^1_0(\Omega)$

The extremal solution $\to$ [Brezis; Is there failure of the IFT?, ...

$\to$ [Cabré: [Extremal solutions & instantaneous compl. B-UP. 2007] (SURVEY)

$\to$ [Cabré Regularity of minimizers... CPAM 2010]

· Examples of stable solutions

Extremal solutions:

$$\boxed{(12)}\quad \begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \quad , \lambda \geq 0 \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$\boxed{(13)}$ with $g(0) > 0$, $g \nearrow$, $\frac{g(s)}{s} \nearrow +\infty$ as $s \uparrow \infty$ (& perhaps g convex)

[$u \equiv 0$ is NOT a sol'n for $\lambda > 0$]

($\nexists$ sol'n for $\lambda > \lambda^*$)

· Examples: $\lambda g(u) = \lambda e^u$, $\lambda(1+u)^p$, $p > 1$.

Prop'n 11 $\exists\, \lambda^* \in (0, +\infty)$ $\forall \lambda \in [0, \lambda^*]$ $\exists\, u_\lambda$ stable (& smallest) sol'n of (12). $u_\lambda \nearrow$ in λ. $u_\lambda \in L^\infty$ if $\lambda < \lambda^*$

Question [Brezis]: Is u^* bdd (regular) or unbdd?

· Example: $\Omega = \underbrace{B_1}_{n \geq 3}$, $\bar{u} = \log\left(\frac{1}{|x|^2}\right) = -2\log|x|$ sol'n of (12) with $\lambda = 2(n-2)$ & $g(u) = e^u$ $(f(u) = 2(n-2)e^u)$.

Linearized opr:

$$-\Delta - \lambda g'(\bar{u}) = -\Delta - 2(n-2)e^{\bar{u}} = -\Delta - \frac{2(n-2)}{|x|^2} \geq 0 \text{ in } H_0^1$$

$\Updownarrow$

$2(n-2) \leq \frac{(n-2)^2}{4}$

$8 \leq n-2$; $n \geq 10$.

$\Downarrow$

$\bar{u}$ stable unbdd !!

Prop'n 12 $\bar{u} = \log(1/|x|^2)$ is stable sol'n of $-\Delta u = 2(n-2)e^u$ if $n \geq 10$.

Thm 12 [Crandall-Rabinowitz]

$f(u)=e^u$ or $f(u)=(1+u)^p$, $p>1$. Then u stable soln of (11) & $n\le 9 \Rightarrow u\in L^\infty$.

Pf Use equ (11) & stability cond with $\xi = e^{\alpha u}-1$ (in case $f(u)=e^u$). $\square$

Thm 13

Thm 14 [Nedev 2000]

$f=\Delta g$ under (13) (assumption on f) $\forall\Omega$ & $n\le 3$ & u stable $\Rightarrow u\in L^\infty$

Thm 15 [Cabré, CPAM 2010]

$\forall f$, u stable soln of (11) & $n\le 4 \Rightarrow u\in L^\infty(\Omega)$

Thm 13 [Cabré-Capella, JFA 2006] (Radial case)

$\forall f$, $\Omega=B_1$, u stable soln of (11) & $n\le 9 \Rightarrow u\in L^\infty(B_1)$

RK : $\forall\Omega\ \forall f$: dims $n=5,6,7,8,9$: still open problem.

Test fcns
- Thm 13 $\Omega=B_1$: $\xi = u_r r^{-\alpha}$
- Thm 14 $n\le 3$: $\xi = h(u)$ for some h depending on f
- Thm 15 $n\le 4$: $\xi = |\nabla u|\varphi(u)$

• <u>Ideas of proof in dimension $n=4$ ($n\le 4$).</u>
Uses the <u>Michael-Simon Sobolev inequality.</u>

$-\Delta u = f(u)$ in Ω

$$\Downarrow$$

$$(\Delta + f'(u))|\nabla u| = \frac{1}{|\nabla u|}\Big\{\sum_{i,j} u_{ij}^2 - \sum_i \Big(\sum_j u_{ij}\frac{u_j}{|\nabla u|}\Big)^2\Big\}$$

$$= \frac{1}{|\nabla u|}\Big\{|\nabla u|^2 \mathcal{B}_u^2 + |\nabla_T|\nabla u||^2\Big\}.$$

where $\mathcal{B}_u^2 = \mathcal{C}_u^2 = \kappa_1^2 + \dots + \kappa_{n-1}^2$ (principal curvatures at $x\in\Omega$ of $\{y : u(y) = u(x)\}$: <u>level set of u through x</u>)
$\mathcal{B}_u^2(x) =$

As in beginning of lectures

$$\int_\Omega f'(u)\,\xi^2 \le \int_\Omega |\nabla \xi|^2 \quad \longleftarrow \quad \xi = c\eta \text{ with } \boxed{c := |\nabla u|}$$

$$\Downarrow$$

$$\int_\Omega c(\Delta c + f'(u)c)\,\eta^2 \le \int_\Omega c^2|\nabla\eta|^2$$

$$\int_\Omega \Big\{|\nabla u|^2\mathcal{B}_u^2 + |\nabla_T|\nabla u||^2\Big\}\eta^2 \le \int_\Omega |\nabla u|^2|\nabla \eta|^2$$

Now $\eta = \eta(x) = \varphi(u(x))$
$\nabla\eta = \dot\varphi(u)\nabla u$

$$\int_\Omega \{|\nabla u|^2 B_u^2 + |\nabla_T |\nabla u||^2\} \varphi(u)^2 \le \int_\Omega |\nabla u|^4 \dot{\varphi}(u)^2$$

$\boxed{\Gamma_s = \{u=s\}}$ & $\boxed{M = \max_\Omega u}$ $\downarrow$ Coarea formula

$$\int_0^M ds \; \varphi^2(s) \int_{\Gamma_s} \underbrace{\frac{|\nabla_T |\nabla u||^2}{|\nabla u|} + |\nabla u| B_u^2}_{\substack{\| \\ \underbrace{4|\nabla_T |\nabla u|^{1/2}|^2 + |B_u |\nabla u|^{1/2}|^2}_{\substack{\vee\!/ \\ c(n)\left(\int_{\Gamma_s} (|\nabla u|^{1/2})^{\frac{2(n-1)}{n-3}}\right)^{\frac{n-3}{n-1}} = \boxed{h_1(s)}}}}} \le \int_0^M ds \; \dot{\varphi}^2(s) \underbrace{\int_{\Gamma_s} |\nabla u|^3}_{\substack{\| \\ \boxed{h_2(s)}}}$$

"related" if $n \le 4$

$\Updownarrow$

$$\frac{2(n-1)}{2(n-3)} \le 3$$

$$n-1 \le 3n-9$$

$\underline{n=4 \Rightarrow h_1(s) = c(n) h_2^{1/3}(s)}$

__Thm 16__ (Michael-Simon '73 & Allard '72) (Sobolev ineq)

$M \subset \mathbb{R}^n$ $(n-1)$-dim'l immersed compact hypersurface without boundary.

$1 \le p < n-1 \Rightarrow \forall v \in C^\infty(M)$

$$\left(\int_M |v|^{p^*}\right)^{1/p^*} \le \underline{\underline{C(n,p)}} \left(\int_M |\nabla v|^p + |H|^p |v|^p\right)^{1/p}$$

where $p^* = \dfrac{p(n-1)}{(n-1)-p}$ & H = mean curv. of M.

—∘—∘— end