UAM Lectures on PDEs and Geometry

Rectifiability, regular manifolds and differential forms in Carnot groups

Bruno Franchi

Università degli studi di Bologna, Italy

A Carnot group $\mathbb{G}$ is a connected simply connected Lie group such that the Lie algebra $\mathfrak{g}$ of the left-invariant vector fields has finite dimension n, and admits a *step κ stratification*, i.e. there exist linear subspaces (so-called layers) $\mathfrak{g}_1, \dots, \mathfrak{g}_\kappa$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_\kappa \neq \{0\}, \tag{1}$$

where $\mathfrak{g}_i = \{0\}$ if $i > \kappa$, and $[\mathfrak{g}_1, \mathfrak{g}_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_i$. The Lie algebra $\mathfrak{g}$ can be endowed with a scalar product that makes the decomposition (1) orthogonal. Through exponential coordinates, the group $\mathbb{G}$ can be identified with $(\mathbb{R}^n, \cdot)$, the Euclidean space $\mathbb{R}^n$ endowed with a (generally non-commutative) group law. Within Carnot groups, we say that aa notion or a property is *intrinsic* if it depends only on the structure of $\mathfrak{g}$.

In Euclidean spaces, a fundamental notion in Geometric Measure Theory is that of *rectifiable set*. Rectifiable sets are obtained, up to a negligible subset, by "gluing up" countable families of C^1 or of Lipschitz submanifolds. Hence, understanding the objects that, within Carnot groups, naturally take the role of C^1 or of Lipschitz submanifolds is preliminary in order to develop a satisfactory theory of *intrinsic* rectifiable sets.

In these lectures we shall discuss alternative notions of regular submanifold in Carnot groups, obtaining the notion of intrinsic graph, and showing in addition the relationship with the so-called Rumin's complex of differential forms, gathering results obtained in collaboration with R. Serapioni and F. Serra Cassano.