Winter Meeting on Nonlocal PDEs and Applications

Universidad Autónoma de Madrid
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Organizers

Matteo Bonforte, María del Mar González and Fernando Quirós

Alessandro Audrito, Politecnico di Torino, IT

The influence of nonlocal diffusion in the structure of the nodal set of solutions to a class of parabolic equations

Abstract: We investigate the nodal properties of solutions $u = u(x,t)$ to a class of *nonlocal* parabolic equations. We characterise the possible blow-ups and we examine the structure of the nodal set of such solutions. More precisely, we prove that their nodal set has at least parabolic Hausdorff codimension one in $\mathbb{R}^N \times \mathbb{R}$, and can be written as the union of a locally smooth part and a singular part, which turns out to possess remarkable stratification properties. Moreover, the asymptotic behaviour of general solutions near their nodal points is classified in terms of a class of explicit polynomials of Hermite and Laguerre type, obtained as eigenfunctions to a Ornstein-Uhlenbeck type operator. Our main results are obtained through a fine blow-up analysis which relies on the monotonicity of an Almgren-Poon type quotient and some new Liouville type results for parabolic equations, combined with more classical results like the Federer Reduction Principle and the Parabolic Whitney Extension.

Begoña Barrios, Universidad de La Laguna, ES

Periodic Solutions for the One-dimensional Fractional Laplacian

Abstract: In this talk we will study the existence of periodic solutions of the non-local problem $(-\ Delta)^s u = f(u)$ in $\ mathbbR$, where $(-\Delta)^s$ is the s -Laplacian with $0 < s < 1$ and f satisfies certain assumptions of regularity and growth. For this, we will link the search for such periodic solutions with the existence of the solutions of a semilinear problem different from the previous one that is obtained using different variational theorems. We lately enlightened the results with the analysis of some examples. In particular, multiplicity results for generalized Benjamin-Ono.

Xavier Cabré, ICREA and Universidad Politecnica de Catalunya, Barcelona, ES

A gradient estimate for nonlocal minimal graphs

Abstract: The talk will be concerned with s-minimal surfaces, that is, hypersurfaces of $\mathbb{R}^n$ with zero nonlocal mean curvature. These are the equations associated to critical points of the fractional s-perimeter. We will present a recent result in collaboration with M. Cozzi in which we establish, in any dimension, a gradient estimate for nonlocal minimal graphs. It leads to their smoothness, a result that was only known for $n = 2$ and $n = 3$ (but without a quantitative bound); in higher dimensions only their continuity had been established. We will also present a work with E. Cinti and J. Serra in which we prove that half spaces are the only stable s-minimal cones in $\mathbb{R}^3$ for s sufficiently close to 1.

Àngel Calsina, Universitat Autònoma de Barcelona, ES

Nonlocal equations in structured population dynamics: the delay formulation

Abstract: Models for physiologically structured populations are often written as partial differential equations of the transport type with nonlocal terms due to the fact that in the reproduction process very seldom offspring have the same individual state as parents and also to that interactions between individuals are nonlocal in the individual state space (which is age, size or other internal variables).

This fact and the wish of a semilinear formulation which enables to deal with qualitative theory, leads to the formulation of the models as Volterra integral equations [4], [2], for the (*history*) of the birth rate and the interaction variables. An interesting issue is that when one considers these interaction variables (the so-called *environment*) as given, the population dynamics and in particular the equation for the birth rate turns out to be linear (the individuals *behave* independently of each other). Nevertheless a major difficulty, which is also present in the classical semilinear formulation of pdes (where one considers the nonlinear part as a perturbation of a generator of a linear semigroup), is that only a clever choice of the state space (and often an enlargement of the latter) allows the perturbation to be well defined and smooth.

In general these delay equations take values in an infinite dimensional Banach space [3]. One example of this happens when the interaction variable is function valued, as in hierarchical competition. We will see an example of competition for light in a forest. Even in this case it may be possible to write a characteristic equation for the stability of the steady states if the births are (size) concentrated (i.e. the birth rate is a scalar). Instead, in cell population models usually the birth rate is a density with respect to the size of the newborns and the infinite dimensionality of the model is more essential [1].

[1] A.Calsina, S. Cuadrado, P. Getto. Delay formulation of models of age and size structured cell populations, in preparation.

[2] Odo Diekmann, P. Getto, Mats Gyllenberg, Stability and bifurcation analysis of Volterra functional equations in the light of suns and stars. *SIAM Journal on Mathematical Analysis* (2007)

[3] O Diekmann, M Gyllenberg, Abstract delay equations inspired by population dynamics. *Functional analysis and evolution equations*, 2007

[4] Odo Diekmann, Mats Gyllenberg, J. A. J. Metz, Shinji Nakaoka, Andre M. de Roos. Daphnia revisited: Local stability and bifurcation theory for physiologically structured population models explained by way of an example, *J. Mat. Biol.* (2010)

José; Alfredo Cañizo, Universidad de Granada, ES

Using the Doeblin and Harris theorem for nonlocal PDE in kinetic theory and biology

Abstract: We will give an overview of the Harris theorem and related results in probability, for which some recent proofs have been given that make them easier to apply in PDE settings. The main objective is to show an exponential rate of relaxation to equilibrium for some PDE, especially nonlocal ones. We will present some recent applications to models for the electrical activity of neurons, and simple models in kinetic theory.

Àngel Castro, CSIC-ICMAT, Madrid, ES

The Muskat problem in unstable regimes

Abstract: In this talk we will consider the Incompressible Porous Media equation with an initial data of Muskat type in the unstable regime. After discussing the physics of the problem, we will show how the covex integration allow us to construct solutions of mixing type in this situation in which the classical Muskat equation is ill-posed. Also, we will present some new results addresses to the construction of solutions in the partial unstable regime.

Hardy Chan, ETH Zürich, CH

Existence of solutions for some fractional elliptic equations

Abstract: We discuss some constructive methods for obtaining solutions for the fractional Allen-Cahn equation and the fractional Yamabe problem.

Eleonora Cinti, Università di Bologna, IT

Some recent results in the study of fractional mean curvature flow

Abstract: We study a geometric flow driven by the fractional mean curvature. The notion of fractional mean curvature arises naturally when performing the first variation of the fractional perimeter functional. More precisely, we show the existence of surfaces which develope neckpinch singularities in any dimension n 2. Interestingly, in dimension n = 2 our result gives a counterexample to Greyson Theorem for the classical mean curvature flow. We also present a very recent result, in the volume preserving case, establishing convergence to a sphere. The results has been obtained in collaboration with C. Sinestrari and E. Valdinoci.

Matteo Cozzi, University of Bath, UK

Rigidity results for nonlocal minimal graphs

ABSTRACT: Nonlocal minimal surfaces are hypersurfaces of Euclidean space that minimize the fractional perimeter, a geometric functional introduced in 2010 by Caffarelli, Roquejoffre & Savin in connection with phase transition problems displaying long-range interactions.

In this talk, I will focus on the class of nonlocal minimal surfaces that can be written as graphs over the whole space R^n. I will present some recent advancements on the classification of these minimizers. Such rigidity results will be a consequence of a weak Harnack-type inequality for non-negative supersolutions of integral equations posed on nonlocal minimal surfaces and on more general hypersurfaces of R^{n+1} having Euclidean volume growth.

The talk will be based on works done in collaboration with X. Cabré (ICREA & UPC Barcelona), A. Farina (Université de Picardie), and L. Lombardini (Università di Milano).

Azahara De la Torre, Universität Freiburg, DE

On higher dimensional singularities for the fractional Yamabe problem

ABSTRACT: We consider the problem of constructing solutions to the fractional Yamabe problem that are singular at a given smooth sub-manifold, for which we establish the classical gluing method of Mazzeo and Pacard for the scalar curvature in the fractional setting. This proof is based on the analysis of the model linearized operator, which amounts to the study of a fractional order ODE, and thus our main contribution here is the development of new methods coming from conformal geometry and scattering theory for the study of non-local ODEs. Note, however, that no traditional phase-plane analysis is available here. Instead, first, we provide a rigorous construction of radial fast-decaying solutions by a blow-up argument and a bifurcation method. Second, we use conformal geometry to rewrite this non-local ODE, giving a hint of what a non-local phase-plane analysis should be. Third, for the linear theory, we use complex analysis and some non-Euclidean harmonic analysis to examine a fractional Schrödinger equation with a Hardy type critical potential. We construct its Green's function, deduce Fredholm properties, and analyze its asymptotics at the singular points in the spirit of Frobenius method. Surprisingly enough, a fractional linear ODE may still have a two-dimensional kernel as in the second order case.

This is a work done in collaboration with Weiwei Ao, Hardy Chan, Marco Fontelos, María del Mar González and Juncheng Wei.

Arturo de Pablo, Univ. Carlos III de Madrid, ES

Anisotropic nonlocal diffusion equations with singular forcing

ABSTRACT: We prove existence, uniqueness and regularity of solutions of nonlocal heat equations associated to anisotropic stable diffusion operators. The main features are that the right-hand side has very few regularity and that the spectral measure can be singular in some directions. The proofs require having good enough estimates for the corresponding heat kernels and their derivatives. The purpose is to apply the results obtained in order to prove regularity for a nonlinear problem of nonlocal porous medium type.

Joint works with F. Quirós and A. Rodríguez.

Francisco Gancedo, Universidad de Sevilla, ES

Recent results on the evolution of incompressible sharp fronts

ABSTRACT: In this talk we consider the dynamics of sharp temperature fronts evolving by incompressible flows. We study two main convection models: Boussinesq and Surface Quasi-geostrophic (SQG) equations. For Boussinesq we show local-in-time and global-in-time regularity results. For SQG models we show blow-up scenarios and conditional regularity results.

José M. Mazón, Universitat de Valencia, ES

The total variation flow in metric random walk spaces

ABSTRACT: A metric random walk space $[X, d, m]$ is a metric space (X, d) together with a family $m = (m_x)_{x\in X}$ of probability measures that encode the jumps of a Markov chain. Important examples of metric random walk spaces are: locally finite weighted connected graphs, finite Markov chains and $[\mathbb{R}^N, d, m^J]$, with d the Euclidean distance and

$$m_x^J(A) := \int_A J(x-y)\mathrm{d}\mathcal{L}^N(y) \qquad \forall A \subset \mathbb{R}^N \text{ borelian},$$

being $J : R^N \to [0; +\infty[$ a measurable, nonnegative and radially symmetric function verifying $\int_{\mathbb{R}^N} \mathrm{d}z = 1$. Also, given a metric measure space (X, d, μ) we can obtain a metric random walk space $[X, d, m^{\mu,\epsilon}]$, the so called *ϵ-step random walk associated to μ*, being

$$m_x^{\mu\epsilon} := \frac{\mu \llcorner B(x,\epsilon)}{\mu(B(x,\epsilon))}$$

Our aim is to study the total variation ow in metric random walk spaces.

The main issues we deal with are:

1. We introduce and study the concepts of total variation and perimeter
2. We define the 1-Laplacian operator and we prove that the Cauchy problem associated with this operator, that is, the total variation ow in metric random walk spaces, has a unique strong solution for any initial datum in L^2.
3. We study the asymptotic behaviour of the total variation ow, proving that in same cases, as for instance for finite graphs, the flow arrive to the media of the initial datum in finite time.
4. We give a variational characterization of the Cheeger constant and its relation with the eigenvalues of the 1-Laplacian.
5. We study the Cheeger and calibrable sets.

María Medina, Universidad de Granada, ES

Helicoidal filaments for the Ginzburg-Landau equation in 3D

Abstract: In this talk we will construct new entire solutions $u : \mathbb{R}^3 \to \mathbb{C}$ of the Ginzburg-Landau equation

$$\Delta u + (1 - |u|^2)u = 0 \text{ in } \mathbb{R}^3.$$

These solutions are screw-symmetric and have two vortex filaments clustering near the e_3-axis.

Xavier Ros-Oton, Universität Zürich, CH

The boundary Harnack principle for nonlocal elliptic equations in Lipschitz domains

Abstract: Motivated by applications to free boundary problems, we study the boundary behavior of solutions to nonlocal elliptic equations. In such context, it is important to understand the regularity of solutions in Lipschitz domains. We will present a new boundary Harnack inequality for general nonlocal elliptic operators in Lipschitz domains, and briefly discuss some applications of such result. This is a joint work with Joaquim Serra.

Luis Vega, BCAM, Bilbao, ES

Regularity of dispersive equations and fractional derivatives

Abstract: In the talk we will review some classical results concerning local and global regularity for solutions of Dispersive Equations. Recent progress on bilinear smoothing estimates for Helmholtz and Klein-Gordon will be also considered.
