Conformal Geometry

Master Thesis

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Abstract

We address the problem of characterizing local conformal flatness for non-regular metrics. This problem has been solved for regular metrics, and it is interesting in the theory of quasiconformal mappings to give similar results for low regular metrics.

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§1. Introduction.

This work is about Conformal Geometry. More concisely we are interested on establishing conditions under which a non-regular metric is locally conformally flat. Let us illustrate the problem. Suppose we have a positive definite matrix function $G : \Omega \to \mathbb{R}^{n\times n}$ defined in some open set $\Omega \subset \mathbb{R}^n$. Here $\mathbb{R}^{n\times n}$ denotes the $n \times n$ square real matrixes. Now consider the system of partial differential equations with unknown $f : \Omega \to \mathbb{R}^n$ given by

$$Df^t(x)Df(x) = J(f,x)^{\frac{2}{n}}G(x) \tag{1}$$

The system 1 is called the Beltrami System. Here $J(f,x) = |Det(Df)|$ denotes the Jacobian determinant of f. Taking determinants we see that if there exists a C^1 solution f to 1 then necessarily the determinant $Det(G) := |G| \equiv 1$ and f is a local C^1 diffeomorphism by the Inverse Function Theorem. We can assume $|G| = 1$ without problem by considering the reescaled metric $det(G)^{-\frac{1}{n}}G$ instead of G. So let us assume that $|G| = 1$.

Suppose that f is a C^1 local diffeomorphism satisfaying the Beltrami system and denote $\lambda(x) := J(f,x)^{\frac{2}{n}}$. Then we see that the metric $G(x)$ is locally the pull-back by the diffeomorphism f of the metric $\lambda^{-1}Id$. In other words, the metric G is the expression in the coordinates $x = f^{-1}(y)$ of the metric $\lambda^{-1}Id$. Therefore a necessary and sufficient condition for the local existence of a solution f to the Beltrami system is that the metric $G(x)$ is locally conformally flat, which means there there exists a C^1 diffeomorphism f so that the pull-back of G via this diffeomorphism is a function multiplied by the identity matrix (and this is the very definition of f being a solution to the Beltrami system).

We see that the problem of determining when a metric G is locally conformally flat is equivalent to proving the local existence of solutions to the Beltrami system. Moreover we see that the more adequate setting to tackle this existence problem is the lenguage of Riemannian Geometry. Therefore along this thesis we will work on smooth Riemannian Manifolds, though as our results will be of local nature we can just think that our manifold is an open set $\Omega \subset \mathbb{R}^n$ equipped with a metric of certain regularity.

By making this simplification we do not have to worry about the differentiable structure of M because open sets have canonical global coordinates. This will be useful since in the future we will have to deal with non-smooth coordinates of M. Non-smooth coordinates do not mix well with differentiable structures. For example, it can happen that a function f defined on the manifold M is very regular expressed in some non-smooth coordinates, and losses regularity when expressed in smooth coordinates.

That is why it is more convenient to have in mind that we are simply considering a function $f(x)$ canonically defined in the standard coordinates x of Ω, and then considering the pull-backs $(\varphi)^*f$ of f under certain local C^1 diffeomorphisms $\varphi : \Omega \to V \subset \mathbb{R}^n$. The same applies for the metric g of the manifold: $g(x) \in \mathbb{R}^{n\times n}$ is canonically defined in the standard coordinates of Ω, and when we say that we express g in other coordinates we are just considering the pull-back $(\varphi)^*g$, which is not g any more but a new metric defined on V. However, despite the fact that $(\varphi)^*g$ is not g, it is still interesting to study properties of the different pull-backs of g under various C^1 coodinates, (for example one may wonder how regular can be the pull-back of a certain non-smooth metric g if we can choose over all the C^1 coordinates).

Thus *one is allowed to identify the metric with its expression in some fixed coordinates* previously given on M (in this case the natural coordinates of Ω). Later on this shall help the reader fixing ideas since we are going to consider the expression of the metric in various coordinates, and it can get confussing.

It took a while before I realised this fact. During a time I considered that the metric g was not identified with a fixed expression in coordinates, and rather was defined in an abstract manifold M.

If one does this, then g is considered to be C^m regular for some $m \in \mathbb{N}$ near a point $x_0 \in M$ if and only if g has C^m regularity expressed in *any* C^{m+1} coordinates near x_0, so one is forced to demand that the coordinates are of a certain regularity depending on what is desired to prove. All this mess can be avoided by fixing some coordinates as the canonical expression of g, and then g is C^m regular if and only if this fixed expression of g is regular. If there are C^1 coordinates so that the expression of g is more regular, we regard this as a property of the pull-backs of g, and not a property of g itself.

Some part on this thesis was written before I got to the conclusion that it is better to forget about abstract manifolds. Therefore, artificial hipothesis asking certain regularity of the coordinates under which the metric is expressed (which are needed when dealing with abstract manifolds) will appear. We recommend the reader to stick to the case of M being Ω and ignore the mentioned artificial hipothesis by supposing that all the coordinates considered are C^1.

I think it is important to set this straight in order to avoid confussion.

After this digression, and coming back to the Beltrami System, note that it can be posed for arbitrarily low regular metrics G, for example for G a metric with measurable entries, by considering $f \in W^{1,2}(\Omega)$ and the weak formulation.

For $n = 2$ it can be shown that the Beltrami system is equivalent to the Beltrami equation $\partial_{\bar{z}} f = \mu(z)\partial_z f$ for some function μ which is derived from the metric G and that satisfies $\|\mu\|_{L^\infty} \leq 1$. This equation has been well studied, and the celebrated Measurable Riemann Mapping Theorem says that for every μ a measurable complex function with $\|\mu\|_{L^\infty} \leq 1$ there exists a weak solution $f \in W^{1,2}$ to the Beltrami equation. Moreover f can be proved to be a quasiconformal homeomorphism. Thus the two dimensional case of the Beltrami system is solved. We will focus on the case $n \geq 3$.

For $n \geq 3$, the classical approach to prove existence of solutions only works if the metric G is very regular, i.e, $G \in C^3$. This is the *Weyl-Schouten Theorem,* a result proved between 1896 and 1921 by Schouten and Weyl. Recently, in 2013, Salo and Liimatainen in the paper [15] have applied modern techniques to get similar results for $C^{1,\alpha}$ metrics, $\alpha > 0$.

The question of solving the Beltrami system for measurable metrics if $n \geq 3$ remains open, and it is of great interest in geometric function theory. For more on the Beltrami system and its aplications, see the monograph [25], Chapther 2.

Our first aim in this thesis is to give a detailed proof of the classic Weyl-Schouten Theorem, that works for C^3 metrics. It is an important theorem in conformal geometry. However the proof is a long and quite hard computation, and normally the literature skip the details. Once this is done, following the paper [15] we tackle the problem of weakening the regularity of g, and we see that an analogous of the Weyl-Schoutem theorem holds for less regular metrics.

1.1. Sketch of the Procedure.

Let us sketch how the classical approach works. If the metric g is regular enough, more concisely $g \in C^3$, some techniques of classic Riemannian Geometry characterize which metrics g are locally conformally flat. This is the so called Weyl-Schouten Theorem that says that a metric g on a Riemannian manifold is locally conformally flat if and only if certain tensors, the Weyl and Cotton tensors, denoted by W and C, vanish. The condition $W = C = 0$ is true if and only if the entries g_{ij} of the metric g satisfy certain identities (pde's) corresponding to the fact that the components of W and C vanish. We shall see that the identities $W = 0$ and $C = 0$ are respectively second and third order pde's on the g_{ij}. This pde's that the g_{ij} satisfy will turn out to be certain integrability conditions that allow the metric g to be locally conformally flat.

More precisely, in dimension $n = 3$ conformal flatness is equivalent to $C = 0$, a condition that involves three derivatives of g. If $n \geq 4$, conformal flatness is equivalent to $W = 0$, a condition involving only two derivatives. However, to see that $W = 0$ indeed implies that g is locally conformally flat we will need to check that in dimension $n \geq 4$ the condition $W = 0$ implies $C = 0$, so we still need $g \in C^3$.

If the metric g is less regular than C^3 this approach does not work. We will follow the techniques of Salo and Liimatainen in the papers [15] and [13]. These techniques will allow us to weaken the hipothesis of C^3 regularity and characterize local conformal flatness for metrics of class $C^{1,\alpha}$ for some $\alpha > 0$. Let us sketch the procedure for dimension $n \geq 4$.

Suppose that we have our metric g defined in Ω and that $g \in C^{1,\alpha}$. As the Weyl tensor involves two derivatives of g, we will be able to define W in the distributional sense by the same formulas than in the regular case. Then we may wonder if the condition $W = 0$ as a distribution implies that g is locally conformally flat.

To see that this is true we will express the metric in some special coordinates, called n-harmonic coordinates. Let us denote $\varphi : V \to \Omega = M$ for any system of n-harmonic coordinates. The φ coordinates satisfy certain pde (the n-Laplacian) and this confer them nice properties. One property is that the pull-back $\varphi^* g$ is also $C^{1,\alpha}$ in V, so of all the possible pull-backs of g, the pull-backs by n-harmonic are the most regulars. This is indeed true also for any system of p-harmonic coordinates for $p > 1$.

Another important property of the n-harmonic coordinates φ is that the condition of being n-harmonic is preserved under conformal change of the metric, so *being n-harmonic is a conformal property rather than a metric property.* This property does not hold for arbitrary p-harmonic coordinates.

That said, consider n-harmonic coordinates $\varphi : V \to \Omega = M$, and the pull-back $\varphi^* g$ defined on V, which is $C^{1,\alpha}$ as mentioned above Its Weyl tensor $\varphi^* W$ keeps being zero. Note that for the metric $\varphi^* g$ the standard coordinates on V are n-harmonic (just by construction). Moreover, as n-harmonic coordinates are a conformal invariant, we can multiply $\varphi^* g$ by a suitable function to obtain a conformally equivalent metric $\tilde{g}$ defined in V so that the standard coordinates on V are n-harmonic for $\tilde{g}$, and besides $|\tilde{g}| = 1$.

On the other hand, we shall prove that the equation $W(\tilde{g})(x) = 0$ for $x \in V$, regarded as a system of pde's on the metric $\tilde{g}$, becomes elliptic under the gauge conditions $|\tilde{g}| = 1$ and $\Delta_n x^k = 0$, being x^k the standard coordinates in V and Δ_n the n-Laplacian.

Therefore by elliptic regularity this imples that $\tilde{g}$ is smooth. Then we apply the classic Weyl Schouten Theorem to $\tilde{g}$ and conclude that $\tilde{g}$ is locally conformally flat. Finally, as both $\tilde{g}$ and g are in the same conformal class, this will imply that g is locally conformally flat also.

Therefore, one could consider that the last section of the Thesis is the most important, since almost all the work done before points towards it. However I find that also the so called technical results are important in mathematics, and interesting by its own right, and that is why an effort is made to prove them as much as possible. When not proved, at least I try to state the result and explain its relevance.

More concisely, in the Thesis we make use of powerful results of elliptic regularity both classic and modern. On the one hand we have the classic Schauder estimates-type regularity results for elliptic equations in Holder spaces.

On the other hand we have the elliptic regularity results provided by the theory of Pseudodifferential Operators. This implies to work with the Fourier transform and with spaces of functions well behaved for singular integral operators. These spaces will be the Zigmund spaces C^r_*. It will require some work to set the main properties of these spaces, which will enable us to establish the elliptic regularity results for pseudodifferential operators.

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1.2. Some Remarks about Notation.

(1) We always assume that $0 \in \mathbb{N}$, the natural numbers.

(2) To express partial derivatives we shall use the well known multiindex notation, though some times we will write $D^k f$ to denote any derivative of f of order k. This notation is clear and shorter, and avoid writing too much in some situations.

(3) We have made an abuse of notation regarding the coordinates on a manifold. In order to refer to coordinate systems on a manifold M we have sometimes used Greek letters such as ϕ, φ, η, and sometimes we have used the letters x and y. This should be understand as follows. The Greek letters normally refer to a parametrization i.e, $\varphi : \Omega \to U \subset M$, and the letters x and y usually are the proper coordinates, so $x = \varphi^{-1} : U \to \Omega$. Sometimes we say that a tensor is expressed in the φ coordinates, and sometimes we say that a tensor is expressed in the x coordinates. Both sentences mean the same, i.e, if the tensor T is defined on M, then its expression in the x or φ coordinates is the pull-back $\varphi^* T$. Note that

$$\varphi^* : \{\text{Things defined on M}\} \to \{\text{Things defined on } \Omega\}$$

For example if T is a $(2,0)$ tensor on M then for X, Y vector fields on Ω we have

$$[\varphi^* T](X,Y) = T(d\varphi(X), d\varphi(Y))$$

We will imposse always on a coordinate system to be C^1 in order to have a well defined pull-back.

(4) When we have derived products of functions we have written for every multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| = \sum_i \alpha_i = m$

$$D^\alpha(fg) = \sum_{|\beta|+|\eta| \le m} D^\beta f D^\eta g$$

This formula is not true, since only some of the summands written above actually appear in $D^\alpha(fg)$. The reader should understand the formula above as saying that $D^\alpha(fg)$ is a sum of some things of type $D^\beta f D^\eta g$ with $|\beta| + |\eta| \le m$.

(5) We point out also that although we work with real valued functions and vector real valued functions troughout this text, we can assume that the functions are complex valued and vector complex valued respectively, with no changes. In the part where we use Fourier analysis this remark is especially important, since complex valued functions are the correct functions to whom apply the Fourier transform. In particular if u is a real function, its Fourier transform $\mathcal{F}[u]$ is complex valued, and in general is not real valued.

(6) In the first part of this Thesis I did not use Einstein summation convention. At first I felt more comfortable with the usual summation symbols, due to inexperience with Einstein notation. Later on, I realised how powerful Einstein notation is, and I began to use it. The most amazing think about Einstein notation is that you can know whether a coordinate expresion is invariant or not at first glance.

For example, for a function u defined on M consider the expression $g^{ab}\partial_a u \partial_b u$, a typical example of Einstein summation. The fact that a and b appear as lower indexes and super indexes mean that in the expression above we sume over a and over b ranging from 1 to $n = dim(M)$. The fact that the sum $g^{ab}\partial_a u \partial_b u$ does not depend on the coordinates can be seen by noting that a and b appear as lower and upper indexes, so we cancel them in the same way we cancel terms in the fraction $\frac{ab}{ab} = 1$. Indeed, $g^{ab}\partial_a u \partial_b u$ is the norm of the gradient of u.

Another example, the expression $\Gamma^j = \Gamma^j_{ab} g^{ab}$ does depend on coordinates, since only a and b cancel out, and j remains, the fraction here would be $\frac{jab}{ab} = j$. Actually, it can be proved that $\Gamma^j = -\Delta x^j$. Note also that in Einstein notation the coordinates x^j are denoted with super indexes. This is done to make things work. For the metric and partial derivatives we use subindexes $g_{ab} = g(\partial_a, \partial_b)$ and for the inverse of the metric we use super indexes g^{ab}. This way, $du = \partial_b u dx^b$ does not depend on coordinates, and neither it does $grad(u) = (g^{ab}\partial_a u)\partial_b$.

In general, given a tensor T, say of type $(4,0)$, we denote $T_{abcd} := T(\partial_a, \partial_b, \partial_c, \partial_d)$, and we raise the index of T putting by $T_{abc}{}^d = T_{abcl}g^{ld}$. Also we denote

$$\nabla_a T_{bcde} := (\nabla_{\partial_a} T)(\partial_b, \partial_c, \partial_d, \partial_e)$$

and for other tensors is analogous.

§2. Integrability Conditions for Overdetermined Systems.

In this section we shall study conditions under which overdetermined system of PDE's admit solutions. As we shall se, this will play a crucial role when proving conformal flatness. In analogy with linear systems of algebraic equations, a system of PDE's is called overdetermined if the system has more equations than unknown functions.

First of all we remind the well known theorem for existence of ordinary differential equations.

Theorem 1. Let $U \subset \mathbb{R}^m$ and open set, $f : U \to \mathbb{R}^n$ a continuous function, and $x_0 \in U$. Let $a > 0$ such that:

(1) $\overline{B}(x_0, 2a) \subset U$.

(2) There exists $L > 0$ such that $|f| \leq L$ in $\overline{B}(x_0, 2a)$.

(3) There exists $K > 0$ such that $|f(x) - f(y)| \leq K|x - y|$ for all $x, y \in \overline{B}(x_0, 2a)$

Now select $b > 0$ such that $b \leq a/L$ and $b < \frac{1}{K}$. Then for all $x \in B(x_0, a)$ there exists a unique $\alpha_x : (-b, b) \to U$ verifying

$$\begin{aligned} \alpha_x'(t) &= f(\alpha_x(t)), \\ \alpha_x(0) &= x, \end{aligned}$$

Proof. See [1], Theorem 2 of Chapter 5. □

Note that in Theorem 1 the domain of definition $(-b, b)$ of the curves α_x can be taken in such a way that it does not depend on the initial values x, as long as this initials values are in $B(x_0, a)$.

Remark 1. If $f : U \to \mathbb{R}^n$ is of class C^1, then for every $x_0 \in U$ and for every $a > 0$ such that $\overline{B}(x_0, 2a) \subset U$, the hypothesis of Theorem 1 are satisfied because we can apply the mean value theorem for the components of f and then use the boundness of the differential Df in $\overline{B}(x_0, 2a)$ to verify the Lipschitz condition (3).

The next theorem we want to prove is about integrability conditions of a first order system of partial differential equations. We begin by an example.

Example 1. Suppose we are given open sets $U \subset \mathbb{R}^2$ and $I \subset \mathbb{R}$ and C^1 functions $f, g : U \times I \to \mathbb{R}$. Let $(x_0, y_0) \in U$. We wonder if there exists a C^1 function $\alpha : W \to I$, where $W \subset U$ is an open neighborhood of (x_0, y_0), such that for every $(x, y) \in W$

$$\begin{aligned} \frac{\partial \alpha}{\partial x}(x, y) &= f(x, y, \alpha(x, y)) \\ \frac{\partial \alpha}{\partial y}(x, y) &= g(x, y, \alpha(x, y)) \end{aligned} \tag{2}$$

If this is the case, α is C^2 and if we put $\tilde{f}(x,y) = f(x,y,\alpha(x,y))$ and $\tilde{g}(x,y) = g(x,y,\alpha(x,y))$, then

$$\frac{\partial \tilde{f}}{\partial y}(x,y) = \frac{\partial^2 \alpha}{\partial y \partial x}(x,y) = \frac{\partial^2 \alpha}{\partial x \partial y}(x,y) = \frac{\partial \tilde{g}}{\partial x}(x,y)$$

and applying the chain rule to $\tilde{f}$ and $\tilde{g}$ we should have

$$\frac{\partial f}{\partial y}(x,y,\alpha(x,y)) + \frac{\partial f}{\partial z}(x,y,\alpha(x,y))\frac{\partial \alpha}{\partial y}(x,y) = \frac{\partial g}{\partial x}(x,y,\alpha(x,y)) + \frac{\partial g}{\partial z}(x,y,\alpha(x,y))\frac{\partial \alpha}{\partial x}(x,y)$$

for all $(x,y) \in U$ and this yields a necessary condition for f and g for the existence of α. This condition is, however, not quite satisfactory, since it appears the unknown α. If we substitute according to equation (2) the partial derivatives of α, we get

$$\begin{aligned} &\frac{\partial f}{\partial y}(x,y,\alpha(x,y)) + \frac{\partial f}{\partial z}(x,y,\alpha(x,y))g(x,y,\alpha(x,y)) \\ &= \frac{\partial g}{\partial x}(x,y,\alpha(x,y)) + \frac{\partial g}{\partial z}(x,y,\alpha(x,y))f(x,y,\alpha(x,y)) \end{aligned}$$

And it still appears the unknown α. But if we want the local existence of such function α around (x_0, y_0) satisfaying equation (2) with *arbitrary* initial conditions $\alpha(x_0,y_0) = z_0$, $z_0 \in I$ then necessarily we must have

$$\frac{\partial f}{\partial y}(x,y,z) + \frac{\partial f}{\partial z}(x,y,z)g(x,y,z) = \frac{\partial g}{\partial x}(x,y,z) + \frac{\partial g}{\partial z}(x,y,z)f(x,y,z)$$

for all $(x,y,z) \in U \times I$.

This indeed give us a good necessary condition depending only on the data of the system (2). This necessary conditions are called *integrability conditions* for the overdetermined system of PDE'S given by (2), (an overdetermined system is a system of equations where there are more equations than variables). Recall that it is indeed reasonable to expect that an overdetermined system of PDE'S needs to be special in order to admit solutions, just as for linear systems of algebraic equations. In the case above, 'special' just means that the system satisfies the integrability conditions.

It is easy to generalize the above calculations for vectorial functions. The next theorem proves that for the system (2) the integrability conditions just obtained are indeed sufficient for local existence of a solution.

Theorem 2. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets, and let $f_1, \ldots, f_m : U \times V \to \mathbb{R}^n$ be C^1 functions. Denote $t = (t_1, \ldots, t_m)$ and $x = (x_1, \ldots, x_n)$ the coordinates in U and V respectively. The following are equivalent:

(a) For every $x_0 \in V$ and $t_0 \in U$ there is a unique function $\alpha = (\alpha^1, \ldots, \alpha^n) : W \to V$, where $W \subset \mathbb{R}^m$ is some open neighborhood of $t_0 \in \mathbb{R}^m$ satisfying

$$\begin{aligned} &\frac{\partial \alpha}{\partial t_j}(t) = f_j(t, \alpha(t)) \\ &\alpha(t_0) = x_0 \end{aligned} \tag{3}$$

for each $t \in U$ and $j = 1, \ldots, m$.

(b) The functions $f_1 = (f_1^1, \ldots, f_1^n), \ldots, f_m = (f_m^1, \ldots, f_m^n)$ satisfy the following integrability conditions for all $(t,x) \in U \times V$:

$$\frac{\partial f_j}{\partial t_i}(t,x) - \frac{\partial f_i}{\partial t_j}(t,x) + \sum_{k=1}^{n} f_i^k(t,x)\frac{\partial f_j}{\partial x_k}(t,x) - \sum_{k=1}^{n} f_j^k(t,x)\frac{\partial f_i}{\partial x_k}(t,x) = 0 \in \mathbb{R}^m$$

for $i, j = 1, \ldots, m$. Note in the equations above that by the partial derivative of a vectorial function we mean taking the partial derivative of each of its components.

Remark 2. Though the integrability conditions may look complicate, they are just as simple as in the previous example in each coordinate, because the vectorial system just mean that

$$\frac{\partial \alpha^l}{\partial t_j}(t) = f_j^l(t, \alpha(t)) \quad \text{for} \quad l = 1, \ldots, n; \;\; j = 1, \ldots, m$$

and if we forget the index l, this is just the previous example, so we imposse

$$\frac{\partial}{\partial t_i}[f_j^l(t, \alpha(t))] = \frac{\partial^2 \alpha^l}{\partial t_i \partial t_j}(t) = \frac{\partial^2 \alpha^l}{\partial t_j \partial t_i}(t) = \frac{\partial}{\partial t_j}[f_i^l(t, \alpha(t))]$$

so by the chain rule,

$$\frac{\partial f_j^l}{\partial t_i}(t, \alpha(t)) + \sum_{k=1}^{n} \frac{\partial f_j^l}{\partial x_k}(t, \alpha(t)) \frac{\partial \alpha^k}{\partial t_i}(t) = \frac{\partial f_i^l}{\partial t_j}(t, \alpha(t)) + \sum_{k=1}^{n} \frac{\partial f_i^l}{\partial x_k}(t, \alpha(t)) \frac{\partial \alpha^k}{\partial t_j}(t)$$

and this yields the integrability conditions in each coordinate if we substitute

$$\frac{\partial \alpha^k}{\partial t_i}(t) \;\text{ by }\; f_i^k(t, \alpha(t)) \;\;,\;\; \frac{\partial \alpha^k}{\partial t_j}(t) \;\text{ by }\; f_j^k(t, \alpha(t)) \;\text{ and }\; \alpha(t) \;\text{ by }\; x$$

so in practice if we want to determine the integrability conditions, rather than identify first who the f_j^l are and then check the integrability conditions, it is easier to imposse the second order partial derivatives to be equal, then substitute every derivative of α you find, and then consider α as a variable rather than a function.

Proof. If condition (a) is true, we can apply the equality of mixed partial derivatives and the generality of the initial data as in the remark to obtain (b).

Now suppose (b). For the sake of notation, we shall assume that $U \subset \mathbb{R}^m$ is an open neighborhood of $0 \in \mathbb{R}^m$ and that $t_0 = 0$.

We first want to define the function α in the t_1 axis. Necessarily it must satisfy

$$\begin{aligned} &\frac{\partial \alpha}{\partial t_1}(t_1, 0, \ldots, 0) = f_1(t_1, 0, \ldots, 0, \alpha(t_1, 0 \ldots, 0)) \\ &\alpha(0) = x_0 \end{aligned} \tag{4}$$

In view of this, we consider the system of ODE'S

$$\begin{aligned} &{\beta_1}'(t) = f_1(t, 0, \ldots, 0, \beta_1(t)) \\ &\beta_1(0) = x_0 \end{aligned} \tag{5}$$

As we know, there is a unique solution β_1 of the last system, defined for $|t| < \varepsilon_1$, being $\varepsilon_1 > 0$ small enough. If we define α in the t_1 axis by putting $\alpha(t_1, 0 \ldots, 0) = \beta_1(t_1)$, then α satisfies (4) as we want.

Now we fix t_1 with $|t_1| < \varepsilon_1$ and consider the system

$$\begin{aligned} &{\beta_2}'(t) = f_2(t_1, t, 0, \ldots, 0, \beta_2(t)) \\ &\beta_2(0) = \alpha(t_1, 0, \ldots, 0) \end{aligned} \tag{6}$$

This system has a unique solution $\beta_2^{t_1}$ defined for $|t| < \varepsilon_2$ for some $\varepsilon_2 > 0$. This solution $\beta_2^{t_1}$ depends on the initial data t_1, so the size of its domain $2\varepsilon_2$ depends also on t_1. However, if we choose ε_1

small enough, all the initial data $\alpha(t_1, 0, \dots, 0) = \beta_1(t_1)$ with $|t_1| < \varepsilon_1$ are as close to x_0 as we want by the continuity of β_1. Now by Theorem 1 we know that all the solutions $\beta_2^{t_1}$ corresponding to initial data close enough can be defined for $t < \varepsilon_2$, being $\varepsilon_2 > 0$ small. In this way we can define for $|t_1| < \varepsilon_1, |t_2| < \varepsilon_2$

$$\alpha(t_1, t_2, 0 \dots, 0) = \beta_2^{t_1}(t_2)$$

and from this definition it is obvious that

$$\begin{aligned} &\frac{\partial \alpha}{\partial t_2}(t_1, t_2, 0 \dots, 0) = f_2(t_1, t_2, 0, \dots, 0, \alpha(t_1, t_2, 0 \dots, 0)) \\ &\alpha(0) = x_0 \end{aligned} \tag{7}$$

We claim in adittion that $\frac{\partial \alpha}{\partial t_1}(t_1, t_2, 0, \dots, 0) = f_1(t_1, t_2, 0 \dots, 0, \alpha(t_1, t_2, 0 \dots, 0))$. Note first that as f_2 is C^1, α is C^2 in their two variables, so this makes sense.

Now, fix t_1 with $|t_1| < \varepsilon_1$ and let

$$g(t) = \frac{\partial \alpha}{\partial t_1}(t_1, t, 0, \dots, 0) - f(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)).$$

We have $g(0) = 0$. By the chain of rule it follows that

$$\begin{aligned} g'(t) &= \frac{\partial^2 \alpha}{\partial t_2 \partial t_1}(t_1, t, 0 \dots, 0) - \frac{\partial f_1}{\partial t_2}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \\ &- \sum_{k=1}^{n} \frac{\partial f_1}{\partial x_k}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \frac{\partial \alpha_k}{\partial t_2}(t_1, t, 0, \dots, 0) \\ &= \frac{\partial}{\partial t_1}[\frac{\partial \alpha}{\partial t_2}(t_1, t, 0 \dots, 0)] - \frac{\partial f_1}{\partial t_2}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \\ &- \sum_{k=1}^{n} \frac{\partial f_1}{\partial x_k}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) {f_2}^k(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \\ &= \frac{\partial}{\partial t_1}[f_2(t_1, t, 0, \dots, 0, \alpha(t_1, t, 0 \dots, 0))] - \frac{\partial f_1}{\partial t_2}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \\ &- \sum_{k=1}^{n} \frac{\partial f_1}{\partial x_k}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) {f_2}^k(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \\ &= \frac{\partial f_2}{\partial t_1}(t_1, t, 0, \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \\ &+ \sum_{k=1}^{n} \frac{\partial f_2}{\partial x_k}(t_1, t, 0, \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \frac{\partial \alpha^k}{\partial t_1}(t_1, t, 0 \dots, 0) \\ &- \frac{\partial f_1}{\partial t_2}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \\ &- \sum_{k=1}^{n} \frac{\partial f_1}{\partial x_k}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) {f_2}^k(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) \end{aligned}$$

and therefore, using the integrability conditions stated as hypothesis we see that

$$g'(t) = \frac{\partial f_2}{\partial t_1}(t_1, t_2, 0, \dots, 0, \alpha(t_1, t, 0 \dots, 0)) - \frac{\partial f_1}{\partial t_2}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0))$$
$$+ \sum_{k=1}^{n} \frac{\partial f_2}{\partial x_k}(t_1, t_2, 0, \dots, 0, \alpha(t_1, t, 0 \dots, 0))[g^k(t) + f_1{}^k(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0))]$$
$$- \sum_{k=1}^{n} \frac{\partial f_1}{\partial x_k}(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)) f_2{}^k(t_1, t, 0 \dots, 0, \alpha(t_1, t, 0 \dots, 0)))$$
$$= \sum_{k=1}^{n} \frac{\partial f_2}{\partial x_k}(t_1, t, 0, \dots, 0, \alpha(t_1, t, 0 \dots, 0)) g^k(t)$$

This shows that the function g is a solution of an homogeneous ODE, which has just one solution fixed the initial data. As $g(0) = 0$, it must be $g(t) = 0$ for every t such that $|t| < \varepsilon_2$.

So we have our solution α defined in the $t_1 t_2$ plane. It is clear that we can continue defining α in this manner until α is defined in some neighborhood $(-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2) \times, \dots, (-\varepsilon_n, \varepsilon_n)$ and this concludes the proof.

□

For obvious reasons, from now on we may not be so explicit about the points where a function is evaluated, unless the evaluated points, as in the previous proof, are not the typical ones. In case we do not write it, it will surely be easy to get it from the context.

§3. Curvature and Local Flatness for C^2 Metrics.

Now we are going to prove a classical theorem: if the Riemann curvature tensor vanishes then the metric is locally flat. Our proof here has the adventage of being purely analytical and one does not get lost in tensor notation. Of course this implies making lots of calculations. Besides, from the proof one can see how the components R^l_{ijk} of the curvature tensor arises naturally as integrability conditions for the existence of coordinates in which the metric is flat. Precisely this is what lead Riemann to the discovery of the curvature tensor, one of the most important tools in differential geometry.

Recall first some definitions and notations.

Remark 3. Let (M, g) be a Riemannian Manifold of dimension n. This mean that M is a smooth manifold but *we do not suppose necessarily that g is smooth.* We will be precise on what regularity we assume on the metric. Fix a local coordinate system $(x_1, \dots, x_n)$, not necessarily smooth either. We denote $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ the matrix of the metric g in this coordinates and g^{ij} the inverse of this matrix. When there is not possible confusion about which coordinate system we are using, we may denote the partial derivatives as $\partial_i := \frac{\partial}{\partial x_i}$. Assume for now that the expression g_{ij} of the metric in these coordinates is regular enough to justify the computations below.

We write ∇ for the Levi-Civita connection with Christoffel symbols Γ^k_{ij} such that

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$$

where the sum is taken for $k = 1, \dots, n$. In general if we do not put limits of summation this will always be assumed. The fact that there exists a unique Levi-Civita connection (symmetric and compatible with the metric) implies that the symbols Γ^k_{ij} can be recovered from the metric. More concisely, if we define

$$[ij, k] = \frac{1}{2}(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k})$$

then we have

$$\Gamma_{ij}^m = \sum_k g^{km}[ij,k]$$

The Riemann curvature tensor R is the $(3,1)$ tensor given by

$$R(X,Y,Z) := R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We will note $r(X,Y,Z,T) = g(R(X,Y)Z,T)$ the $(4,0)$ Riemann curvature tensor. The components R_{ijk}^l of R in coordinates are given by

$$R(\partial_i,\partial_j)\partial_k = \sum_l R_{ijk}^l \frac{\partial}{\partial x_l}$$

It is convenient to obtain the expression of R_{ijk}^l in local coordinates $(x_1,\ldots,x_n)$. We have first

$$\begin{aligned}
\nabla_{\partial_i}\nabla_{\partial_j}\partial_k &= \nabla_{\partial_i}(\sum_l \Gamma_{jk}^l \partial_l) = \sum_l (\frac{\partial \Gamma_{jk}^l}{\partial x_i}\frac{\partial}{\partial x_l} + \Gamma_{jk}^l \nabla_{\partial_i}\partial_l) \\
&= \sum_l \frac{\partial \Gamma_{jk}^l}{\partial x_i}\partial_l + \sum_{l,m} \Gamma_{jk}^l \Gamma_{il}^m \partial_m = \sum_l \frac{\partial \Gamma_{jk}^l}{\partial x_i}\partial_l + \sum_{m,l} \Gamma_{jk}^m \Gamma_{im}^l \partial_l = \sum_l (\frac{\partial \Gamma_{jk}^l}{\partial x_i} + \sum_m \Gamma_{jk}^m \Gamma_{im}^l)\partial_l
\end{aligned}$$

therefore

$$\begin{aligned}
R(\partial_i,\partial_j)\partial_k &= \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k \\
&= \sum_l (\frac{\partial \Gamma_{jk}^l}{\partial x_i} - \frac{\partial \Gamma_{ik}^l}{\partial x_j} + \sum_m \Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l)\frac{\partial}{\partial x_l}
\end{aligned}$$

We conclude that

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x_i} - \frac{\partial \Gamma_{ik}^l}{\partial x_j} + \sum_m (\Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l) \tag{8}$$

Note that the functions R_{ijk}^l involve second order derivatives of the metric g, so it seems that the curvature only can be defined in these coordinates $(x_1,\ldots,x_n)$ (at least in a classical sense) if the functions g_{ij} are C^2.

In the following Remark we define non-smooth functions on the manifold M, and discuss what happens if a tensor is expresseed in a system of non-smooth coordinates. We will see that there will be an important difference between how regular is a tensor expressed in some given and fixed coordinates and how regular is that tensor regarded as a tensor defined on the manifold M.

Remark 4. Given a smooth manifold M of dimension n, its smooth structure is given by

$$\mathcal{S}^\infty = \{\phi_\alpha^{-1} : A_\alpha \to U_\alpha \subset M : \alpha \in \Lambda, A_\alpha \text{ an open subset of } \mathbb{R}^n \}$$

The functions ϕ_α^{-1} are called *parametrizations* and the functions ϕ_α are called *coordinates* . We know the definition of $C^\infty(M)$, the smooth functions on M, as the functions $f : M \to \mathbb{R}^m$ such that for every $\phi_\alpha \in \mathcal{S}^\infty$ the localization or pull-back $(\phi_\alpha^{-1})^* f = f \circ \phi_\alpha^{-1} : A_\alpha \to \mathbb{R}^m$ is smooth as a function defined in $A_\alpha \subset \mathbb{R}^n$.

We can define less regular functions in an analogous way. For a nonnegative integer k, and given a function $f : M \to \mathbb{R}$, we say that $f \in C^k(M)$ if for every $\phi_\alpha \in \mathcal{S}^\infty$, $f \circ \phi_\alpha^{-1} \in C^k(A_\alpha)$ in the classical sense. We call $f \circ \phi_\alpha^{-1}$ an *expression of f in coordinates* , which is the same as the pull-back $(\phi^{-1})^* f$. So we can define C^k diffeomorphisms between manifolds as biyective mappings whose expression on any system of smooth coordinates is a C^k diffeomorphism between open sets of $\mathbb{R}^n$. Then we define C^k coordinates on M as C^k diffeomorfisms between open sets of M an open sets of $\mathbb{R}^n$.

It is straighforward that with this definition the composition of two C^k functions is C^k, so if $f \in C^k$ and $(x_1, \ldots, x_n)$ is a C^k system of local coordinates, then f is C^k when expressed in that coordinates. The minimun regularity we demand on a system of coordinates is C^1 if we want to work in a manifold, because otherwise we can not make sense of $\frac{\partial}{\partial x_i}$ so the tangent spaces are not defined. Note that to define the curvature tensor in coordinates, we need C^2 regularity since we need to make use of $\frac{\partial^2}{\partial x_j \partial x_i}$.

We can extend the definition of C^k regularity to tensors. Given $m \in \mathbb{N}$ and a tensor T of type $(m, 0)$, we say that $T \in C^k(M)$ if for every smooth vector fields $X_1, \ldots, X_m$, the function $T(X_1, \ldots X_m) \in C^k(M)$. Other way to see this is that when we express T in smooth coordinates, we have that

$$T = \sum_{i_1, \ldots, i_m} a_{i_1 \ldots i_m} dx_{i_1} \otimes \ldots dx_{i_m}$$

and $T \in C^k(M)$ if and only if the functions $a_{i_1 \ldots i_m} \in C^k(M)$.

About the Hipothesis of Regularity on this Work.

In this thesis we are seeking for local results about conformal flatness, and these results will be obtained by working in not necessarily smooth coordinates. Imagine for example that we work in a C^1 coordinate system $\varphi^{-1} : A \to U$ for $A \subset \mathbb{R}^n$, $U \subset M$ and we deduce that the expression of a function $f \in C(M)$ given by $(\varphi^{-1})^* f = f \circ \varphi^{-1}$ is smooth in the φ coordinates. Then the only thing we can say about f (as a function defined on M) is that $f \in C^1(M)$. However, it is interesting the fact that $(\varphi^{-1})^* f$ is smooth, even if this does not imply that f is smooth as a function in M.

Therefore, across all this thesis, when we suppose that a function f or an $(m, 0)$-tensor $T(X_1, \ldots, X_m)$ defined on M are C^k expressed in some coordinates η with variable regularity (depending on the particular need of derivatives we have supposed $\eta \in C^4$ or $\eta \in C^2$) we can change this hipothesis and simply suppose that in some fixed C^1 coordinates $\varphi : U \to A$, with $U \subset M$ and $A \subset \mathbb{R}^n$ open sets, the pull-backs $(\varphi^{-1})^* f$ and $(\varphi^{-1})^* T$ are C^k in the Riemannian manifold (A, g'), being $g' := (\varphi^{-1})^* g$ the pull-back of g, given by

$$g'_x(u, v) = [(\varphi^{-1})^* g]_x(u, v) := g(d_x \varphi^{-1}(u), d_x \varphi^{-1}(v)).$$

The definition of $(\varphi^{-1})^* T$ is analogous. This way, under the mentioned assumptions that $(\varphi^{-1})^* f$ and $(\varphi^{-1})^* T$ are C^k in (A, g'), if additionally we know that the coordinates φ are C^{k+1}, then the vector fields ∂_i are C^k, so both f and T are also C^k defined on M. Note that we would only need $\varphi \in C^k$ to conclude that f is C^k on M since f does not involve partial derivatives.

From this, we see that all the results we will obtain are valid for tensors on M if the coordinates φ on which we work are regular enough. If φ is not so regular we simply obtain information about a concrete expression in coordinates of f and T, and not about the function f or the tensor T defined on M. Nevertheless we do obtain information about the function $(\varphi^{-1})^* f$ and the tensor $(\varphi^{-1})^* T$ defined in the Riemannian manifold (A, g') and this is also interesting.

The next Lemma will be the key tool to prove that if the curvature tensor vanishes then the manifold is locally flat.

Lemma 1. Let U be an open set in $\mathbb{R}^n$ with a C^1 metric g. Let us denote $(y_1, \ldots, y_n)$ the standard coordinate system of $\mathbb{R}^n$. Let $(x_1, \ldots, x_n)$ be an arbitrary coordinate system. Suppose that the metric satisfies $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij}$, i.e, the metric is flat in the coordinates $(x_1, \ldots, x_n)$. Then the coordinates $(x_1, \ldots, x_n)$ satisfy this overdetermined system: for all $\lambda = 1, \ldots, n$

$$\frac{\partial}{\partial y_k}[(\frac{\partial x_\lambda}{\partial y_1}, \ldots, \frac{\partial x_\lambda}{\partial y_n})] = (\sum_\gamma \frac{\partial x_\lambda}{\partial y_\gamma} \Gamma^\gamma_{1k}, \ldots, \sum_\gamma \frac{\partial x_\lambda}{\partial y_\gamma} \Gamma^\gamma_{nk})$$

Proof. Let us write $g_{ij} = g(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$. Then the metric can be written $g = \sum_{i,j} g_{ij} dy_i \otimes dy_j$. On the other hand we have the functions $x_\alpha(y_1, \ldots, y_n)$ so

$$dx_\alpha = \sum_i \frac{\partial x_\alpha}{\partial y_i} dy_i.$$

The metric is flat in the $(x_1, \ldots, x_n)$ coordinates if and only if $g = \sum_\alpha dx_\alpha \otimes dx_\alpha$. If we substitute dx_α in terms of dy_i, we see that this is equivalent to

$$\begin{aligned} g &= \sum_\alpha (\sum_i \frac{\partial x_\alpha}{\partial y_i} dy_i) \otimes (\sum_j \frac{\partial x_\alpha}{\partial y_j} dy_j) \\ &= \sum_{i,j} (\sum_\alpha \frac{\partial x_\alpha}{\partial y_i} \frac{\partial x_\alpha}{\partial y_j}) dy_i \otimes dy_j \end{aligned}$$

so the metric is flat in the (x_i) coordinates if and only if for every i, j we have

$$g_{ij} = \sum_\alpha \frac{\partial x_\alpha}{\partial y_i} \frac{\partial x_\alpha}{\partial y_j}. \tag{9}$$

This is a system of PDE'S, but this system does not accomodate to Frobenious theorem. However, some manipulations will allow us to put it that way. First of all we can give an equivalent formulation of system (9) which will be useful later. Consider the matrixes $A = (a_{ij}) = (\frac{\partial x_i}{\partial y_j})$ and $G = (g_{ij})$. Then (9) says that

$$A^t \cdot A = G$$

which is equivalent to $G^{-1} = A^{-1} \cdot (A^t)^{-1}$ and hence to

$$A \cdot G^{-1} \cdot A^t = I$$

where I is the identity matrix. If we read this last matrix identity, it just mean that

$$\sum_{i,j} g^{ij} \frac{\partial x_\mu}{\partial y_i} \frac{\partial x_\nu}{\partial y_j} = \delta_{\mu\nu} \tag{10}$$

Now we will derive from system 9 another easier to work with. We first differentiate respect to y_k and obtain

$$\frac{\partial g_{ij}}{\partial y_k} = \sum_\alpha \frac{\partial^2 x_\alpha}{\partial y_k \partial y_i} \frac{\partial x_\alpha}{\partial y_j} + \sum_\alpha \frac{\partial x_\alpha}{\partial y_i} \frac{\partial^2 x_\alpha}{\partial y_k \partial y_j}$$

Now we write the same equation changing the indexes (i, j, k) for (j, k, i) and for (k, i, j) (note this is a cyclic permutation) and get

$$\frac{\partial g_{jk}}{\partial y_i} = \sum_\alpha \frac{\partial^2 x_\alpha}{\partial y_i \partial y_j} \frac{\partial x_\alpha}{\partial y_k} + \sum_\alpha \frac{\partial x_\alpha}{\partial y_j} \frac{\partial^2 x_\alpha}{\partial y_i \partial y_k}$$

$$\frac{\partial g_{ki}}{\partial y_j} = \sum_\alpha \frac{\partial^2 x_\alpha}{\partial y_j \partial y_k} \frac{\partial x_\alpha}{\partial y_i} + \sum_\alpha \frac{\partial x_\alpha}{\partial y_k} \frac{\partial^2 x_\alpha}{\partial y_j \partial y_i}.$$

and when we sum, cancellations occur in such a way that

$$[jk, i] = \frac{1}{2}(\frac{\partial g_{ij}}{\partial y_k} - \frac{\partial g_{jk}}{\partial y_i} + \frac{\partial g_{ki}}{\partial y_j}) = \sum_\alpha \frac{\partial x_\alpha}{\partial y_i} \frac{\partial^2 x_\alpha}{\partial y_k \partial y_j}.$$

Now we have the following identities

$$\sum_{\gamma} \Gamma_{jk}^{\gamma} \frac{\partial x_{\lambda}}{\partial y_{\gamma}} = \sum_{i,\gamma} g^{i\gamma} \frac{\partial x_{\lambda}}{\partial y_{\gamma}} [jk, i] = \sum_{i,\gamma,\alpha} g^{i\gamma} \frac{\partial x_{\lambda}}{\partial y_{\gamma}} \frac{\partial x_{\alpha}}{\partial y_i} \frac{\partial^2 x_{\alpha}}{\partial y_k \partial y_j}$$
$$= \sum_{\alpha} \frac{\partial^2 x_{\alpha}}{\partial y_k \partial y_j} (\sum_{i,\gamma} \frac{\partial x_{\alpha}}{\partial y_i} \frac{\partial x_{\lambda}}{\partial y_{\gamma}} g^{i\gamma}) = \sum_{\alpha} \frac{\partial^2 x_{\alpha}}{\partial y_j \partial y_k} \delta_{\alpha\lambda} = \frac{\partial^2 x_{\lambda}}{\partial y_j \partial y_k}$$

where we have used equation 10. This prove what we wanted.

□

Remark 5. With the notation of the previous lemma, let $\alpha_{\lambda} = (\frac{\partial x_{\lambda}}{\partial y_1}, \dots, \frac{\partial x_{\lambda}}{\partial y_n})$, $\alpha_{\lambda} : \mathbb{R}^n \to \mathbb{R}^n$. We have just seen that for all $\lambda = 1, \dots, n$, the function α_{λ} satisfies the system of PDE'S given by

$$\frac{\partial \alpha_{\lambda}}{\partial y_k}(y) = f_k(y, \alpha_{\lambda}(y)) \quad , k = 1, \dots, n \tag{11}$$

where $f_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$f_k(y, z) = (f_k^1(y, z), \dots, f_k^n(y, z)) = (\sum_{\gamma} z_{\gamma} \Gamma_{1k}^{\gamma}(y), \dots, \sum_{\gamma} z_{\gamma} \Gamma_{nk}^{\gamma}(y)).$$

We have n solutions $\alpha_1, \dots \alpha_n$ of this system so that $\alpha_1(y), \dots \alpha_n(y)$ are linearly independent as vectors in $\mathbb{R}^n$ in some neighborhood U of say $y_0 \in \mathbb{R}^n$, since they are the rows of the differential of the chart $(x_1, \dots, x_n)$. As the $f_k(y, z)$ are linear in the variable z it follows that linear combinations of solutions of the system 11 are also solutions. In particular we have a solution α of the system 11 satisfying any initial condition $\alpha(y_0) = v_0 \in \mathbb{R}^n$. Then the functions f_k must satisfy the integrability conditions

$$\frac{\partial f_k}{\partial y_l}(y, z) - \frac{\partial f_l}{\partial y_k}(y, z) + \sum_{\mu=1}^{n} f_l^{\mu}(y, z) \frac{\partial f_k}{\partial z_{\mu}}(y, z) - \sum_{\mu=1}^{n} f_k^{\mu}(y, z) \frac{\partial f_l}{\partial z_{\mu}}(y, z) = 0 \in \mathbb{R}^n$$

for $k, l = 1 \dots, n$. In this case, looking at the components $f_k^j(y, z) = \sum_{\gamma} \Gamma_{jk}^{\gamma}(y) z_{\gamma}$ this integrability conditions are

$$\sum_{\gamma} \frac{\partial \Gamma_{jk}^{\gamma}}{\partial y_l}(y) z_{\gamma} - \sum_{\gamma} \frac{\partial \Gamma_{jl}^{\gamma}}{\partial y_k}(y) z_{\gamma} + \sum_{\mu,\gamma} \Gamma_{jk}^{\mu} \Gamma_{\mu l}^{\gamma} z_{\gamma} - \sum_{\mu,\gamma} \Gamma_{jl}^{\mu} \Gamma_{\mu k}^{\gamma} z_{\gamma} = 0 \tag{12}$$

for $j, k, l = 1, \dots n$, for all $y \in U, z \in \mathbb{R}^n$. By linearity in z, equation 12 is true for all $z \in \mathbb{R}^n$ if and only if it is true for $z = e_i$ the canonical basis of $\mathbb{R}^n$, and putting $z = e_{\gamma}$ for each γ we obtain

$$R_{lkj}^{\gamma} = \frac{\partial \Gamma_{jk}^{\gamma}}{\partial y_l}(y) - \frac{\partial \Gamma_{jl}^{\gamma}}{\partial y_k} + \sum_{\mu} (\Gamma_{jk}^{\mu} \Gamma_{\mu l}^{\gamma} - \Gamma_{jl}^{\mu} \Gamma_{\mu k}^{\gamma}) = 0 \quad , \gamma = 1, \dots, n \tag{13}$$

and this is precisely to say that $R_{lkj}^{\gamma} = 0$ for all $l, k, j, \gamma = 1, \dots, n$. This calculation shows how the components of the curvature tensor arises naturally (well, as natural as the calculations above) as something that must vanish for the metric to be plane. Now the point is that the process we have done is reversible in the sense that if R vanishes we can recover the coordinate system $(x_1, \dots, x_n)$ in which the metric is plane.

Definition 1. We say that a Riemannian manifold (M, g), with $g \in C^2$, is locally flat if around each point there are C^3 coordinates $(x_1, \dots, x_n)$ in which the metric is flat, i.e, $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = g_{ij} = \delta_{ij}$.

Theorem 3. Let (M, g) be a Riemanian manifold with $g \in C^2$. Then (M, g) is locally flat if and only if the curvature tensor R vanishes.

Proof. Suppose that around every point of M there are C^3 coordinates in which the metric is flat. Then working on these coordinates $g_{ij} \in C^2$, and $\Gamma_{ij}^k = 0$ for all i, j, k and then $R_{ijk}^l = 0$ for all i, j, k, l, so $R = 0$ as $(3, 1)$ tensor. Note that we need the coordinates to be at least C^3 in order to differentiate the metric twice.

Now suppose $R = 0$. Since this is a local question we can assume $M = U$ is an open neighborhood of $0 \in \mathbb{R}^n$ with the standard coordinate system $(y_1, \ldots, y_n)$ and the components of the metric $g_{ij} : U \to \mathbb{R}$ are C^2 functions. If we look at the proof of the lemma and the remark below, we just want to go back and recover the coordinates $(x_1, \ldots, x_n)$.

Step 1: First we claim that there are C^2 functions $h = (h^1, \ldots, h^n)$ with any desired initial conditions $(h^1(0), \ldots, h^n(0))$ satisfayng the equation

$$\frac{\partial}{\partial y_k}[(h^1, \ldots, h^n)] = (\sum_\gamma h^\gamma \Gamma_{1k}^\gamma, \ldots, \sum_\gamma h^\gamma \Gamma_{nk}^\gamma) \quad ; \quad k = 1, \ldots, n \tag{14}$$

This is because the integrability conditions of this system are precisely $R_{ijk}^l = 0$ as we just saw in the remark 5 and lemma 1 . We can choose then n solutions $h_1, \ldots, h_n$ of that system such that $X_1 = h_1(0), \ldots, X_n = h_n(0)$ are n orthonormal vectors respect to the metric $g^{ij}(0)$.

Step 2. If a function $h = (h^1, \ldots, h^n)$ satisfies the system 14, then there is a function $x : U \to \mathbb{R}$ (maybe in a smaller neighborhood of 0) such that

$$\frac{\partial x}{\partial y_j}(y) = h^j(y) \tag{15}$$

In terms of tensor lenguage, this is the same as saying that the 1-form

$$\eta = h^1 dy_1 + \cdots + h^n dy_n$$

is exact. The existence of the function x is because the integrability conditions of the system 15 above are as simple as

$$\frac{\partial h^j}{\partial y_k} = \frac{\partial h^k}{\partial y_j} \text{ for } k, j = 1, \ldots, n$$

and this is true because h satisfies the system 14, and then

$$\frac{\partial h^j}{\partial y_k} = \sum_\gamma \Gamma_{jk}^\gamma h^\gamma = \sum_\gamma \Gamma_{kj}^\gamma h^\gamma = \frac{\partial h^k}{\partial y_j}$$

since (by the simmetry of the connection) $\Gamma_{jk}^\gamma = \Gamma_{kj}^\gamma$.

Now choose functions x_α, $\alpha = 1, \ldots, n$, with $h_\alpha^j = \frac{\partial x_\alpha}{\partial y_j}$. Then the functions x_α satisfy

$$\frac{\partial^2 x_\alpha}{\partial y_j \partial y_k} = \sum_\gamma \Gamma_{jk}^\gamma \frac{\partial x_\alpha}{\partial y_\gamma}.$$

Note that these are the equations of lemma 1. So we have a vectorial function given by $(x_1, \ldots, x_n)$ such that its differential at 0 is the matrix whose rows are the vectors $X_1, \ldots, X_n$, and by the inverse function theorem we conclude that $(x_1, \ldots, x_n)$ is a coordinate system in a neighborhood of 0.

Step 3. We claim that $(x_1, \dots, x_n)$ is the desired coordinate system. It is sufficient to see equation 10 of lemma 1, i.e. that

$$g(dx_\mu, dx_\nu) := \sum_{i,j} g^{ij} \frac{\partial x_\mu}{\partial y_i} \frac{\partial x_\nu}{\partial y_j} = \delta_{\mu\nu}.$$

We know this is true at 0 because of the choice of the initial conditions, so if we show that all its partial derivatives are 0, we are done.

$$\frac{\partial}{\partial y_k}(\sum_{i,j} g^{ij} \frac{\partial x_\mu}{\partial y_i} \frac{\partial x_\nu}{\partial y_j}) = \sum_{i,j} \frac{\partial g^{ij}}{\partial y_k} \frac{\partial x_\mu}{\partial y_i} \frac{\partial x_\nu}{\partial y_j} + \sum_{i,j} g^{ij} \frac{\partial^2 x_\mu}{\partial y_k \partial y_i} \frac{\partial x_\nu}{\partial y_j} + \sum_{i,j} g^{ij} \frac{\partial x_\mu}{\partial y_i} \frac{\partial^2 x_\nu}{\partial y_k \partial y_j}$$

$$= \sum_{i,j} \frac{\partial g^{ij}}{\partial y_k} \frac{\partial x_\mu}{\partial y_i} \frac{\partial x_\nu}{\partial y_j} + \sum_{i,j} g^{ij} (\sum_\gamma \Gamma^\gamma_{ik} \frac{\partial x_\mu}{\partial y_\gamma}) \frac{\partial x_\nu}{\partial y_j} + \sum_{i,j} g^{ij} \frac{\partial x_\mu}{\partial y_i} (\sum_\gamma \Gamma^\gamma_{jk} \frac{\partial x_\nu}{\partial y_\gamma})$$

we switch the summation index γ by i and j respectively in the second and third summand to obtain

$$\frac{\partial}{\partial y_k}(\sum_{i,j} g^{ij} \frac{\partial x_\mu}{\partial y_i} \frac{\partial x_\nu}{\partial y_j}) = \sum_{i,j} \frac{\partial x_\mu}{\partial y_i} \frac{\partial x_\nu}{\partial y_j} [\frac{\partial g^{ij}}{\partial y_k} + \sum_\gamma (g^{\gamma j} \Gamma^i_{\gamma k} + g^{\gamma i} \Gamma^j_{\gamma k})] = 0$$

by the following claim.

Claim. We have the identity

$$0 = \frac{\partial g^{ij}}{\partial y_k} + \sum_l (g^{lj} \Gamma^i_{lk} + g^{il} \Gamma^j_{lk}).$$

To see this recall the definition of the symbols

$$[lk, m] = \frac{1}{2}(\frac{\partial g_{lm}}{\partial y_k} + \frac{\partial g_{km}}{\partial y_l} - \frac{\partial g_{lk}}{\partial y_m}).$$

From this it is clear that

$$[lk, m] + [mk, l] = \frac{\partial g_{lm}}{\partial y_k}; \quad \Gamma^j_{lk} = \sum_m g^{jm} [lk, m]$$

Recall that if we differentiate the identity $\sum_m g^{mj} g_{lm} = \delta_{lj}$, we get

$$\sum_m \frac{\partial g^{mj}}{\partial y_k} g_{lm} + g^{mj} \frac{\partial g_{lm}}{\partial y_k} = 0.$$

Now we compute what we wanted

$$\sum_l g^{il} \Gamma^j_{lk} + g^{lj} \Gamma^i_{lk} = \sum_l g^{il} \sum_m g^{jm} [lk, m] + \sum_l g^{jl} \sum_m g^{im} [lk, m]$$

$$= \sum_{l,m} g^{il} g^{jm} [lk, m] + \sum_{m,l} g^{il} g^{jm} [mk, l] = \sum_{l,m} g^{il} g^{jm} ([lk, m] + [mk, l])$$

$$= \sum_l g^{il} \sum_m g^{jm} \frac{\partial g_{lm}}{\partial y_k} = -\sum_l g^{il} \sum_m \frac{\partial g^{jm}}{\partial y_k} g_{lm}$$

$$= -\sum_m \frac{\partial g^{jm}}{\partial y_k} \sum_l g^{il} g_{lm} = -\sum_m \frac{\partial g^{jm}}{\partial y_k} \delta_{im} = -\frac{\partial g^{ij}}{\partial y_k}$$

and this gives the claim and prove the theorem.

□

So we have seen that the curvature tensor characterizes when a Riemannian manifold is locally flat when the metric is regular enough to define the curvature. We observe that this proof is motivated by the lemma 1 and remark 5 where we saw how the system 14 arises when looking for local flatness of the metric.

§4. Curvature and Local Conformal Flatness for C^3 Metrics.

In this section we will be interested in a weaker condition than local flatness. Now we wonder when a Riemannian manifold is locally conformally flat.

Definition 2. A Riemannian manifold (M, g), is locally conformally flat if for every point $p \in M$ there exists a C^4 coordinate system $(x_1, \ldots, x_n) : U \subset M \to V \subset \mathbb{R}^n$ in a neighborhood U of p and a function $u : U \to \mathbb{R}$ such that, when expressed in coordinates, $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})(x) = g_{ij}(x) = e^{2u(x)}\delta_{ij}$ for every $x \in V$. In other words, $g = e^{2u} \sum_i dx_i \otimes dx_i$.

Remark 6.

(1) We observe that this is the same as saying that there exists a positive function $\lambda : U \to \mathbb{R}$ such that $g = \lambda \sum_i dx_i \otimes dx_i$, but the exponential formulation will prove to be useful for computations. Note also that the definition is valid for any matric, with no regularity restrictions.

(2) Directly form the definition, we see that a Riemannian manifold (M, g) with $g \in C^2$ is locally conformally flat if and only if for every point $p \in M$ there exists a C^4 coordinate system $x = (x_1, \ldots, x_n) : U \to V \subset \mathbb{R}^n$ in a neighborhood U of p and a continuous function $u : U \to \mathbb{R}$ such that, expressed in the x coordinates, we have that the metric $g' := e^{2u}g = \sum_i dx_i \otimes dx_i$ is diagonal. As $g \in C^2$ and the coordinate system x is C^4, we see that g_{ij} are C^2, and, this implies that $u = -\frac{1}{2}log(g_{ij})$ is also C^2, so the metric g' (defined only locally) is C^2. Therefore we can apply 3 to conclude that (M, g) is locally conformally flat if and only if the metric g' has curvature tensor $R' = 0$.

(3) Note that, if we add that $g \in C^3$ to the hypothesis of the definition, the vector fields $\frac{\partial}{\partial x_i}$ are C^3 since the coordinates are C^4, so the functions g_{ij} are C^3, and then $u \in C^3$.

In view of the previous remark, our strategy for detecting conformal flatness of a Riemannian manifold (M, g) (in pressence of enough regulariry) will be to define a new metric $g' = e^{2u}g$ for a certain unknown function u and see if we can find u such that the curvature R' associated to g' vanishes. This will boil down to solve a differential equation for the unknown u involving second partial derivatives of u. The integrability conditions for this system will require one more derivative, so we need at least C^3 regularity to solve the system. This is the reason why we require that regularity in the definition.

To study the expression of the curvatore under conformal change, we first need to discuss how the Riemann curvature tensor can split into a sum of others tensors, and this lead us to the study of tensors having similar algebraic properties as the curvature tensor.

4.1. Descomposition of Curvature Tensors

Definition 3. Let E be a vector space, and E^* its dual. We define

$$\otimes^n E^* = \{m \text{ - } multilinear\ linear\ forms\ \beta : E^n \to \mathbb{R}\}$$

$$S^n E^* = \{symmetric\ m \text{ - } multilinear\ forms\ \sigma : E^n \to \mathbb{R}\}$$

$$\Lambda^n E^* = \{antisymmetric\ m \text{ - } multilinear\ forms\ \alpha : E^n \to \mathbb{R}\}.$$

Let $a, b \in E^*$. We define its tensorial product $a \otimes b$ such that

$$(a \otimes b)(x, y) = a(x)b(y)$$

so we have the tensorial product

$$\otimes : E^* \times E^* \to \otimes^2 E^* : (a, b) \mapsto a \otimes b$$

We define its symmetric product $a \circ b$ such that

$$(a \circ b)(x, y) = \frac{1}{2}[a(x)b(y) + a(y)b(x)]$$

so we have the symmetric product

$$\circ : E^* \times E^* \to S^2 E^* : (a, b) \mapsto a \circ b$$

We define its antisymmetric product $a \wedge b$ such that

$$(a \wedge b)(x, y) = \frac{1}{2}[a(x)b(y) - a(y)b(x)]$$

so we have the antisymmetric product

$$\wedge : E^* \times E^* \to \Lambda^2 E^* : (a, b) \mapsto a \wedge b$$

Suppose E is finite dimensional. If $\beta \in \otimes^2 E^*$ then β is a linear combination of tensorial products of elements of E^*. To see this, choose a basis $\{e_1, \ldots, e_n\}$ of E and let $\{e_1^*, \ldots, e_n^*\}$ be the dual basis. Then

$$\beta = \sum_{i,j} \beta(e_i, e_j) e_i^* \otimes e_j^*$$

because both coincide when evaluated in each (e_i, e_j).

It also holds that if $\sigma \in S^2 E^*$ then σ is a linear combination of symmetric products of elements of E^*. Indeed, by the previous line and taking into account that $\sigma(e_i, e_j) = \sigma(e_j, e_i)$ we have

$$\sigma = \sum_{i,j} \sigma(e_i, e_j) e_i^* \otimes e_j^* = \sum_i \sigma(e_i, e_i) e_i^* \circ e_i^* + \sum_{i<j} 2\sigma(e_i, e_j) e_i^* \circ e_j^*$$

For last, if $\alpha \in \Lambda^2 E^*$ then α is a linear combination of antisymmetric products of elements of E^*. Indeed, by the previous line and taking into account that $\sigma(e_i, e_j) = -\sigma(e_j, e_i)$ we have

$$\sigma = \sum_{i,j} \sigma(e_i, e_j) e_i^* \otimes e_j^* = \sum_{i<j} 2\sigma(e_i, e_j) e_i^* \wedge e_j^*$$

This fact, though basic, is important because it explains the notation, and usually it is not explicitly stated. Observe that although we have only defined the products in the case $m = 2$ for simplicity, all these products can be defined for any m, with analogous results, but the proof is longer.

We are interested in the following space

Definition 4. We define the space

$$S^2(\Lambda^2 E^*) := \{r \in \otimes^4 E^* : r(x, y, z, t) = r(z, t, x, y) = -r(y, x, z, t) = -r(x, y, t, z)\}$$

Remark 7. Note that the notation suggests that the space $\Lambda^2 E^*$ could be regarded as the dual of some vector space $\Lambda^2 E$. This is indeed true. The space $\Lambda^2 E$ is defined in a similar way as $E \otimes E$. We remind how space $E \otimes E$ is defined.

We define the *free* vector space $E \times E$ over $\mathbb{R}$, denoted by $\mathcal{F}(E \times E)$, whose elements are finite 'formal linear combinations' of elements of $E \times E$ where the sum and product by scalars are *not* the sum and product by scalars that E has in every coordinate, but are just formal operations. More explicitly, given generic elements $a, b \in \mathcal{F}(E \times E)$

$$a = \sum_{i,j} \lambda_{ij}(e_i, e_j); \qquad b = \sum_{k,l} \alpha_{kl}(e_k, e_l)$$

then we define $a + b$ as

$$a + b = \sum_{i,j} \lambda_{ij}(e_i, e_j) + \sum_{k,l} \alpha_{kl}(e_k, e_l); \qquad \lambda a = \sum_{i,j} \lambda\lambda_{ij}(e_i, e_j)$$

and we force this sum to be commutative and distributive respect scalar multiplication so $\mathcal{F}(E \times E)$ is a vector space. Note that this is the same as considering the vector space of mappings $E \times E \to \mathbb{K} : (e_i, e_j) \mapsto \lambda_{ij}$ such that λ_{ij} is non zero only for a finite number of (e_i, e_j) equipped with its usual operations.

Now, we define

$$E \otimes E := \frac{\mathcal{F}(E \times E)}{A}$$

i.e, the quotient space of $\mathcal{F}(E \times E)$ by the subespace A generated by elements of type

$$(x_1 + x_2, y) - (x_1, y) - (x_2, y); \quad (x, y_1 + y_2) - (x, y_1) - (x, y_2); \quad (\lambda x, y) - (x, \lambda y); \quad (x, \lambda y) - \lambda(x, y).$$

This is the same as considering in $\mathcal{F}(E \times E)$ the equivalence relation generated by the relations

$$(x_1 + x_2, y) \sim (x_1, y) + (x_2, y); \quad (x, y_1 + y_2) \sim (x, y_1) + (x, y_2); \quad (\lambda x, y) \sim (x, \lambda y) \sim \lambda(x, y).$$

If we add to this equivalenve relation the condition $(x, y) \sim -(y, x)$ (i.e, if we include elements of the type $(x, y) + (y, x)$ as generators of A), we get the space $\Lambda^2 E$. We denote $x \wedge y \in \Lambda^2 E$ the equivalence class of $(x, y) \in E \times E$. The relations above just mean that

$$(x_1 + x_2) \wedge y = x_1 \wedge y + x_2 \wedge y \;\; ; \qquad x \wedge (y_1 + y_2) = x \wedge y_1 + x \wedge y_2$$
$$(\lambda x) \wedge y = x \wedge (\lambda y) = \lambda(x \wedge y) \;\; ; \qquad x \wedge y = -y \wedge x.$$

Now we claim that $\Lambda^2 E^* = (\Lambda^2 E)^*$. To see this, given $\alpha \in \Lambda^2 E^*$ we define $\alpha' \in (\Lambda^2 E)^*$ by

$$\alpha'(\sum_{\gamma,\mu} \lambda_{\gamma\mu}(x_\gamma \wedge y_\mu)) = \sum_{\gamma,\mu} \lambda_{\gamma\mu}\alpha(x_\gamma, y_\mu).$$

and it is obvious that α' is well defined if and only if α is antisymmetric and bilinear.

In the other way, given $\alpha' \in (\Lambda^2 E)^*$ we define $\alpha \in \Lambda^2 E^*$ by $\alpha(x, y) = \alpha'(x \wedge y)$, which is antisymmetric and bilinear by definition of the equivalence class $x \wedge y$.

Now we claim that the space $S^2(\Lambda^2 E^*)$ are just the symmetric bilinear forms defined on $\Lambda^2 E \times \Lambda^2 E$. In other words, we claim that

$$S^2(\Lambda^2 E^*) = S^2((\Lambda^2 E)^*).$$

This should be true if we want the notation to be consistent because we defined first $S^2(\Lambda^2 E^*)$ in an apparently arbitrary manner, and later it turned out that $\Lambda^2 E^* = (\Lambda^2 E)^*$, but ,as we saw before,

$S^2((\Lambda^2 E)^*)$ has a natural definition, so both definitions should agree. To see this, given $r \in S^2(\Lambda^2 E^*)$ we define $r' \in S^2((\Lambda^2 E)^*)$ by

$$r'(\sum_{\mu,\nu} \lambda_{\mu\nu}(x_\mu \wedge y_\nu), \sum_{\eta,\gamma} \delta_{\eta\gamma}(z_\eta \wedge t_\gamma)) = \sum_{\nu,\mu,\eta,\gamma} \lambda_{\mu\nu}\delta_{\eta\gamma} r(x_\mu, y_\nu, z_\eta, t_\gamma)$$

which is obviusly bilinear and symmetric because $r(x, y, z, t) = r(z, t, x, y)$.

In the other way, given $r' \in S^2((\Lambda^2 E)^*)$ we define $r \in S^2(\Lambda^2 E^*)$ by

$$r(x, y, z, t) = r'(x \wedge y, z \wedge t)$$

and it is straighforward to see that $r(x, y, z, t) = -r(y, x, z, t)$, $r(x, y, z, t) = -r(x, y, t, z)$ and $r(x, y, z, t) = r(z, t, x, y)$, so r satisfies the requirements to be in $S^2(\Lambda^2 E^*)$.

As a consequence of this, $r \in S^2(\Lambda^2 E^*)$ if and only if r is a linear combination of symmetric products of elements of $\Lambda^2 E^*$, where the symmetric product of $\Lambda^2 E^*$ is just the inherited from the symmetric product on $(\Lambda^2 E)^*$, i.e, for every $\alpha_1, \alpha_2 \in \Lambda^2 E^*$,

$$(\alpha_1 \circ \alpha_2)(x, y, z, t) = \frac{1}{2}[\alpha_1(x, y)\alpha_2(z, t) + \alpha_1(z, t)\alpha_2(x, y)]$$

so the elements of the type $\alpha_1 \circ \alpha_2$ as above are the typical elements in $S^2(\Lambda^2 E^*)$.

We know that the curvature tensor $r \in S^2(\Lambda^2 E^*)$ but r satisfies one more algebraic property which can not be derived from the others, the Bianchi identity given by

$$r(x, y, z, t) + r(y, z, x, t) + r(z, x, y, t) = 0$$

so it seems natural the following.

Definition 5. We define the bianchi map by

$$b : S^2(\Lambda^2 E^*) \to \otimes^4 E^* : r \mapsto b(r); \quad b(r)(x, y, z, t) = \frac{1}{3}[r(x, y, z, t) + r(y, z, x, t) + r(z, x, y, t)]$$

and we call $\mathcal{C} := ker(b)$ the space of curvature tensors on E.

There is another important map called the Ricci contraction, which we now define.

Definition 6. Fix a metric g on the vector space E. Then we define the Ricci contraction c by

$$c : \mathcal{C} \to S^2 E^* : r \mapsto c(r); \quad c(r)(y, z) = \sum_{i=1}^{n} r(e_i, y, z, e_i)$$

if $\{e_1, \ldots, e_n\}$ is an orthonormal basis of (E, g). To see that it is well defined we must check that $c(r)(y, z)$ does not depend on the basis chosen and that it is symmetric. It does not depend on the basis because if we define R by

$$R : E^3 \to E : (x, y, z) \mapsto R(x, y)z := R(x, y, z)$$

by the condition $g(R(x, y)z, t) = r(x, y, z, t)$ for all $t \in E$, then it is claer that R is a linear map between vector spaces, and the map

$$R(\cdot, y)z : E \to E : x \mapsto R(x, y)z$$

is an endomorphism of E, so its trace does not depend on the basis. Now it is clear that the trace of $R(\cdot, y)z$ respect to the basis $\{e_1, \dots, e_n\}$ is

$$\sum_i g(R(e_i, y)z, e_i) = \sum_i r(e_i, y, z, e_i) = c(y, z)$$

which proves that the definition of c does not depend on the basis. Now we must see that $c(r)$ is symmetric. Indeed,

$$c(r)(y, z) = \sum_{i=1}^{n} r(e_i, y, z, e_i) = \sum_{i=1}^{n} r(z, e_i, e_i, y) = -\sum_{i=1}^{n} r(e_i, z, e_i, y)$$
$$= \sum_{i=1}^{n} r(e_i, z, y, e_i) = c(r)(z, y).$$

where we have used all the properties of the tensors in $S^2(\Lambda^2 E^*)$.

Definition 7. We call $\mathcal{W} = ker(c)$ the space of Weyl tensors on E, which are the curvature tensors with Ricci contraction zero. Since Ricci contraction is a kind of trace in the first and fourth variables, we will say that the Weyl tensors are those curvature tensors with zero trace.

Finally we define the trace of a symmetric $(2, 0)$-tensor.

Definition 8. We define tha trace map as $Tr : S^2(E^*) \to \mathbb{R}$ such that if $s \in S^2(E^*)$ then

$$Tr(s) = \sum_i s(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis of E. This is well defined since if we consider the $(1, 1)$-tensor S given by the condition $g(S(x), y) = s(x, y)$ then

$$Tr(s) = \sum_i s(e_i, e_i) = \sum_i g(S(e_i), e_i)$$

is just the trace of the endomorphism S with respect to the basis $\{e_i\}$, and this trace we know that does not depend on the basis.

Remark 8. In fact, we do not need the symmetry to define the trace. It can be defined in exactly the same way for non-symmetric two-tensors. The only difference is that if we choose the $(1, 1)-$tensor S' given by $g(S'(x), y) = s(y, x)$ then $S' = S$ if and only if s is symmetric. But in any case S and S' have the same trace as endomorphisms, so the final result does not depend on any choice and it is well defined then.

Recall that the Ricci contraction takes a curvature tensor and returns a symmetric (2,0) tensor. Now we want to reverse the process. For this, we define the following.

Definition 9. We define the Kulkarni-Nomizu product $\owedge : S^2(E) \times S^2(E) \to \mathcal{C}$ such that if $h, k \in S^2(E)$, then

$$(h \owedge k)(x, y, z, t) = h(x, z)k(y, t) - h(x, t)k(y, z) + k(x, z)h(y, t) - k(x, t)h(y, z)$$

This product is well defined, i.e, $(h \owedge k) \in \mathcal{C}$. We check the Bianchi identity and leave the rest, which are even easier.

$$\begin{aligned}
&(h \owedge k)(x,y,z,t) + (h \owedge k)(y,z,x,t) + (h \owedge k)(z,x,y,t)\\
&= [h(x,z)k(y,t) - h(x,t)k(y,z) + k(x,z)h(y,t) - k(x,t)h(y,z)]\\
&+[h(y,x)k(z,t) - h(y,t)k(z,x) + k(y,x)h(z,t) - k(y,t)h(z,x)]\\
&+[h(z,y)k(x,t) - h(z,t)k(x,y) + k(z,y)h(x,t) - k(z,t)h(x,y)]\\
&= [A - B + C - D] + [E - C + F - A] + [D - F + B - E] = 0
\end{aligned}$$

Besides, it is obvious that $h \owedge k = k \owedge h$ so $\owedge$ is commutative, and it is bilinear since $(h_1 + h_2) \owedge k = h_1 \owedge k + h_2 \owedge k$ and the same for the second entry. So it qualifies as a (commutative) product.

Now we can prove the result involving the descomposition of a curvature tensor.

Proposition 1. Let (E,g) be a vector espace E with a metric g, $dim(E) \geq 3$. Then $\mathcal{C} = \mathcal{W} \bigoplus (g \owedge S^2(E^*))$

Proof. Given a curvature tensor $r \in \mathcal{C}$, we must find a symmetric $(2,0)$-tensor s and a Weyl tensor w such that

$$r(x,y,z,t) = w(x,y,z,t) + g(x,z)s(y,t) - g(x,t)s(y,z) + s(x,z)g(y,t) - s(x,t)g(y,z)$$

Assuming this is true, we take Ricci contraction and obtain

$$\begin{aligned}
&c(r)(y,z) = \sum_i r(e_i,y,z,e_i)\\
&= 0 + \sum_i [g(e_i,z)s(y,e_i) - g(e_i,e_i)s(y,z) + s(e_i,z)g(y,e_i) - s(e_i,e_i)g(y,z)]\\
&= s(y,z) - ns(y,z) + s(y,z) - g(y,z)\sum_i s(e_i,e_i) = (2-n)s(y,z) - g(y,z)Tr(s)
\end{aligned}$$

where we used that, as $\{e_i\}$ is an orthonormal basis, $z = \sum_i g(e_i,z)e_i$. Now we take the trace and obtain

$$Tr(c(r)) = (2-n)Tr(s) - nTr(s) = (2-2n)Tr(s)$$

Thinking of r being the Riemann curvature tensor, we call $Scal := Tr(c(r))$ and we get that if the descomposition desired is true, necessarily

$$Tr(s) = \frac{Scal}{2-2n}.$$

Then, returning to the first equality, we must have

$$s(y,z) = \frac{1}{2-n}[c(r)(y,z) + g(y,z)Tr(s)] = \frac{1}{2-n}[c(r)(y,z) + \frac{Scal}{2-2n}g(y,z)]$$

which is symmetric because r is a curvature tensor. Besides this two identities are consistent since, with this definition of s, we have indeed that

$$Tr(s) = \frac{1}{2-n}[Scal + \frac{Scal}{2-2n}n] = \frac{[(2-2n)+n]Scal}{(2-n)(2-2n)} = \frac{Scal}{2-2n}$$

So s is completely determined by r, and $s = \frac{1}{2-n}[c(r) + \frac{Scal}{2-2n}g]$.

We have obtained who must be s, and now we just set $w := r - g \owedge s$, which is a curvature tensor because is the sum of two curvature tensors. Obviously, $r = w + g \owedge s$, and we must check that w is a Weyl tensor. But this follows directly if we look at how s was obtained, since s was defined so that $c(r) = c(s \owedge g)$, and then $c(w) = 0$.

So every curvature tensor r admits a descomposition $r = w + s \owedge g$ for some Weyl tensor w and some symmetric $(2,0)$-tensor s. Besides, we saw when we obtained s that it is uniquely determined, and so is w, so this descomposition is unique. This proves that $\mathcal{W}$ and $g \owedge S^2(E^*)$ have zero intersection as vector spaces, because if $r \in \mathcal{W} \cap (g \owedge S^2(E^*))$, then r admits the two descompositions$r = r + 0 = 0 + r$ so $r = 0$. We conclude that $\mathcal{C} = \mathcal{W} \bigoplus (g \owedge S^2(E^*))$, as we wanted. $\square$

Remark 9. If $dim(E) = 2$ the proof above does not work, but this case is easier, since every curvature tensor can be descomposed $r = -\frac{1}{2} Scal(g \owedge g)$. To see this, it is enough to see that

$$r(x,y,x,y) = -\tfrac{1}{2} Scal(g \owedge g)(x,y,x,y) = -\frac{Scal}{2}(g(x,x)g(y,y) - g(x,y)g(x,y))$$

if x, y are a basis of E. This is because, by the simmetries of curvature tensors, this implies that both coincide evaluated on any permutation of the basis $\{x, y\}$, and hence on E by linearity. To check this, we remind that the number

$$K = \frac{r(x,y,x,y)}{g(x,x)g(y,y) - g(x,y)g(x,y)}$$

is independet of the basis chosen, and it is called the sectional curvature. The denominator is not zero because x, y are not proportional, so the Cauchy Schwartz inequality is strict. Besides, we know that that for an orthonormal basis $\{e_1, e_2\}$ of E,

$$\begin{aligned} Scal &= Tr(c(r)) = c(r)(e_1,e_1) + c(r)(e_2,e_2) = r(e_1,e_1,e_1,e_1) + r(e_2,e_1,e_1,e_2) \\ &+ r(e_1,e_2,e_2,e_1) + r(e_2,e_2,e_2,e_2) = -2r(e_1,e_2,e_1,e_2) = -2K \end{aligned}$$

which yields the claim.

In dimension 3 the situation is simpler. We show in the next proposition that in this case the only curvature tensor with zero Ricci contraction is the zero tensor, so in dimension 3 the space of Weyl tensors is $\mathcal{W} = \{0\}$.

Proposition 2. If $dim(E) = 3$ then $\mathcal{W} = 0$, i.e, all the curvature tensors with Ricci contraction zero are zero.

Proof. Let $w \in \mathcal{W}$, and pick $\{e_1, e_2, e_3\}$ an orthonormal basis. We know that, as w has Ricci contraction zero, for all x, y

$$c(w)(x,y) = w(e_1,x,y,e_1) + w(e_2,x,y,e_2) + w(e_3,x,y,e_3) = 0.$$

Let's see $w = 0$ restricted to the basis $\{e_1, e_2, e_3\}$. Remind that $w(x,x,y,z) = w(y,z,x,x) = 0$ by the symmetries of curvature tensors. We have

$$\begin{aligned} r(w)(e_1,e_1) &= w(e_2,e_1,e_1,e_2) + w(e_3,e_1,e_1,e_3) = 0 \\ r(w)(e_1,e_2) &= w(e_3,e_1,e_2,e_3) = 0 \\ r(w)(e_1,e_3) &= w(e_2,e_1,e_3,e_2) = 0 \\ r(w)(e_2,e_2) &= w(e_1,e_2,e_2,e_1) + w(e_3,e_2,e_2,e_3) = 0 \\ r(w)(e_2,e_3) &= w(e_1,e_2,e_3,e_1) = 0 \\ r(w)(e_3,e_3) &= w(e_1,e_3,e_3,e_1) + w(e_2,e_3,e_3,e_2) = 0 \end{aligned}$$

and then it follows that

$$r(w)(e_1,e_1)+r(w)(e_2,e_2)=2w(e_2,e_1,e_1,e_2)+r(w)(e_3,e_3)=2w(e_2,e_1,e_1,e_2)=0$$

and from this we conclude that $w(e_3,e_1,e_1,e_3)=w(e_2,e_3,e_3,e_2)=0$.

This shows that when we substitute in w only two members of the basis, the result is zero. If we substitute the three members, one of them must appear twice, and then modulo sign we have three options $w(e_1,e_2,e_1,e_3), w(e_2,e_1,e_2,e_3), w(e_3,e_1,e_3,e_2)$, all of them zero because $r(w)(e_1,e_2)=r(w)(e_1,e_3)=r(w)(e_2,e_3)=0$. So w restricted to the basis is zero, and hence $w=0$ by linearity. □

Remark 10. All these results about curvature tensors have been proved in a general setting. However, the particular case in which we are interested is when we have (M,g) a Riemannian manifold with $dim(M)\geq 3$ and r is its Riemann curvature tensor. Then the Ricci contraction $c(r)$ of r is the well known Ricci tensor so we call $Ric := c(r)$. The trace $Tr(c(r)) = Tr(Ric)$ of the Ricci tensor is called the scalar curvature, and we write $Scal := Tr(Ric)$. The proposition below just show us that if we define

$$s := \frac{1}{2-n}[Ric + \frac{Scal}{2-2n}g]; \quad w := r - g \owedge s$$

then s is a symmetric $(2,0)$-tensor, w is a Weyl tensor, and $r = w + s \owedge g$.

Definition 10. Given a Riemannian manifold (M,g). With notations as in the previous Remark 10, we call w *the Weyl tensor* and s the *Schouten tensor* . Note that they are unique fixed g.

Remark 11. If the opposite sign convention for the curvature r is chosen, then Ricci tensor is defined by contracting on the second and fourth entries instead of the first and fourth. This way, whathever choice of the sign of the curvature is picked, we obtain the same Ricci tensor, and the same scalar curvature. However, if we choose the opposite sign convention for the curvature r then the Schouten and the Weyl tensors change sign, since if $r = w + s \owedge g$ then $-r = -w + (-s) \owedge g$. So do not be susprised if you find these tensors defined with the opposite sign in other references (this will be because the opposite sign for the curvature has been chosen).

4.2. Conformal Change of the Curvature Tensor

Now we can discuss how the curvature changes under conformal transformation. We begin with a definition.

Definition 11. Let (M,g) be a Riemannian manifold, and let $u\in C^k$ for $k\geq 1$. We define *the gradient of* u as the vector field $grad(u)$ such that

$$g(X, grad(u)) = du(x)$$

for every vector field X in M. Working in coordinates it is easy to see that $grad(u)\in C^{k-1}$.

In the next Lemma we show how the Levi-Civita conection changes when the metric is conformally perturbed.

Lemma 2. Let (M,g) be a Riemannian manifold. Suppose $g\in C^2$ and let u be a C^2 function defined in some neighborhood. Set $g' = e^{2u}g\in C^2$. Denote by ∇ and ∇' the Levi-Civita connections of g and g' respectively. Let $B(X,Y) = \nabla'_X Y - \nabla_X Y$. Then B has the expression

$$B(X,Y) = du(X)Y + du(Y)X - g(X,Y)grad(u)$$

Proof. First we claim that as ∇ and ∇' are symmetric conections, then B is a symmetric $(2,1)$ tensor. Indeed for $f \in C^\infty(M)$ and X, Y vector fields on M, we have

$$\begin{aligned} B(X, fY) &= \nabla'_X fY - \nabla_X fY = X(f)Y + f\nabla'_X Y - X(f)Y - f\nabla_X Y \\ &= f(\nabla'_X Y - \nabla_X Y) = fB(X,Y) \end{aligned}$$

That B is $C^\infty(M)$ linear in its first entry is because the connections are, so is a linear combination of connections. Let us see B is symmetric.

$$B(X,Y) = \nabla'_X Y - \nabla_X Y = \nabla'_Y X - [X,Y] - \nabla_Y X + [X,Y] = B(Y,X).$$

Now we want to find an explicit expression of B.

As ∇' is the Levi-Civita connection for g', we have for every X, Y, Z vector fields on M

$$\begin{aligned} 0 &= (\nabla'_X g')(Y,Z) = \nabla'_X(g'(Y,Z)) - g'(\nabla'_X Y, Z) - g'(Y, \nabla'_X Z) \\ &= X(e^{2u}g(Y,Z)) - e^{2u}[g(\nabla'_X Y, Z) + g(Y, \nabla'_X Z)] \\ &= 2X(u)e^{2u}g(Y,Z) + e^{2u}X(g(Y,Z)) - e^{2u}[g(\nabla_X Y + B(X,Y), Z) + g(Y, \nabla_X Z + B(X,Z))] \\ &= 2X(u)e^{2u}g(Y,Z) + e^{2u}[\underline{g(\nabla_X Y, Z)} + \underline{g(Y, \nabla_X Z)}] \\ &\quad -e^{2u}[\underline{g(\nabla_X Y, Z)} + g(B(X,Y), Z) + \underline{g(Y, \nabla_X Z)} + g(Y, B(X,Z))] \\ &= e^{2u}[2X(u)g(Y,Z) - g(B(X,Y),Z) - g(Y, B(X,Z))] \end{aligned}$$

We make this calculation 3 times switching cyclically X, Y, Z and conclude

$$\begin{aligned} g(B(X,Y),Z) + g(Y, B(X,Z)) &= 2X(u)g(Y,Z) \\ g(B(Y,Z),X) + g(Z, B(Y,X)) &= 2Y(u)g(Z,X) \\ g(B(Z,X),Y) + g(X, B(Z,Y)) &= 2Z(u)g(X,Y) \end{aligned}$$

If we make the signed sum and use the symmetry of B, we have

$$\begin{aligned} &2[X(u)g(Y,Z) + Y(u)g(Z,X) - Z(u)g(X,Y)] \\ &= g(B(X,Y),Z) + g(Y,B(X,Z)) + g(B(Y,Z),X) \\ &+ g(Z,B(Y,X)) - g(B(Z,X),Y) - g(X,B(Z,Y)) = 2g(B(X,Y),Z) \end{aligned}$$

so we conclude that

$$\begin{aligned} g(B(X,Y),Z) &= X(u)g(Y,Z) + Y(u)g(Z,X) - Z(u)g(X,Y) \\ &= X(u)g(Y,Z) + Y(u)g(Z,X) - g(Z, grad(u))g(X,Y) \\ &= g(X(u)Y + Y(u)X - g(X,Y)grad(u), Z) \end{aligned}$$

Then, necessarily

$$B(X,Y) = X(u)Y + Y(u)X - g(X,Y)grad(u)$$

and this proves the Lemma. □

Definition 12. Let (M,g) be a Riemannian manifold, and let $u \in C^k$ for $k \geq 2$. We define the *Hessian of* u as the symmetric $(2,0)$-tensor acting on vector fields X, Y by $Hess(u)(X,Y) := g(\nabla_X grad(u), Y)$.

It is symmetric since

$$
\begin{aligned}
&Hess(u)(X,Y) := g(\nabla_X grad(u), Y) = X(g(grad(u),Y)) - g(grad(u), \nabla_X Y)\\
&= X(Y(u)) - (\nabla_X Y)(u) = Y(X(u)) + ([X,Y])(u) - (\nabla_X Y)(u)\\
&= Y(X(u)) - (\nabla_Y X)(u) = \frac{1}{2}[X(Y(u)) - (\nabla_X Y)(u) + Y(X(u)) - (\nabla_Y X)(u)]
\end{aligned}
$$

Also, working in coordinates we easily see that $Hess(u) \in C^{k-2}$.

The next Lemma will be crucial, and shows how the curvature tensor changes when the metric is conformally perturbed.

Lemma 3. Let (M,g) be a Riemannian manifold and let $g' = e^{2u}g$. Denote r and r' the curvature tensors associated to g and g' respectively. Then

$$r' = e^{2u}(r + b_u \owedge g)$$

where b_u is the symmetric $(2,0)$-tensor given by

$$b_u(x,y) = Hess(u)(x,y) - du(x)du(y) + \frac{1}{2}g(grad(u), grad(u))g(x,y)$$

Proof. Denote by ∇ and ∇' the Levi-Civita connections of g and g' respectively. Let $B = \nabla' - \nabla$. First we will see how are related the $(3,1)$ curvature tensors of g and g', denoted respectively by R and R'. We will use color underline to show the terms that cancel out.

$$
\begin{aligned}
&R'(X,Y)Z = \nabla'_X \nabla'_Y Z - \nabla'_Y \nabla'_X Z - \nabla'_{[X,Y]} Z\\
&= \nabla'_X(\nabla_Y Z + B(Y,Z)) - \nabla'_Y(\nabla_X Z + B(X,Z)) - \nabla_{[X,Y]}Z - B([X,Y],Z)\\
&= [\nabla_X + B(X,\cdot)](\nabla_Y Z + B(Y,Z)) - [\nabla_Y + B(Y,\cdot)](\nabla_X Z + B(X,Z))\\
&-\nabla_{[X,Y]}Z - B([X,Y],Z)\\
&= \nabla_X\nabla_Y Z + B(X,\nabla_Y Z) + \nabla_X(B(Y,Z)) + B(X,B(Y,Z)) - \nabla_Y\nabla_X Z - B(Y,\nabla_X Z)\\
&-\nabla_Y(B(X,Z)) - B(Y,B(X,Z)) - \nabla_{[X,Y]}Z - B([X,Y],Z)\\
&= \nabla_X\nabla_Y Z + \underline{B(X,\nabla_Y Z)} + (\nabla_X B)(Y,Z) + \underline{B(\nabla_X Y, Z)} + \underline{B(Y,\nabla_X Z)}\\
&+B(X,B(Y,Z)) - \nabla_Y\nabla_X Z - \underline{B(Y,\nabla_X Z)} - (\nabla_Y B)(X,Z) - \underline{B(\nabla_Y X, Z)}\\
&-\underline{B(X,\nabla_Y Z)} - B(Y,B(X,Z)) - \nabla_{[X,Y]}Z - \underline{B([X,Y],Z)}\\
&= R(X,Y)Z + (\nabla_X B)(Y,Z) + B(X,B(Y,Z)) - (\nabla_Y B)(X,Z) - B(Y,B(X,Z))
\end{aligned}
$$

The blue terms cancel since $\nabla_X Y - \nabla_Y X = [X,Y]$. We conclude that

$$
\begin{aligned}
&R'(X,Y)Z - R(X,Y)Z = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(X,B(Y,Z)) - B(Y,B(X,Z))\\
&= A_1 - A_2 + A_3 - A_4
\end{aligned}
$$

Until now everything has been formal, and we have not used the special form that B has. In the previous lemma, we saw that for every X,Y vector fields on M,

$$B(X,Y) = du(X)Y + du(Y)X - g(X,Y)grad(u) = X(u)Y + Y(u)X - g(X,Y)grad(u).$$

Then

$$A_1 = (\nabla_X B)(Y,Z) = \nabla_X(B(Y,Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$
$$= \nabla_X[Y(u)Z + Z(u)Y - g(Y,Z)grad(u)] - [(\nabla_X Y)(u)Z + Z(u)\nabla_X Y - g(\nabla_X Y, Z)grad(u)]$$
$$-[Y(u)\nabla_X Z + (\nabla_X Z)(u)Y - g(Y, \nabla_X Z)grad(u)]$$
$$= X(Y(u))Z + \underline{Y(u)\nabla_X Z} + X(Z(u))Y + \underline{Z(u)\nabla_X Y}$$
$$-[\underline{g(\nabla_X Y, Z)} + \underline{g(Y, \nabla_X Z)}]grad(u) - g(Y,Z)\nabla_X grad(u) - (\nabla_X Y)(u)Z$$
$$-\underline{Z(u)\nabla_X Y} + \underline{g(\nabla_X Y, Z)grad(u)} - \underline{Y(u)\nabla_X Z} - (\nabla_X Z)(u)Y + \underline{g(Y, \nabla_X Z)grad(u)}$$
$$= X(Y(u))Z + X(Z(u))Y - g(Y,Z)\nabla_X grad(u) - (\nabla_X Y)(u)Z - (\nabla_X Z)(u)Y$$

so if W is another vector field we have

$$g(A_1, W) = g((\nabla_X B)(Y,Z), W) = [XY(u) - (\nabla_X Y)(u)]g(Z,W)$$
$$+[XZ(u) - (\nabla_X Z)(u)]g(Y,W) - g(Y,Z)g(\nabla_X grad(u), W)$$
$$= g(Z,W)Hess(u)(X,Y) + g(Y,W)Hess(u)(X,Z) - g(Y,Z)Hess(u)(X,W)$$

The term $g(A_2, W)$ is obtained by just switching the letters X and Y in the expression of $g(A_1, W)$, so

$$g(A_2, W) = g((\nabla_Y B)(X,Z), W)$$
$$= g(Z,W)Hess(u)(Y,X) + g(X,W)Hess(u)(Y,Z) - g(X,Z)Hess(u)(Y,W)$$

We conclude that

$$g(A_1 - A_2, W)$$
$$= g(Y,W)Hess(u)(X,Z) - g(Y,Z)Hess(u)(X,W)$$
$$-g(X,W)Hess(u)(Y,Z) + g(X,Z)Hess(u)(Y,W)$$

We go now with the term A_3

$$A_3 = B(X, B(Y,Z)) = X(u)B(Y,Z) + B(Y,Z)(u)X - g(X, B(Y,Z))grad(u)$$
$$= X(u)(Y(u)Z + Z(u)Y - g(Y,Z)grad(u))$$
$$+[Y(u)Z(u) + Z(u)Y(u) - g(Y,Z)(grad(u))(u)]X$$
$$-[Y(u)g(X,Z) + Z(u)g(X,Y) - g(Y,Z)g(X, grad(u))]grad(u)$$
$$= X(u)Y(u)Z + X(u)Z(u)Y - \underline{X(u)g(Y,Z)grad(u)}$$
$$+2Y(u)Z(u)X - g(Y,Z)g(grad(u), grad(u))X$$
$$-Y(u)g(X,Z)grad(u) - Z(u)g(X,Y)grad(u) + \underline{g(Y,Z)X(u)grad(u)}$$
$$= X(u)Y(u)Z + X(u)Z(u)Y + [2Y(u)Z(u) - g(Y,Z)|grad(u)|_g^2]X$$
$$-[Y(u)g(X,Z) + Z(u)g(X,Y)]grad(u)$$

and then

$$g(B(X, B(Y,Z)), W) = X(u)Y(u)g(Z,W) + X(u)Z(u)g(Y,W) + 2Y(u)Z(u)g(X,W)$$
$$-g(Y,Z)|grad(u)|_g^2 g(X,W) - Y(u)g(X,Z)W(u) - Z(u)g(X,Y)W(u) = g(A_3, W)$$

The term $g(A_4, W)$ is obtained switching the letters X and Y in the expression of $g(A_3, W)$, so

$$g(A_4, W) = Y(u)X(u)g(Z,W) + Y(u)Z(u)g(X,W) + 2X(u)Z(u)g(Y,W)$$
$$-g(X,Z)|grad(u)|_g^2 g(Y,W) - X(u)g(Y,Z)W(u) - Z(u)g(Y,X)W(u)$$

So it follows that

$$\begin{aligned}
g(A_3 - A_4, W) &= X(u)Z(u)g(Y,W) + 2Y(u)Z(u)g(X,W) \\
&-g(Y,Z)|grad(u)|_g^2 g(X,W) - Y(u)g(X,Z)W(u) - Y(u)Z(u)g(X,W) \\
&-2X(u)Z(u)g(Y,W) + g(X,Z)|grad(u)|_g^2 g(Y,W) + X(u)g(Y,Z)W(u)
\end{aligned}$$

So finally we get

$$\begin{aligned}
& g(R'(X,Y)Z,W) = g(R(X,Y)Z,W) + g(A_1 - A_2, W) + g(A_3 - A_4, W) \\
& = r(X,Y,Z,W) \\
& +g(Y,W)Hess(u)(X,Z) - g(Y,Z)Hess(u)(X,W) \\
& -g(X,W)Hess(u)(Y,Z) + g(X,Z)Hess(u)(Y,W) \\
& +\underline{X(u)Z(u)g(Y,W)} + \underline{2Y(u)Z(u)g(X,W)} - g(Y,Z)|grad(u)|_g^2 g(X,W) - Y(u)g(X,Z)W(u) \\
& -\underline{Y(u)Z(u)g(X,W)} - \underline{2X(u)Z(u)g(Y,W)} + g(X,Z)|grad(u)|_g^2 g(Y,W) + X(u)g(Y,Z)W(u) \\
& = r(X,Y,Z,W) + (Hess(u) \owedge g)(X,Y,Z,W) \\
& -X(u)Z(u)g(Y,W) + Y(u)Z(u)g(X,W) - Y(u)W(u)g(X,Z) + X(u)W(u)g(Y,Z) \\
& -g(Y,Z)|grad(u)|_g^2 g(X,W) + g(X,Z)|grad(u)|_g^2 g(Y,W) \\
& = r(X,Y,Z,W) + (Hess(u) \owedge g)(X,Y,Z,W) - [(du \otimes du) \owedge g](X,Y,Z,W) \\
& +[(\frac{1}{2}|grad(u)|_g^2 g) \owedge g](X,Y,Z,W)
\end{aligned}$$

and then

$$\begin{aligned}
r'(X,Y,Z,W) &= g'(R'(X,Y)Z,W) = e^{2u} g(R'(X,Y)Z,W) \\
&= e^{2u}[r(X,Y,Z,W) + (b_u \owedge g)(X,Y,Z,W)
\end{aligned}$$

where $b_u = hess(u) - du \otimes du + \frac{1}{2}|grad(u)|_g^2 g$, as we wanted. □

Corollary 1. Let (M,g) be a Riemannian manifold with $g \in C^2$, and let w and s be the Weyl and Schouten tensors defined in 10. Put $g' := e^{2u}g$ for some function $u \in C^2$. Denote also K for the sectional curvature of (M,g). We have

(1) The Weyl and Scouten tensors w' and s' of g' are given by

$$w' = e^{2u}w; \quad s' = s + b_u = s + hess(u) - du \otimes du + \frac{1}{2}|grad(u)|_g^2 g$$

(2) The $(3,1)$ Weyl tensors W and W' of g and g' are equal, $W = W'$, so W is conformally invariant.

(3) The sectional curvature K' of (M,g') is given by

$$K'(\sigma(x,y)) = e^{-2u}[K(\sigma(x,y)) - Tr(b_u|_{\sigma(x,y)})]$$

Proof. With the same notation as below, let r' the Riemann curvature tensor of g'. Suppose $dim(M) \geq 3$. Then we know that r' descomposes in a unique way as $r' = w' + s' \owedge g'$, being w' the Weyl tensor of r' and s' the Schouten tensor of r'. Let $r = w + s \owedge g$ be the corresponding descomposition for the Riemann curvature tensor r associated to g. Then by the previous lemma

$$r' = w' + s' \owedge g' = e^{2u}(r + b_u \owedge g) = e^{2u}(w + s \owedge g + b_u \owedge g) = e^{2u}w + (s + b_u) \owedge g'$$

Now, $e^{2u}w$ is a Weyl tensor because w is, so by uniqness of the descomposition, we deduce

$$w' = e^{2u}w; \quad s' = s + b_u = s + hess(u) - du \otimes du + \frac{1}{2}|grad(u)|_g^2 g$$

and this concludes how the Weyl and Schouten tensors change under conformal change of the metric.

If we consider the $(3,1)$-Weyl tensors W and W' of g and g', we have

$$g'(W'(x,y)z,t)) = e^{2u}g(W'(x,y)z,t)) = w'(x,y,z,t) = e^{2u}w(x,y,z,t) = e^{2u}g(W(x,y)z,t)$$

and we conclude that $W' = W$ under conformal change of the metric.

Let us see how the sectional curvature K changes. Given linearly independet vectors $x, y \in T_pM$, denote $\sigma(x,y)$ the plane spanned by them. Let $\{e_1, e_2\}$ be an orthonormal basis respect to g of $\sigma(x,y)$. Denote $K(\sigma(x,y))$ and $K'(\sigma(x,y))$ for the sectional curvatures of g and g' respectively. Then

$$\begin{aligned}
&r'(e_1,e_2,e_1,e_2) = e^{2u}(r + b_u \owedge g)(e_1,e_2,e_1,e_2)\\
&= e^{2u}[r(e_1,e_2,e_1,e_2) + b_u(e_1,e_1)g(e_2,e_2) - b_u(e_1,e_2)g(e_2,e_1) + g(e_1,e_1)b_u(e_2,e_2) - g(e_1,e_2)b_u(e_2,e_1)]\\
&= e^{2u}[r(e_1,e_2,e_1,e_2) + b_u(e_1,e_1) + b_u(e_2,e_2)] = e^{2u}[r(e_1,e_2,e_1,e_2) + Tr(b_u|_{\sigma(x,y)})]
\end{aligned}$$

and therefore

$$\begin{aligned}
K'(\sigma(x,y)) &= \frac{r'(e_1,e_2,e_1,e_2)}{g'(e_1,e_1)g'(e_2,e_2) - g'(e_1,e_2)^2}\\
&= \frac{e^{2u}[r(e_1,e_2,e_1,e_2) + Tr(b_u|_{\sigma(x,y)})]}{e^{4u} - 0} = e^{-2u}[K(\sigma(x,y)) - Tr(b_u|_{\sigma(x,y)})]
\end{aligned}$$

This proves the claim. □

Note that if the opposite sign for the curvature tensor is chosen, the Weyl and the Schouten tensors change sign, so the transformation behaviour in this case would be $-w' = e^{2u}(-w)$ and $-s' = -s - b_u$, as can be seen in the references.

Definition 13. Let (M,g) be a Riemannian manifold, and let $u \in C^k$ for $k \geq 2$. We define Δu, the Laplace-Beltrami operator (which we may just call Laplacian), which is defined as

$$\Delta : C^k(M) \to C^{k-2}(M) : u \mapsto \Delta u := Tr(Hess(u)).$$

Working in coordinates it is easy to see that $\Delta u \in C^{k-2}(M)$.

Remark 12. From the previous Corollary 1 we see in particular that if $dim(M) = 2$, then

$$Tr(b_u|_{\sigma(x,y)}) = Tr(b_u) = Tr(Hess(u)) - du(e_1)^2 - du(e_2)^2 + |grad(u)|_g^2 = Tr(Hess(u)) = \Delta u$$

since $du(e_1)^2 + du(e_2)^2 = g(grad(u), e_1)^2 + g(grad(u), e_2)^2 = |grad(u)|_g^2$.

We conclude that if $dim(M) = 2$ then

$$K'(\sigma(x,y)) = e^{-2u}[K(\sigma(x,y)) - \Delta(u))$$

In addition, we know from remark 9 that if $dim(M) = 2$ the curvature has a simple expression, more precisely $r = K(g \owedge g)$ and $r' = K'(g' \owedge g')$, so under conformal change

$$r' = e^{-2u}[K - \Delta u](e^{2u}g \owedge e^{2u}g) = e^{2u}[K - \Delta u](g \owedge g)$$

Definition 14. We define the divergence of a vector field X as $div(X) = Tr(\nabla_{(\cdot)}X)$, i.e, the trace of the endomorphism $Y \mapsto \nabla_Y X$. This way we have the divergence operator

$$div : \Gamma(TM) \to C^\infty(M) : X \mapsto div(X)$$

where $\Gamma(TM)$ is the set of C^∞ sections of the tangent bundle of M, which is the same as the set of C^∞ vector fields on M. Therefore we have

$$div(X) = \sum_i g(\nabla_{e_i}X, e_i) \quad \text{for every orthonormal basis } \{e_i\}$$

Remark 13. Let us see what is the expression of the gradient, divergence and Laplacian in local coordinates. First note that, although usually the gradient is defined for smooth functions, it makes sense for C^1 functions. This way, the divergence makes sense for C^1 vector fields, and the Laplacian for C^2 functions.

Secondly we recall an equivalent formula for the Laplacian, given by

$$\Delta u = Tr(Hess(u)) = \sum_i Hess(u)(e_i, e_i) = \sum_i g(\nabla_{e_i} grad(u), e_i) = div(grad(u)).$$

Now let $u \in C^1$, and write, for unknown a_i

$$grad(u) = \sum_i a_i \frac{\partial}{\partial x_i}.$$

To obtain who the a_i are, we must impose $du(X) = g(X, grad(u))$ for every X vector field. By linearity, it is enough to check that in any basis, and we choose the basis $\{\frac{\partial}{\partial x_i}\}$. Now,

$$du\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial u}{\partial x_j} = g\left(grad(u), \frac{\partial}{\partial x_j}\right) = \sum_i a_i g_{ij}$$

as this hols for every $j = 1, \dots, n$, we have the matricial system

$$\left(\frac{\partial u}{\partial x_j}\right) = (g_{ij})(a_i) \qquad \text{equivalent to} \qquad (a_i) = (g^{ij})(\frac{\partial u}{\partial x_j})$$

and we get that

$$a_i = \sum_j g^{ij}\frac{\partial u}{\partial x_j} \qquad \text{so} \qquad grad(u) = \sum_i (\sum_j g^{ij}\frac{\partial u}{\partial x_j})\frac{\partial}{\partial x_i}$$

Now let X be a C^1 vector field. In local coordinates,

$$X = \sum_i a_i \frac{\partial}{\partial x_i} \qquad \text{and then} \qquad \nabla_{\frac{\partial}{\partial x_j}} X = \sum_i \left(\frac{\partial a_i}{\partial x_j} + \sum_k a_k \Gamma^i_{kj}\right)\frac{\partial}{\partial x_i}$$

so we conclude that

$$div(X) = \sum_j \left(\frac{\partial a_j}{\partial x_j} + \sum_k a_k \Gamma^j_{kj}\right) = \sum_j \frac{\partial a_j}{\partial x_j} + \sum_{k,j} a_k \Gamma^j_{kj}$$

Suming up, if $u \in C^2$, we have

$$\begin{aligned} \Delta u = div(grad(u)) &= \sum_j \frac{\partial}{\partial x_j}[\sum_l g^{jl}\frac{\partial u}{\partial x_l}] + \sum_{k,j}(\sum_l g^{kl}\frac{\partial u}{\partial x_l})\Gamma^j_{kj} \\ &= \sum_{j,l}(g^{jl}\frac{\partial^2 u}{\partial x_l \partial x_j} + \frac{\partial g^{jl}}{\partial x_j}\frac{\partial u}{\partial x_l}) + \sum_{k,j,l} g^{kl}\frac{\partial u}{\partial x_l}\Gamma^j_{kj} \end{aligned} \tag{16}$$

and this is the expression in local coordinates of the Laplacian.

Now we can solve our problem and determine necessary and suficient conditions for a manifold to be locally conformally flat, at least in presence of enough regularity. We start with surfaces.

Theorem 4. Let (M, g) be a two dimensional Riemannian manifold with $g \in C^{2,\alpha}$ for some $0 < \alpha \leq 1$ (see 21 for the definition of Holder spaces). Then (M, g) is locally conformally flat.

Proof. (Sketch of proof). Let (x, U) ve a smooth coordinate system. By hipothesis, $g_{ij} \in C^{2,\alpha}$ in these coordinates. We saw above that if we put $g' = e^{2u}g$ for some unkwon function u, then $K' = K + \Delta u$, being K and K' the sectional curvatures of (M, g) and (M, g'). We mentioned above that in surfaces the sectional curvature K determines the curvature tensor r. Then $r' = 0$ if and only if $K' = 0$, and this is equivalent to solve the following equation for $x \in U$:

$$\Delta u(x) = \sum_{j,l}(g^{jl}\frac{\partial^2 u}{\partial x_l \partial x_j} + \frac{\partial g^{jl}}{\partial x_j}\frac{\partial u}{\partial x_l}) + \sum_{k,j,l} g^{kl}\frac{\partial u}{\partial x_l}\Gamma^j_{kj} = -K(x)$$

We have a linear PDE for u, and if we prove the local existence of a solution $u \in C^2$, we are done, because in that case we know by Theorem 3 that in some coordinate system (maybe different) we have $g'_{ij} = \delta_{ij}$ so $g = e^{-2u}\delta_{ij}$. As the coefficients of this system are C^α, it can be proved that there exists a solution $u \in C^{2,\alpha}(\Omega)$, though we shall not do it here. □

4.3. Conformal Flatness for Regular Metrics. Weyl-Schouten Theorem

We will focus on detecting local conformal flatness in the case where (M, g) has dimension $n \geq 3$, and g is regular enough to justify all the computations. We shall see that the regularity we will need is $g \in C^3$, since all our computations will require at most three derivatives of g.

Note that a necessary condition for (M, g) to be locally conformally flat is that its Weyl tensor vanishes. Indeed, if (M, g) is locally conformally flat, then there exists a C^3 coordinate system $(x_1, \ldots, x_n)$ an a C^2 function u such that the metric $g' = e^{2u}g$ satisfies $g'_{ij} = \delta_{ij}$ in this coordinate system. As g and g' are C^2 is this coordinates, we can define its curvature and Weyl tensors r, w and w', r' respectively. As $r' = 0$ then $w' = 0$. We saw that under conformal change the Weyl tensor transforms as $w' = e^{2u}w$ so w must be zero.

We already saw that the Weyl tensor is always zero when $dim(M) = 3$, so this necessary condition is always satisfied in dimension three. We will see that this condition is not enough for a three manifold, but it is required that another tensor vanishes (note that otherwise every three manifold would be locally conformally flat). However, if $dim(M) \geq 4$ the condition $w = 0$ will turn out to be sufficient, though in an indirect way as we will see.

Suppose then that we have our Riemannian manifold (M, g) and that the Weyl tensor $w = 0$. We look for a C^3 function u locally defined on M such that the metric $g' = e^{2u}g$ has curvature tensor $r' = 0$.

As $w = 0$, then $w' = 0$, so by the descomposition $r' = w' + s' \owedge g$ we see that $r' = 0$ is equivalent to $s' = 0$. Then, by the transformations formulas given in 1 we must find an u such that locally it solves the equation

$$s' = s + b_u = s + hess(u) - du \otimes du + \frac{1}{2}|grad(u)|^2_g g = 0 \tag{17}$$

and this is the system that we must solve to prove that (M, g) is conformally flat. We note first that in system 17 there are no terms depending on u, but everything depends on the derivatives of u. Then it looks reasonable to make a change $du = \alpha$. Before substitution, we must relate first $Hess(u)$ and du. One may expect from calculus on $\mathbb{R}^n$ that $\nabla du = Hess(u)$. Indeed, this is true.

Lemma 4. Let (M,g) be a Riemannian manifold, and let $u \in C^2$. Then we have $\nabla du = Hess(u)$.

Proof. We compute

$$\begin{aligned}
&(\nabla_X du)(Y) = X(du(Y)) - du(\nabla_X Y) = X(g(grad(u),Y)) - g(grad(u),\nabla_X Y)\\
&= g(\nabla_X grad(u),Y) + g(grad(u),\nabla_X Y) - g(grad(u),\nabla_X Y) = g(\nabla_X grad(u),Y)\\
&= Hess(u)(X,Y)
\end{aligned}$$

and this proves the Lemma. □

From the lemma above we see that system (17) really is the same as

$$\nabla du - du \otimes du + \frac{1}{2}|grad(u)|_g^2 g = -s \tag{18}$$

For simplicity of natotion, before going further solving system (18), it is convenient to establish the following definition.

Definition 15. Let (M,g) be a Riemannian manifold. We define the metric on 1-forms γ, β by putting $g(\gamma,\beta) = g(\gamma^\#, \beta^\#)$ being

$$\begin{aligned}
&b : TM \to T^*M : x \mapsto g(x,\cdot) = x^b\\
&\# = b^{-1} : T^*M \to TM : \alpha \mapsto \alpha^\# \text{ such that } g(\alpha^\#,\cdot) = \alpha(\cdot)
\end{aligned}$$

the musical isomorphisms. We denote this isomorphisms b and # to show that they respectively lower and raise the index, as b and # lower and raise the frequency of notes in music.

Coming back to equation (18), we make the substitution $du = \alpha$, and noting that $\alpha^\# = du^\# = grad(u)$, we get

$$\nabla\alpha - \alpha \otimes \alpha + \frac{1}{2}|\alpha|_g^2 g = -s \tag{19}$$

We must check whether we lose information with the change $du = \alpha$, i.e, if every α satisfayng the system (19) is exact. If we see this, we can reverse the process and solve the system (18). We remind that by definition, a one form α on M is exact if there exists a function u defined on M such that $\alpha = du$.

By the well known Poincaré Lemma swe know that if $d\alpha = 0$ then α is locally exact. But we do not need to use that result, since we can give a direct proof of this using our integrability conditions.

Proposition 3. Let α be a 1-form on a manifold M and suppose $d\alpha = 0$. Then for every $x \in M$ there exists $U \subset M$ an open set, and there exists a function $u : U \to \mathbb{R}$ such that $\alpha = du$

Proof. Suppose that $\alpha = a_1 dx_1 + \cdots + a_n dx_n$ in coordinates, so

$$\begin{aligned}
&d\alpha = (\frac{\partial a_1}{\partial x_2} dx_2 \wedge dx_1 + \cdots + \frac{\partial a_1}{\partial x_n} dx_n \wedge dx_1) + \cdots + (\frac{\partial a_n}{\partial x_1} dx_1 \wedge dx_n + \cdots + \frac{\partial a_n}{\partial x_{n-1}} dx_{n-1} \wedge dx_n)\\
&= \sum_{i<j} (\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j}) dx_i \wedge dx_j = 0 \quad \text{if and only if} \quad \frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} = 0 \quad \text{for every } i \neq j
\end{aligned}$$

and these are just the integrebility conditions for the system

$$\frac{\partial u}{\partial x_i} = a_i, \quad i = 1, \ldots, n$$

so there exists a locally defined such function u, and obviusly $du = \alpha$. □

Then, coming back to our system (19), let us see that the change $du = \alpha$ does not lose information. In view of the Proposition 3 below, we must show that $d\alpha = 0$.

Indeed, as α satisfies the system (19), we have

$$\nabla\alpha = \alpha \otimes \alpha - \frac{1}{2}|\alpha|_g^2 g - s$$

and then $\nabla\alpha$ is symmetric, i.e,

$$(\nabla\alpha)(X,Y) = (\nabla_X\alpha)(Y) = \alpha(X)\alpha(Y) - \frac{1}{2}|\alpha|_g^2 g(X,Y) - p(x,y) = (\nabla\alpha)(Y,X) = (\nabla_Y\alpha)(X)$$

This implies that $d\alpha = 0$, as the next proposition shows.

Proposition 4. Let α be a 1-form on a manifold M. Then α is exact if and only if $\nabla\alpha$ is symmetric.

Proof. Let $\alpha = a_1dx_1 + \cdots + a_ndx_n$ in coordinates.

Suppose first that $\nabla\alpha$ is symmetric. Then

$$\begin{aligned}(d\alpha)(\partial_i,\partial_j) &= \frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} = \partial_i[\alpha(\partial_j)] - \partial_j[\alpha(\partial_i)] \\ &= (\nabla_{\partial_i}\alpha)(\partial_j) + \alpha(\nabla_{\partial_i}\partial_j) - (\nabla_{\partial_j}\alpha)(\partial_i) - \alpha(\nabla_{\partial_j}\partial_i) = \alpha([\partial_i,\partial_j]) = 0\end{aligned}$$

and then $d\alpha = 0$.

Suppose now that $d\alpha = 0$ so for all i,j we have

$$(d\alpha)(\partial_i,\partial_j) = \frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} = 0.$$

Then, noting that $\nabla_i\partial_j = \nabla_j\partial_i$, it follows

$$(\nabla_{\partial_i}\alpha)(\partial_j) = \frac{\partial a_j}{\partial x_i} - \alpha(\nabla_{\partial_i}\partial_j) = \frac{\partial a_i}{\partial x_j} - \alpha(\nabla_{\partial_j}\partial_i) = (\nabla_{\partial_j}\alpha)(\partial_i)$$

and then $\nabla\alpha$ is symmetric as we wanted to prove. ☐

So we have seen that in order to solve the system (17), we can focus on solving the simpler system given by

$$\nabla\alpha - \alpha \otimes \alpha + \frac{1}{2}|\alpha|_g^2 g = s \tag{20}$$

To do this we shall express system (20) in local coordinates. First note that if we express in cordintes $\alpha = \sum_i a_i dx_i$ we have

$$\begin{aligned}(\nabla\alpha)(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}) &:= (\nabla_{\frac{\partial}{\partial x_i}}\alpha)(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_i}[\alpha(\frac{\partial}{\partial x_j})] - \alpha(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}) \\ &= \frac{\partial a_j}{\partial x_i} - \alpha(\sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}) = \frac{\partial a_j}{\partial x_i} - \sum_k \Gamma_{ij}^k a_k\end{aligned}$$

So system (20) reads as

$$\frac{\partial a_j}{\partial x_i} = \sum_k \Gamma_{ij}^k a_k + a_i a_j - \frac{1}{2}|\alpha|_g^2 g_{ij} - s_{ij} := f_i^j(x, a_1(x), \ldots, a_n(x)).$$

We know how to compute the integrability conditions for this system. Let's call for the moment

$$b := \alpha \otimes \alpha - \frac{1}{2}|\alpha|_g^2 g - s \quad \text{and} \quad b_{ij} = b(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) \tag{21}$$

The integrability conditions are obtained by forcing that

$$\frac{\partial^2 a_j}{\partial x_l \partial x_i} = \sum_k \frac{\partial \Gamma_{ij}^k}{\partial x_l} a_k + \sum_k \Gamma_{ij}^k \frac{\partial a_k}{\partial x_l} + \frac{\partial b_{ij}}{\partial x_l} = \frac{\partial^2 a_j}{\partial x_i \partial x_l} = \sum_k \frac{\partial \Gamma_{lj}^k}{\partial x_i} a_k + \sum_k \Gamma_{lj}^k \frac{\partial a_k}{\partial x_i} + \frac{\partial b_{lj}}{\partial x_i} \tag{22}$$

We substitute in (22)

$$\frac{\partial a_k}{\partial x_i} \quad \text{by} \quad \sum_m \Gamma_{ik}^m a_m + b_{ik}$$

to obtain

$$\sum_k \frac{\partial \Gamma_{ij}^k}{\partial x_l} a_k + \sum_{k,m} \Gamma_{ij}^k \Gamma_{lk}^m a_m + \sum_k \Gamma_{ij}^k b_{lk} + \frac{\partial b_{ij}}{\partial x_l} = \sum_k \frac{\partial \Gamma_{lj}^k}{\partial x_i} a_k + \sum_{k,m} \Gamma_{lj}^k \Gamma_{ik}^m a_m + \sum_k \Gamma_{lj}^k b_{ik} + \frac{\partial b_{lj}}{\partial x_i} \tag{23}$$

and switching index of summation, we finally get

$$\sum_k \left[\frac{\partial \Gamma_{ij}^k}{\partial x_l} - \frac{\partial \Gamma_{lj}^k}{\partial x_i} + \sum_m (\Gamma_{ij}^m \Gamma_{lm}^k - \Gamma_{lj}^m \Gamma_{im}^k) \right] a_k + \sum_k \Gamma_{ij}^k b_{lk} - \sum_k \Gamma_{lj}^k b_{ik} + \frac{\partial b_{ij}}{\partial x_l} - \frac{\partial b_{lj}}{\partial x_i} = 0 \tag{24}$$

Now, we note that

$$\begin{aligned}
&(\nabla_{\partial_l} b)(\partial_i, \partial_j) - (\nabla_{\partial_i} b)(\partial_l, \partial_j) \\
&= \frac{\partial b_{ij}}{\partial x_l} - \underline{b(\nabla_{\partial_l}\partial_i, \partial_j)} - b(\partial_i, \nabla_{\partial_l}\partial_j) - \frac{\partial b_{lj}}{\partial x_i} + \underline{b(\nabla_{\partial_i}}\partial_l, \partial_j) + b(\partial_l, \nabla_{\partial_i}\partial_j) \\
&= \frac{\partial b_{ij}}{\partial x_l} - b(\partial_i, \sum_k \Gamma_{lj}^k \partial_k) - \frac{\partial b_{lj}}{\partial x_i} + b(\partial_l, \sum_k \Gamma_{ij}^k \partial_k) \\
&= \sum_k \Gamma_{ij}^k b_{lk} - \sum_k \Gamma_{lj}^k b_{ik} + \frac{\partial b_{ij}}{\partial x_l} - \frac{\partial b_{lj}}{\partial x_i}
\end{aligned}$$

Then, by the computation above and the expression of R_{ijk}^l given in 8, we see that equation (24) is the same as

$$\begin{aligned}
&\sum R_{lij}^k a_k + (\nabla_{\partial_l} b)(\partial_i, \partial_j) - (\nabla_{\partial_i} b)(\partial_l, \partial_j) \\
&= \alpha(R(\partial_l, \partial_i)\partial_j) + (\nabla_{\partial_l} b)(\partial_i, \partial_j) - (\nabla_{\partial_i} b)(\partial_l, \partial_j) = 0
\end{aligned}$$

which is equivalent by tensoriality to

$$\alpha(R(X,Y)Z) + (\nabla_X b)(Y,Z) - (\nabla_Y b)(X,Z) = 0 \tag{25}$$

for every vector fields X, Y, Z

Note that the conditions given in (25) are not the integrability conditions yet. This is just the condition for the second order partial derivatives of the functions a_i to be the same. Observe that b contains terms which depend on the a_i, so ∇b will have terms depending on derivatives of the a_i. Now we must insert the real expression of b given in (21), and substitute again the partial derivatives of the a_i as we did before to obtain equations on which only terms on the a_i appears, without derivatives. Then we set the $a_i's$ as variables, and we are done, those are the integrability conditions for the system (20) (we explained why the integrability conditions are given by this procedure in Remark 2).

The previous somewhat strange manipulation was done to obtain (25), which will allow us to work without coordinates, and this will make clearer the calculations. Therefore, instead of substituting in coordinates the partial derivatives of the $a_i's$ as we have done until now, we will substitute $\nabla\alpha$ by $b = \alpha\otimes\alpha - \frac{1}{2}|\alpha|_g^2 g - s$ whenever $\nabla\alpha$ appears. Note that both substitutions are equivalent. Now we compute. When terms are underlined on the same color is bacause they cancel, or because they can be summed conveniently.

$$
\begin{aligned}
&(\nabla_X b)(Y,Z) = X(b(Y,Z)) - b(\nabla_X Y, Z) - b(Y, \nabla_X Z)\\
&= X[\alpha(Y)\alpha(Z) - \tfrac{1}{2}g(\alpha^{\#},\alpha^{\#})g(Y,Z) - s(Y,Z)]\\
&-\alpha(\nabla_X Y)\alpha(Z) + \tfrac{1}{2}|\alpha|_g^2 g(\nabla_X Y, Z) + s(\nabla_X Y, Z)\\
&-\alpha(Y)\alpha(\nabla_X Z)) + \tfrac{1}{2}|\alpha|_g^2 g(Y, \nabla_X Z) + s(Y, \nabla_X Z)\\
&= \underline{\alpha(Z)X(\alpha(Y))} + \underline{\alpha(Y)X(\alpha(Z))} - g(\nabla_X\alpha^{\#},\alpha^{\#})g(Y,Z)\\
&\underline{-\tfrac{1}{2}g(\alpha^{\#},\alpha^{\#})(g(\nabla_X Y, Z) + g(Y,\nabla_X Z))} - \underline{X(s(Y,Z))} - \underline{\alpha(\nabla_X Y)\alpha(Z)}\\
&+\underline{\tfrac{1}{2}|\alpha|_g^2 g(\nabla_X Y, Z)} + \underline{s(\nabla_X Y, Z)} - \underline{\alpha(Y)\alpha(\nabla_X Z))} + \underline{\tfrac{1}{2}|\alpha|_g^2 g(Y,\nabla_X Z)} + \underline{s(Y,\nabla_X Z)}\\
&= \alpha(Z)(\nabla_X\alpha)(Y) + \alpha(Y)(\nabla_X\alpha)(Z) - g(\nabla_X\alpha^{\#},\alpha^{\#})g(Y,Z) - (\nabla_X s)(Y,Z) = (*)
\end{aligned}
$$

Now we note that $\nabla_X\alpha^{\#} = (\nabla_X\alpha)^{\#}$, since

$$
\begin{aligned}
&(\nabla_X\alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y) = X(g(\alpha^{\#},Y)) - g(\nabla_X Y,\alpha^{\#})\\
&= g(\nabla_X\alpha^{\#},Y) + g(\alpha^{\#},\nabla_X Y) - g(\nabla_X Y,\alpha^{\#}) = g(\nabla_X\alpha^{\#},Y)
\end{aligned}
$$

then coming back to $(*)$ we obtain

$$
\begin{aligned}
&(\nabla_X b)(Y,Z) = (*) = \alpha(Z)(\nabla_X\alpha)(Y) + \alpha(Y)(\nabla_X\alpha)(Z) - \nabla_X\alpha(\alpha^{\#})g(Y,Z) - (\nabla_X s)(Y,Z)\\
&= \alpha(Z)[\alpha(X)\alpha(Y) - \tfrac{1}{2}|\alpha|_g^2 g(X,Y) - s(X,Y)] + \alpha(Y)[\alpha(X)\alpha(Z) - \tfrac{1}{2}|\alpha|_g^2 g(X,Z) - s(X,Z)]\\
&-g(Y,Z)[\alpha(X)\alpha(\alpha^{\#}) - \tfrac{1}{2}|\alpha|_g^2 g(X,\alpha^{\#}) - s(X,\alpha^{\#})] - (\nabla_X s)(Y,Z)\\
&= 2\alpha(X)\alpha(Y)\alpha(Z) - \tfrac{1}{2}\alpha(Z)|\alpha|_g^2 g(X,Y) - \alpha(Z)s(X,Y) - \tfrac{1}{2}\alpha(Y)|\alpha|_g^2 g(X,Z) - \alpha(Y)s(X,Z)\\
&-g(Y,Z)[\alpha(X)|\alpha|_g^2 - \tfrac{1}{2}|\alpha|_g^2\alpha(X) - s(X,\alpha^{\#})] - (\nabla_X s)(Y,Z)\\
&= 2\alpha(X)\alpha(Y)\alpha(Z) - \tfrac{1}{2}\alpha(Z)|\alpha|_g^2 g(X,Y) - \alpha(Z)s(X,Y) - \tfrac{1}{2}\alpha(Y)|\alpha|_g^2 g(X,Z) - \alpha(Y)s(X,Z)\\
&-\tfrac{1}{2}g(Y,Z)\alpha(X)|\alpha|_g^2 + g(Y,Z)s(X,\alpha^{\#}) - (\nabla_X s)(Y,Z)
\end{aligned}
$$

in these last expressions we have already substituted all the terms involving partial derivatives of α. Note that when making the commutator $(\nabla_X b)(Y,Z) - (\nabla_Y b)(X,Z)$, all symmetric terms in (X,Y)

cancel so we do not even write them. For the integrability conditions, we have

$$
\begin{aligned}
&0 = (\nabla_X b)(Y,Z) - (\nabla_Y b)(X,Z) + \alpha(R(X,Y)Z)\\
&= \underline{-\tfrac{1}{2}\alpha(Y)|\alpha|_g^2 g(X,Z)} - \alpha(Y)s(X,Z) - \underline{\tfrac{1}{2}g(Y,Z)\alpha(X)|\alpha|_g^2} + g(Y,Z)s(X,\alpha^{\#}) - (\nabla_X s)(Y,Z)\\
&+\underline{\tfrac{1}{2}\alpha(X)|\alpha|_g^2 g(Y,Z)} + \alpha(X)s(Y,Z) + \underline{\tfrac{1}{2}g(X,Z)\alpha(Y)|\alpha|_g^2} - g(X,Z)s(Y,\alpha^{\#}) + (\nabla_Y s)(X,Z)\\
&+\alpha(R(X,Y)Z)\\
&= -g(Y,\alpha^{\#})s(X,Z) + g(Y,Z)s(X,\alpha^{\#}) + g(X,\alpha^{\#})s(Y,Z) - g(X,Z)s(Y,\alpha^{\#})\\
&+(\nabla_Y s)(X,Z) - (\nabla_X s)(Y,Z) + r(X,Y,Z,\alpha^{\#})\\
&= (g \owedge s)(X,Y,\alpha^{\#},Z) - r(X,Y,\alpha^{\#},Z) + (\nabla_Y s)(X,Z) - (\nabla_X s)(Y,Z)\\
&= -w(X,Y,\alpha^{\#},Z) + (\nabla_Y s)(X,Z) - (\nabla_X s)(Y,Z).
\end{aligned}
$$

Finally we have to put the functions a_i as variables, which is the same as putiing $\alpha^{\#} := T$ as an arbitrary vector field. Thus we have finally obtained the integrability conditions of system (20), which are

$$(\nabla_Y s)(X,Z) - (\nabla_X s)(Y,Z) - w(X,Y,T,Z) = 0 \tag{26}$$

for every vector fields X, Y, Z, T. This motivates the following definition.

Definition 16. Let (M, g) be a Riemannian manifold and let s be its Schouten tensor. We define the *Cotton tensor* as $c(X,Y,Z) := (\nabla_X s)(Y,Z) - (\nabla_Y s)(X,Z)$.

Note that, by the simmetries of w, the integrability conditions in (26) read now as $w(X,Y,Z,T) = c(X,Y,Z)$ for every vector fields X, Y, Z, T. Now we can state the main Theorem in this section.

Theorem 5. Let (M, g) be a Riemannian manifold with zero Weyl tensor w. Let s and c be its Schouten and Cotton tensors. Then (M, g) is locally conformally flat if and only if $(\nabla_Y s)(X,Z) - (\nabla_X s)(Y,Z) = 0$ for every vector fields X, Y, Z. So we are saying that (M, g) with zero Weyl tensor is locally conformally flat if and only if its Cotton tensor vanishes, i.e, $c(X,Y,Z) = 0$ for every vector fields X, Y, Z.

Proof. We make a review of the proof which has been done above. As we have seen, (M, g) is locally conformally flat if and only if the system

$$\nabla du - du \otimes du + \frac{1}{2}|grad(u)|_g^2 g = -s \tag{27}$$

admits solution locally. As we saw above, this happens if and only if the system

$$\nabla \alpha - \alpha \otimes \alpha + \frac{1}{2}|\alpha|_g^2 g = -s \tag{28}$$

admits solution locally. We proved that the integrability conditions for the system (28) are $w(X,Y,Z,T) = c(X,Y,Z)$ for every vector fields X, Y, Z, T. We are supposing that $w = 0$, so system (28) is locally solvable if and only if the integrability conditions are satisfied, if and only if the Cotton tensor $c = 0$. $\square$

Note that it could happen that $c = 0$ but $w \neq 0$, so we need both to be zero for conformal flatness. However, in dimension greater than 3, it happens that the condition $w = 0$ implies that $c = 0$. To see this we need to make some hard computations.

In Riemannian geometry there is a very useful trick to make computations that involve tensors. The idea is that fixed a point $p \in M$, we can construct coordinates around p that behave nicely at p.

Given this, if we want to prove that two tensors are equal, it will be enough to prove that they are equal when evaluated at each fixed point p, and then we can use this special coordinates (depending on p) to see the equality at p. Then, as p is arbitrary, the tensors must be equal. To construct these coordinates, we need some definitions first.

Definition 17. Let (M, g) be a Riemannian manifold, and let $p \in M$.

(1) A curve $\gamma : [0, 1] \to M$ is called a *geodesic* if the mapping $t \to \nabla_{\gamma'(t)}\gamma'(t)$ is identically zero. This condition expressed in coordinates becomes a second order linear system of differential equations for the coordinates $\gamma_j(t)$ of γ, which has a unique solution $(\gamma_1(t), \dots, \gamma_n(t))$ in we fix the initial data $\gamma(0) = p$ and $\gamma'(0) = v$ for some $v \in T_pM$.

(2) The *exponential map at* p is the map $exp_p : T_pM \to M : v \mapsto \gamma_{p,v}(1)$ where $\gamma_{p,v}(t)$ is the unique geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$. It is straighforward to see that the differential $d_0exp_p : T_pM \to T_pM$ is the identity, so, by the Inverse Function Theorem (IFT) on manifolds, the exponential defines a diffeomorphism between a neighborhood U of $0 \in T_pM$ and a neighborhhod V of $p \in M$.

Remark 14. In the definition of (1), note that in order to define $\nabla_{\gamma'(t)}\gamma'(t)$ we would need to extend $\gamma'(t)$ to a vector field on M and see that it is independent of the extension, which is easy to see. The intuitive reason is that $\nabla_X Y$ express the rate of change (in some sense) of Y along X, so it is to be expected that the value of $\nabla_X Y$ at p only depens on the values of Y along any curve whose tangent vector at p is $X(p)$. In particular the value of $\nabla_{\gamma'(t)}\gamma'(t)$ at the point $p = \gamma(t)$ only depens on the values of $Y = \gamma'(t)$ at any curve whose tangent vector at p is $\gamma'(t)$, and of course one such curve is $\gamma(t)$, where $Y = \gamma'(t)$ is indeed defined, so the definition is correct.

On the other hand, in the definition of (2), note that we are not sure whether the geodesic $\gamma_{p,v}(t)$ is defined at $t = 1$. However, Theorem 1 says that there exists $\varepsilon > 0$ small enough and $c > 0$ so that for all $|v| < \varepsilon$, then $\gamma_{p,v}(c)$ is defined. By reescaling the parameter, we see that $\gamma_{p,cv}(1) = \gamma_{p,v}(c)$, so if $|v| < \frac{\varepsilon}{c}$ the exponential map is well defined.

Lemma 5. Let (M, g) be a Riemannian manifold. For each point $p \in M$ there exists a neighborhood of p on which there exists:

(1) A system of *normal coordinates at p* , i.e., a coordinates system $(x_1, \dots, x_n)$ such that

$$(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j})(p) = 0 \quad \text{for each} \quad i, j = 1, \dots, n \tag{29}$$

(2) A *geodesic frame based at p* , i.e., an orthonormal frame $E_1, \dots, E_n$ such that

$$(\nabla_{E_i} E_j)(p) = 0 \quad \text{for each} \quad i, j = 1, \dots, n \tag{30}$$

Proof. To see (1), select an orthonormal basis $\{v_1, \dots, v_n\}$ of T_pM. Define $\pi : T_pM \to \mathbb{R}^n$ as a linear isometry sending v_i to the corresponding vector e_i of the canonical basis of $\mathbb{R}^n$. Now consider $x := \pi \circ (\exp_p)^{-1} : M \to \mathbb{R}^n$, whose differential at p is π, so x defines a system of coordinates. Besides, it is easy to see that condition (29) is satisfied. For details, see [1], Vol.I, page 169.

The assertion (2) is in Exercise 7, Chapter 3 of [2]. The proof is not difficult, but it would require more definitions (covariant differentiation and parallel transport) and it is not of our interest going into there in this work. □

Note that in a geodesic frame we have $g_{ij} = \delta_{ij}$, but do not have in general $[e_i, e_j] = 0$, since in general this frame can not be expressed as the partial derivatives of some coordinate system. So one can not have it all: either the brackets are zero or the metric is the identity.

Lemma 6. (Second Bianchi Identity) Let (M,g) be a Riemannian manifold, and r its curvature tensor. For every vector fields X,Y,Z,V,W we have

$$(\nabla_W r)(X,Y,Z,V)+(\nabla_Z r)(X,Y,V,W)+(\nabla_V r)(X,Y,W,Z)=0 \tag{31}$$

Proof. First note that

$$\begin{aligned}&(\nabla_W r)(X,Y,Z,V)=W(r(X,Y,Z,V))-r(\nabla_W X,Y,Z,V)-r(X,\nabla_W Y,Z,V)\\ &-r(X,Y,\nabla_W Z,V)-r(X,Y,Z,\nabla_W V)\\ &=W(r(Z,V,X,Y))-r(Z,V,\nabla_W X,Y)-r(Z,V,X,\nabla_W Y)\\ &-r(\nabla_W Z,V,X,Y)-r(Z,\nabla_W V,X,Y)=(\nabla_W r)(Z,V,X,Y)\end{aligned}$$

so, changing X by V and Y by W, equation 31 is equivalent to

$$(\nabla_Z r)(X,Y,V,W)+(\nabla_X r)(Y,Z,V,W)+(\nabla_Y r)(Z,X,V,W)=0. \tag{32}$$

Now, since ∇r is a tensor, it is enough to prove 32 evaluated in a generic but fixed point $p\in M$. For it, we choose a system of normal coordinates $(x_1,\dots,x_n)$ around p, and we have

$$\begin{aligned}&(\nabla_{\partial_k} r)(\partial_i,\partial_j,\partial_l,\partial_m)(p)=(\partial_k[g(R(\partial_i,\partial_j)\partial_l,\partial_m)])(p)\\ &=g(\nabla_{\partial_k}\nabla_{\partial_i}\nabla_{\partial_j}\partial_l-\nabla_{\partial_k}\nabla_{\partial_j}\nabla_{\partial_i}\partial_l,\partial_m)(p)\end{aligned}$$

so if we make the cyclic permutation on the tuple of index (i,j,k) we obtain

$$(\nabla_{\partial_k} r)(\partial_i,\partial_j,\partial_l,\partial_m)(p)+(\nabla_{\partial_i} r)(\partial_j,\partial_k,\partial_l,\partial_m)(p)+(\nabla_{\partial_j} r)(\partial_k,\partial_i,\partial_l,\partial_m)(p)=g(*,\partial_m)(p)$$

where $*$ equals to

$$\begin{aligned}&[\underline{\nabla_{\partial_k}\nabla_{\partial_i}\nabla_{\partial_j}\partial_l}-\underline{\nabla_{\partial_k}\nabla_{\partial_j}\nabla_{\partial_i}\partial_l}+\underline{\nabla_{\partial_i}\nabla_{\partial_j}\nabla_{\partial_k}\partial_l}-\underline{\nabla_{\partial_i}\nabla_{\partial_k}\nabla_{\partial_j}\partial_l}+\underline{\nabla_{\partial_j}\nabla_{\partial_k}\nabla_{\partial_i}\partial_l}-\underline{\nabla_{\partial_j}\nabla_{\partial_i}\nabla_{\partial_k}\partial_l}](p)\\ &=[R(\partial_k,\partial_i)\nabla_{\partial_j}\partial_l+R(\partial_j,\partial_k)\nabla_{\partial_i}\partial_l+R(\partial_i,\partial_j)\nabla_{\partial_k}\partial_l](p)=0\end{aligned}$$

and by tensoriality this gives 32 when evaluated at p. □

Of course, the use of normal coordinates and geodesic frames is not necessary, but it simplifies the calculations.

Definition 18. Given a Riemannian manifold (M,g), the divergence of a symmetric $(2,0)$-tensor s is the $(1,0)$-tensor given by

$$div(s)(x)=\sum_i(\nabla_{e_i}s)(x,e_i)=Tr(\nabla s_{(\cdot)}(x,\cdot))$$

where $\{e_i\}$ is any orthonormal basis. It is independent of the basis because is the trace of the $(2,0)$-tensor $Tr(\nabla s_{(\cdot)}(x,\cdot))$.

If s is not symmetric the definition splits into two, because in this case $Tr(\nabla s_{(\cdot)}(x,\cdot))$ and $Tr(\nabla s_{(\cdot)}(\cdot,x))$ can be different so we have two divergences depending on the choice.

We are interested in a particular form that takes the Second Bianchi Identity when contracted.

Corollary 2. (Once contracted Bianchi Identity) Let (M,g) a Riemannian manifold and let X,Y,V be arbitrary fixed vector fields. Define the $(2,0)$-tensor

$$\nabla_{(\cdot)}r(X,Y,V,\cdot):(Z,W)\mapsto(\nabla_Z r)(X,Y,V,W)$$

then we have that

$$Tr(\nabla r(\cdot,X,Y,V,\cdot))=(\nabla_X Ric)(Y,V)-(\nabla_Y Ric)(X,V) \tag{33}$$

Proof. We shall see (33) at a generic point $p \in M$. Choose a geodesic frame $\{E_i\}$ centered at p. Recall that in this frame we have by tensoriality, that

$$(\nabla_X E_i)(p) = 0$$

for every vector field X. Now we remind the Second Bianchi Identity

$$(\nabla_Z r)(X, Y, V, W) + (\nabla_X r)(Y, Z, V, W) + (\nabla_Y r)(Z, X, V, W) = 0. \tag{34}$$

Set $Z = W = E_i$ in (34) and sum over i to obtain

$$0 = (\nabla_{E_i} r)(X, Y, V, E_i) + (\nabla_X r)(Y, E_i, V, E_i) + (\nabla_Y r)(E_i, X, V, E_i) = A + B + C$$

Note first that $A = Tr(\nabla r(\cdot, X, Y, V, \cdot))$. Besides, when evaluating everything at p, we have

$$\begin{aligned}
B = (\nabla_X r)(Y, E_i, V, E_i) &= X[r(Y, E_i, V, E_i)] - r(\nabla_X Y, E_i, V, E_i) \\
&-r(Y, \nabla_X E_i, V, E_i) - r(Y, E_i, \nabla_X V, E_i) - r(Y, E_i, V, \nabla_X E_i) \\
&= X[r(Y, E_i, V, E_i)] - r(\nabla_X Y, E_i, V, E_i) - r(Y, E_i, \nabla_X V, E_i) \\
&= -X[Ric(Y, V)] + Ric(\nabla_X Y, V) + Ric(Y, \nabla_X V) = -(\nabla_X Ric)(Y, V).
\end{aligned}$$

Finally, by an analogous calculation,

$$C = (\nabla_Y r)(E_i, X, V, E_i) = (\nabla_Y Ric)(X, V)$$

and this yields (33). □

Corollary 3. (Twice Contracted Second Bianchi Identity) Let (M, g) a Riemannian manifold, Ric and $Scal$ its Ricci tensor and scalar curvature respectively. Then

$$div(Ric) = \frac{1}{2}\nabla Scal = \frac{1}{2}d(Scal).$$

Proof. We take E_i a geodesic frame based at a generic point $p \in M$. From the Second Bianchi Identity, putting in equation 31 $X = V = E_i$ and $Y = Z = E_j$, and summing over i, j, we have

$$0 = \sum_{i,j} (\nabla_W r)(E_i, E_j, E_j, E_i) + (\nabla_{E_j} r)(E_i, E_j, E_i, W) + (\nabla_{E_i} r)(E_i, E_j, W, E_j)$$

on the other hand, taking into account that $(\nabla_{E_i} E_j)(p) = 0$ for all i, j, we have (everything is evaluated at p)

$$div(Ric)(W) = \sum_i (\nabla_{E_i} Ric)(W, E_i) = \sum_i E_i[Ric(W, E_i)] - Ric(\nabla_{E_i} W, E_i)$$

$$= \sum_{i,j} E_i[r(E_j, W, E_i, E_j)] - r(E_j, \nabla_{E_i} W, E_i, E_j) = \sum_{i,j} (\nabla_{E_i} r)(E_j, W, E_i, E_j)$$

and we have too that (evaluated at p)

$$(\nabla Scal)(W) = \nabla_W Scal = \sum_{i,j} W[r(E_i, E_j, E_j, E_i)] = \sum_{i,j} (\nabla_W r)(E_i, E_j, E_j, E_i)$$

since by tensoriality, for general W, $(\nabla_W E_i)(p) = 0$ for each i.

Recall now that along the way we have proved without explicit mention that for each X fixed, the tensor $(\nabla_X r)$ has the same symmetries that r. So we conclude that

$$0 = (\nabla Scal)(W) - 2div(Ric)(W)$$

which yields the result. □

Now we prove that if $dim(M) \geq 4$ then $w = 0$ imply $c = 0$. First we need two lemmas.

Lemma 7. Given a $(2,0)$-tensor s on a Riemannian manifold (M, g) then $Tr(\nabla_Y s) = d(Tr(s))(Y)$ for every vector field Y.

Proof. To see this, we first show that $Tr(\nabla_Y s) = d(Tr(s))(Y)$ for every $(2,0)$-tensor. For this we use our geodesic frame $\{E_i\}$ based at p, that verifies $(\nabla_Y E_i)(p) = 0$. We have, evaluating at p

$$Tr(\nabla_Y s) = \sum_i (\nabla_Y s)(E_i, E_i) = \sum_i Y[s(E_i, E_i)] = Y[tr(s)] = d(Tr(s))(Y)$$

and this proves the Lemma. □

Lemma 8. For the Schouten tensor s of a Riemannian Manifold (M, g) we have

$$d(Tr(s))(Y) = div(s)(Y) = \frac{d(Scal)(Y)}{2 - 2n}$$

for every vector field Y.

Proof. $d(Tr(s))(Y) = div(s)(Y)$. For this, we first recall the definition of the Schouten tensor

$$s := \frac{1}{2-n}[Ric + \frac{Scal}{2-2n} g].$$

Recall the contracted second bianchi identity $div(Ric) = \frac{1}{2}d(Scal)$

$$\begin{aligned}
div(s)(Y) &= \frac{1}{2-n} div(Ric)(Y) + \frac{1}{2-n}\frac{1}{2-2n} div(Scal \cdot g)(Y) \\
&= \frac{1}{2-n}\frac{1}{2} d(Scal)(Y) + \frac{1}{2-n}\frac{1}{2-2n}\sum_i (\nabla_{E_i}(Scal \cdot g))(Y, E_i) \\
&= \frac{1}{2(2-n)} d(Scal)(Y) + \frac{1}{2-n}\frac{1}{2-2n}\sum_i E_i[Scal \cdot g(Y, E_i)] - (Scal \cdot g)(\nabla_{E_i} Y, E_i) \\
&= \frac{1}{2(2-n)} d(Scal)(Y) + \frac{1}{2-n}\frac{1}{2-2n}\sum_i g(Y, E_i) E_i[Scal] \\
&= \frac{1}{2(2-n)} d(Scal)(Y) + \frac{1}{2-n}\frac{1}{2-2n} Y[Scal] \\
&= \frac{d(Scal)(Y)}{2(2-n)} + \frac{d(Scal)(Y)}{2(2-n)(1-n)} = \frac{d(Scal)(Y)}{2(1-n)}.
\end{aligned}$$

On the other hand,

$$Tr(s) = \frac{1}{2-n}[Tr(Ric) + \frac{Scal}{2-2n} Tr(g)] = \frac{Scal}{2-n} + \frac{n \cdot Scal}{(2-2n)(2-n)} = \frac{Scal}{2-2n}.$$

Finally we conclude that

$$d(Tr(s))(Y) = \frac{d(Scal)(Y)}{2-2n} = div(s)(Y)$$

which proves the lemma. □

Lemma 9. Let s be the Schouten tensor of a Riemannian manifold M. Then we have

$$\begin{aligned}
&Tr((\nabla_{(\cdot)} s \owedge g)(Y, Z, V, \cdot)) + Tr((\nabla_Y s \owedge g)(Z, \cdot, V, \cdot)) \\
&+ Tr((\nabla_Z s \owedge g)(\cdot, Y, V, \cdot)) = (3-n) c(Y, Z, V)
\end{aligned} \tag{35}$$

Proof. We consider a geodesic frame $\{E_i\}$ based at some arbitrary point $p \in M$. Then we have

$$\begin{aligned} Tr((\nabla_{(\cdot)} s \owedge g)(Y, Z, V, \cdot)) &= \sum_i (\nabla_{E_i} s \owedge g)(Y, Z, V, E_i) := A \\ Tr((\nabla_Y s \owedge g)(Z, \cdot, V, \cdot)) &= \sum_i (\nabla_Y s \owedge g)(Z, E_i, V, E_i) := B \\ Tr((\nabla_Z s \owedge g)(\cdot, Y, V, \cdot)) &= \sum_i (\nabla_Z s \owedge g)(E_i, Y, V, E_i) := C \end{aligned} \tag{36}$$

We compute each of the sumands separately. First the term A.

$$\begin{aligned} A &= \sum_i (\nabla_{E_i} s \owedge g)(Y, Z, V, E_i) \\ &= \sum_i (\nabla_{E_i} s)(Y, V) g(Z, E_i) - (\nabla_{E_i} s)(Y, E_i) g(Z, V) \\ &+ g(Y, V)(\nabla_{E_i} s)(Z, E_i) - g(Y, E_i)(\nabla_{E_i} s)(Z, V) \\ &= (\nabla_Z s)(Y, V) - g(Z, V) div(s)(Y) + g(Y, V) div(s)(Z) - (\nabla_Y s)(Z, V) \end{aligned}$$

the last equality because in this frame $Z = \sum_i g(Z, E_i) E_i$ and $Y = \sum_i g(Y, E_i) E_i$. Now we compute B.

$$\begin{aligned} B &= \sum_i (\nabla_Y s \owedge g)(Z, E_i, V, E_i) \\ &= \sum_i (\nabla_Y s)(Z, V) g(E_i, E_i) - (\nabla_Y s)(Z, E_i) g(E_i, V) \\ &+ g(Z, V)(\nabla_Y s)(E_i, E_i) - g(Z, E_i)(\nabla_Y s)(E_i, V) \\ &= n(\nabla_Y s)(Z, V) - (\nabla_Y s)(Z, V) + g(Z, V) Tr(\nabla_Y s) - (\nabla_Y s)(Z, V) \\ &= (n-2)(\nabla_Y s)(Z, V) + g(Z, V) Tr(\nabla_Y s) \end{aligned}$$

Finally we compute C.

$$\begin{aligned} C &= \sum_i (\nabla_Z s \owedge g)(E_i, Y, V, E_i) \\ &= \sum_i (\nabla_Z s)(E_i, V) g(Y, E_i) - (\nabla_Z s)(E_i, E_i) g(Y, V) \\ &+ g(E_i, V)(\nabla_Z s)(Y, E_i) - g(E_i, E_i)(\nabla_Z s)(Y, V) \\ &= (\nabla_Z s)(Y, V) - Tr(\nabla_Z s) g(Y, V) + (\nabla_Z s)(Y, V) - n(\nabla_Z s)(Y, V) \\ &= (2-n)(\nabla_Z s)(Y, V) - Tr(\nabla_Z s) g(Y, V) \end{aligned}$$

so we find that

$$\begin{aligned} 0 &= A + B + C \\ &= \underline{(\nabla_Z s)(Y, V)} - g(Z, V) div(s)(Y) + g(Y, V) div(s)(Z) - \underline{(\nabla_Y s)(Z, V)} \\ &+ \underline{(n-2)(\nabla_Y s)(Z, V)} + g(Z, V) Tr(\nabla_Y s) \\ &+ \underline{(2-n)(\nabla_Z s)(Y, V)} - Tr(\nabla_Z s) g(Y, V) \\ &= (3-n)(\nabla_Z s)(Y, V) + (n-3)(\nabla_Y s)(Z, V) \\ &+ g(Z, V)[Tr(\nabla_Y s) - div(s)(Y)] + g(Y, V)[div(s)(Z) - Tr(\nabla_Z s)] \\ &= (n-3)((\nabla_Y s)(Z, V) - (\nabla_Z s)(Y, V)) = (n-3) c(Y, Z, V) \end{aligned}$$

where in the last equality we have used that $Tr(\nabla_Y s) - div(s)(Y) = 0$, as proved in lemma 7 and 8. $\square$

Lemma 10. Let (M, g) be a Riemannian manifold, w its Weyl tensor and s its Schouten tensor. Then we have the following identities

$$(\nabla_Y s)(Z, V) = \frac{1}{2-n}[(\nabla_Y Ric)(Z, V) + \frac{(dScal)(Y)}{2-2n} g(Z, V)] \tag{37}$$

$$Tr((\nabla_Y w)(Z, \cdot, V, \cdot)) - Tr((\nabla_Z w)(Y, \cdot, V, \cdot)) = 0 \tag{38}$$

Proof. We compute first (37). Recall that by definition

$$s(Z, V) = \frac{1}{2-n}[Ric(Z, V) + \frac{Scal}{2-2n} g(Z, V)]$$

so it follows that

$$\begin{aligned}
&(\nabla_Y s)(Z, V) = Y[s(Z, V)] - s(\nabla_Y Z, V) - s(Z, \nabla_Y V) \\
&= \frac{1}{2-n}[(\nabla_Y Ric)(Z, V) + \frac{Y[Scal]}{2-2n} g(Z, V)] + \frac{1}{2-n}\{\frac{Scal}{2-2n} Y[g(Z, V)]\} \\
&-\frac{1}{2-n}[\frac{Scal}{2-2n} g(\nabla_Y Z, V) - \frac{Scal}{2-2n} g(Z, \nabla_Y V)] \\
&= \frac{1}{2-n}[(\nabla_Y Ric)(Z, V) + \frac{Y[Scal]}{2-2n} g(Z, V)]
\end{aligned}$$

since $Y[g(Z, V)] = g(\nabla_Y Z, V) + g(Z, \nabla_Y V)$.

Now, to see (38), let $\{E_i\}$ be a geodesis frame centered at an arbitrary point $p \in M$. Then

$$\begin{aligned}
&Tr((\nabla_Y w)(Z, \cdot, V, \cdot)) = \sum_i (\nabla_Y w)(Z, E_i, V, E_i) \\
&= \sum_i (\nabla_Y r)(Z, E_i, V, E_i) - (\nabla_Y s \owedge g)(Z, E_i, V, E_i) \\
&= \sum_i (\nabla_Y r)(Z, E_i, V, E_i) - (\nabla_Y s)(Z, V) g(E_i, E_i) \\
&+(\nabla_Y s)(Z, E_i) g(E_i, V) - g(Z, V)(\nabla_Y s)(E_i, E_i) + g(Z, E_i)(\nabla_Y s)(E_i, V)] \\
&= \sum_i [(\nabla_Y r)(Z, E_i, V, E_i)] + (-n+2)(\nabla_Y s)(Z, V) - g(Z, V) Tr(\nabla_Y s) \\
&= \sum_i [(\nabla_Y r)(Z, E_i, V, E_i)] + (\nabla_Y Ric)(Z, V) + \frac{Y[Scal]}{2-2n} g(Z, V) - g(Z, V)\frac{Y[Scal]}{2-2n} \\
&= \sum_i [(\nabla_Y r)(Z, E_i, V, E_i)] + (\nabla_Y Ric)(Z, V)
\end{aligned}$$

by lemma 8. Then, switching Z and Y and substracting we have

$$\begin{aligned}
&\sum_i (\nabla_Y w)(Z, E_i, V, E_i) - \sum_i (\nabla_Z w)(Y, E_i, V, E_i) \\
&= \sum_i [(\nabla_Y r)(Z, E_i, V, E_i) - (\nabla_Z r)(Y, E_i, V, E_i)] + (\nabla_Y Ric)(Z, V) - (\nabla_Z Ric)(Y, V) \\
&= \sum_i [(\nabla_Y r)(Z, E_i, V, E_i) + (\nabla_Z r)(E_i, Y, V, E_i)] + (\nabla_Y Ric)(Z, V) - (\nabla_Z Ric)(Y, V) \\
&= \sum_i [-(\nabla_{E_i} r)(Y, Z, V, E_i)] + (\nabla_Y Ric)(Z, V) - (\nabla_Z Ric)(Y, V) = 0
\end{aligned}$$

where we have used the Second Bianchi Identity and the once contracted Second Bianchi Identity in the two final steps, given in Lemma 6 and Corollary 2. □

Remark 15. Note that, as the tensors $\nabla_Y w$ and w has the same symmetries, equation (38) is equivalent to

$$Tr((\nabla_Y w)(Z,\cdot,V,\cdot)) + Tr((\nabla_Z w)(\cdot,Y,V,\cdot)) = 0 \tag{39}$$

Lemma 11. Let s and t be two $(2,0)$-tensors on a Riemannian manifold (M,g). Then for every vector field X

$$\nabla_X(s \owedge t) = \nabla_X s \owedge t + s \owedge \nabla_X t \tag{40}$$

Proof. We compute in normal coordinates with partial derivatives $\{\partial_i\}_i$, based at an arbitrary point $p \in M$. Recall that in these coordinates we have $(\nabla_{\partial_i}\partial_j)(p) = 0$ so evaluating at p we have

$$\partial_i[s(\partial_j,\partial_k)] = (\nabla_{\partial_i} s)(\partial_j,\partial_k)$$

and the same is valid fot the tensor t. We conclude that, evaluating at p,

$$\begin{aligned}
&(\nabla_{\partial_i}(s \owedge t))(\partial_j,\partial_k,\partial_l,\partial_m) = \partial_i[(s \owedge t)(\partial_j,\partial_k,\partial_l,\partial_m)]\\
&= \partial_i[s(\partial_j,\partial_l)t(\partial_k,\partial_m) - s(\partial_j,\partial_m)t(\partial_k,\partial_l) + t(\partial_j,\partial_l)s(\partial_k,\partial_m) - t(\partial_j,\partial_m)s(\partial_k,\partial_l)]\\
&= (\nabla_{\partial_i} s \owedge t + s \owedge \nabla_{\partial_i} t)(\partial_j,\partial_k,\partial_l,\partial_m)
\end{aligned}$$

and by tensoriality we have (40). □

Lemma 12. Let (M,g) be a Riemnnian manifold and let w and c be its Weyl and Cotton tensors. Then

$$Tr(\nabla_{(\cdot)} w)(Y,Z,V,\cdot) = (n-3)c(Y,Z,V) \tag{41}$$

for every vector fields Y,Z,V.

Proof. First note that by equation (40) we have $(\nabla_X)(s \owedge g) = (\nabla_X s) \owedge g$. Now we use the Second Bianchi identity to get

$$\begin{aligned}
0 &= (\nabla_X r)(Y,Z,V,W) + (\nabla_Y r)(Z,X,V,W) + (\nabla_Z r)(X,Y,V,W)\\
&= (\nabla_X s \owedge g)(Y,Z,V,W) + (\nabla_Y s \owedge g)(Z,X,V,W) + (\nabla_Z s \owedge g)(X,Y,V,W)\\
&+(\nabla_X w)(Y,Z,V,W) + (\nabla_Y w)(Z,X,V,W) + (\nabla_Z w)(X,Y,V,W).
\end{aligned}$$

We put $X = W = E_i$, sum over i, and obtain

$$0 = (3-n)c(Y,Z,V) + Tr(\nabla_{(\cdot)} w)(Y,Z,V,\cdot)$$

where we used Lemma 9 and Lemma 10 above. □

Theorem 6. (Weyl-Schouten) Let (M,g) be a Riemannian manifold with $g \in C^3$ and $dim(M) \geq 3$.
(1) If $dim(M) \geq 4$, (M,g) is locally conformally flat if and only if its Weyl tensor w vanishes.
(2) If $dim(M) = 3$, (M,g) is locally conformally flat if and only if its Cotton tensor c vanishes.

Proof. To see (1), suppose $dim(M) \geq 4$. If $w = 0$, by equation (41), we see that $c = 0$, and by theorem 5 this implies that (M,g) is locally conformaly flat. On the other hand, if (M,g) is locally conformaly flat we know, by the conformal transformation of the Weyl tensor, that $w = 0$.

To see (2), suppose $dim(M) = 3$. We know that if in this case the Weyl tensor is always zero. Then, if $c = 0$ we apply Theorem 5 and conclude that (M,g) is locally conformaly flat. On the other hand, if (M,g) is locally conformaly flat, the integrability conditions of the system expressing this fact were obtained in the proof of Theorem 5, and these conditions just mean $c = 0$. □

Remark 16. Note that, although the Weyl tensor requires $g \in C^2$ to be defined, here we need $g \in C^3$ because we need the Cotton tensor to vanish in order to solve the system in theorem 5, and the Cotton tensor involves third derivatives of g.

We have seen in Theorem 6 that if $dim(M) = 3$ being conformally flat is equivalent to the condition that the Cotton tensor c vanishes, since these are the integrability conditions of the system related to conformal flatness. This suggests that the Cotton tensor may have, at least in dimension 3, a nice transformation behaviour under conformal change of the metric. This is what we shall prove now. For the moment we will not use this transformation behaviour, but later it will be used.

Proposition 5. (Conformal change of the Cotton tensor) Let (M, g) be a Riemannian manifold and c its Cotton tensor. Under conformal change of the metric $g' := e^{2u}g$ the Cotton tensor c' of g' satisfies

$$c'(X, Y, Z) = c(X, Y, Z) - w(X, Y, Z, grad(u))$$

Proof. This will be a long computation. We saw before that under conformal change the Schouten tensor and the covariant derivative change in the following way

$$\begin{aligned}
s' &= s + b_u = s + hess(u) - du \otimes du + \frac{1}{2}|grad(u)|_g^2 g \\
\nabla'_X Y &= \nabla_X Y + B(X, Y) = \nabla_X Y + du(X)Y + du(Y)X - g(X, Y)grad(u)
\end{aligned}$$

For any $(2, 0)$ tensor t let us denote $\mathcal{C}(t)$ and $\mathcal{C}'(t)$ for the $(3, 0)$-tensors given by

$$\begin{aligned}
\mathcal{C}(t)(X, Y, Z) &:= (\nabla_X t)(Y, Z) - (\nabla_Y t)(X, Z) \\
\mathcal{C}'(t)(X, Y, Z) &:= (\nabla'_X t)(Y, Z) - (\nabla'_Y t)(X, Z)
\end{aligned}$$

The letter $\mathcal{C}$ is chosen since it is some kind of conmutator of ∇. Note that in particular the Cotton tensor is derived from the schouten tensor in this way, so $c = \mathcal{C}(s)$ and $c' = \mathcal{C}'(s')$. That said, we just compute

$$\begin{aligned}
&(\nabla'_X s')(Y, Z) = X[s'(Y, Z)] - s'(\nabla'_X Y, Z) - s'(Y, \nabla'_X Z) \\
&= X[s(Y, Z) + b_u(Y, Z)] - (s + b_u)(\nabla_X Y + B(X, Y), Z) - (s + b_u)(Y, \nabla_X Z + B(X, Z)) \\
&= X[s(Y, Z)] - s(\nabla_X Y, Z) - s(Y, \nabla_X Z) + X[b_u(Y, Z)] - b_u(\nabla_X Y, Z) - b_u(Y, \nabla_X Z) \\
&- s(B(X, Y), Z) - s(Y, B(X, Z)) - b_u(B(X, Y), Z) - b_u(Y, B(X, Z)) \\
&= (\nabla_X s)(Y, Z) + (\nabla_X b_u)(Y, Z) - s(B(X, Y), Z) - s(Y, B(X, Z)) - b_u(B(X, Y), Z) - b_u(Y, B(X, Z))
\end{aligned}$$

So interchanging the rolls of X and Y and substracting we have

$$\begin{aligned}
c'(X, Y, Z) &= c(X, Y, Z) + \mathcal{C}(b_u)(X, Y, Z) + [s(X, B(Y, Z)) - s(Y, B(X, Z))] \\
&+ [b_u(X, B(Y, Z)) - b_u(Y, B(X, Z))] \\
&= c(X, Y, Z) + \mathcal{C}(b_u)(X, Y, Z) + E(X, Y, Z) + F(X, Y, Z)
\end{aligned} \tag{42}$$

Being $E(X, Y, Z) := s(X, B(Y, Z)) - s(Y, B(X, Z))$ and $F(X, Y, Z) := b_u(X, B(Y, Z)) - b_u(Y, B(X, Z))$. We will patiently compute each of the four terms. First we shall compute each of the sumands of $\mathcal{C}(b_u) = \mathcal{C}(Hess(u)) - \mathcal{C}(du \otimes du) - \frac{1}{2}\mathcal{C}(|grad(u)|_g^2 g)$. First we have

$$\begin{aligned}
&(\nabla_X Hess(u))(Y, Z) = X[g(\nabla_Y grad(u), Z)] - g(\nabla_Z grad(u), \nabla_X Y) - g(\nabla_Y grad(u), \nabla_X Z) \\
&= g(\nabla_X \nabla_Y grad(u), Z) + g(\nabla_Y grad(u), \nabla_X Z) - g(\nabla_Z grad(u), \nabla_X Y) - g(\nabla_Y grad(u), \nabla_X Z) \\
&= g(\nabla_X \nabla_Y grad(u), Z) - g(\nabla_Z grad(u), \nabla_X Y)
\end{aligned}$$

Again, interchanging the rolls of X and Y and substracting we have

$$\mathcal{C}(Hess(u))(X,Y,Z) = g(\nabla_X\nabla_Y grad(u) - \nabla_Y\nabla_Z grad(u), Z) - g(\nabla_Z grad(u), \nabla_X Y - \nabla_Y X)$$
$$= g(R(X,Y)grad(u)), Z) + g(\nabla_{[X,Y]}grad(u), Z) - g(\nabla_Z grad(u), [X,Y])$$
$$= g(R(X,Y)grad(u)), Z) + Hess(u)([X,Y],Z) - Hess(u)(Z,[X,Y]) = -R(X,Y,Z,grad(u))$$

For the second summand we have

$$(\nabla_X du\otimes du)(Y,Z) = X[du(Y)du(Z)] - du(\nabla_X Y)du(Z) - du(Y)du(\nabla_X Z)$$
$$= X[g(grad(u),Y)]du(Z) + du(Y)X[g(grad(u),Z)] - g(\nabla_X Y, grad(u))du(Z) - du(Y)g(\nabla_X Z, grad(u))$$
$$= [g(\nabla_X grad(u),Y) + \underline{g(grad(u),\nabla_X Y)}]du(Z) + du(Y)[g(\nabla_X grad(u),Z) + \underline{g(grad(u),\nabla_X Z)}]$$
$$-\underline{g(\nabla_X Y, grad(u))du(Z)} - \underline{du(Y)g(\nabla_X Z, grad(u))}$$
$$= Hess(u)(X,Y)du(Z) + du(Y)Hess(u)(X,Z)$$

So interchanging the rolls of X and Y and substracting

$$\mathcal{C}(du\otimes du)(X,Y,Z) = du(Y)Hess(u)(X,Z) - du(X)Hess(u)(Y,Z)$$

For the third summand we have

$$\frac{1}{2}(\nabla_X g(grad(u),grad(u))g)(Y,Z) = g(\nabla_X grad(u),grad(u))g(Y,Z) + \underline{\frac{1}{2}|grad(u)|_g^2 X[g(Y,Z)]}$$
$$-\underline{\frac{1}{2}|grad(u)|_g^2 g(\nabla_X Y,Z)} - \underline{\frac{1}{2}|grad(u)|_g^2 g(Y,\nabla_X Z)} = Hess(u)(X,grad(u))g(Y,Z)$$

And this implies that

$$\frac{1}{2}\mathcal{C}(|grad(u)|_g^2 g)(X,Y,Z) = Hess(u)(X,grad(u))g(Y,Z) - Hess(u)(Y,grad(u))g(X,Z)$$

Summing up the expressions of the three sumands we see that the first term is

$$\mathcal{C}(b_u)(X,Y,Z) = \mathcal{C}(Hess(u))(X,Y,Z) - \mathcal{C}(du\otimes du)(X,Y,Z) + \frac{1}{2}\mathcal{C}(|grad(u)|_g^2 g)(X,Y,Z)$$
$$= -r(X,Y,Z,grad(u)) + du(X)Hess(u)(Z,Y) - du(Y)Hess(u)(X,Z)$$
$$+Hess(u)(X,grad(u))g(Y,Z) - Hess(u)(Y,grad(u))g(X,Z)$$

This ends with the first term. Now we go for the term F

$$b_u(X,B(Y,Z)) = Hess(u)(X, du(Y)Z + du(Z)Y - g(Y,Z)grad(u))$$
$$-du(X)du\left(du(Y)Z + du(Z)Y - \underline{g(Y,Z)grad(u)}\right)$$
$$+\frac{1}{2}|grad(u)|_g^2 g(X, du(Y)Z + du(Z)Y - \underline{g(Y,Z)grad(u)})$$
$$= du(Y)Hess(u)(X,Z) + du(Z)Hess(X,Y) - g(Y,Z)Hess(u)(X,grad(u))$$
$$-2du(X)du(Y)du(Z) + \frac{1}{2}|grad(u)|_g^2\{du(Y)g(X,Z) + du(Z)g(X,Y) + g(Y,Z)du(X)\}$$

By the simmetry in X, Y of the summands of the last line, this yields that

$$F(X,Y,Z) := b_u(X,B(Y,Z)) - b_u(Y,B(X,Z)) = du(Y)Hess(u)(X,Z)$$
$$-du(X)Hess(u)(Y,Z) + g(X,Z)Hess(u)(Y,grad(u)) - g(Y,Z)Hess(u)(X,grad(u))$$

Note that $\mathcal{C}(b_u)(X,Y,Z) + F(X,Y,Z) = -r(X,Y,Z,grad(u))$ since a lot of cancellations occur. Now we compute the term E.

$$\begin{aligned} s(X,B(Y,Z)) &= s(X,du(Y)Z + du(Z)Y - g(Y,Z)grad(u)) \\ &= du(Y)s(X,Z) + du(Z)s(X,Y) - g(Y,Z)s(X,grad(u)) \end{aligned}$$

Using that $du(X) = g(grad(u),X)$ we see that

$$\begin{aligned} &E(X,Y,Z) := s(X,B(Y,Z)) - s(Y,B(X,Z)) = g(Y,grad(u))s(X,Z) - g(X,grad(u))s(Y,Z) \\ &-g(Y,Z)s(X,grad(u)) + g(X,Z)s(Y,grad(u)) = (s \owedge g)(X,Y,Z,grad(u)) \end{aligned}$$

Finally we use that $-r = -w - s \owedge g$, so coming back to (42) we see that

$$\begin{aligned} &c'(X,Y,Z) = c(X,Y,Z) + \mathcal{C}(b_u)(X,Y,Z) + E(X,Y,Z) + F(X,Y,Z) \\ &= c(X,Y,Z) - w(X,Y,Z,grad(u)) \end{aligned}$$

This at last proves the Proposition. □

Corollary 4. In a Riemannian manifold (M,g) of dimension 3 the Cotton tensors is conformally invariant. Note that this is not true in dimension ≥ 4 since in this case the Weyl tensor does not necessarily vanish.

4.4. Consequences of the Weyl-Schouten Theorem.

We shall see now how the Weyl-Schouten tensor applies to the Riemannian Manifold with constant sectional curvature. We need first a lemma.

Lemma 13. Let (M,g) a Riemannian manifold with constant sectional curvature $K \equiv k$. Then

$$r = k(g \owedge g) = -\frac{Scal}{(n-1)n}(g \owedge g)$$

Proof. The second equality follows since, for an orthonormal frame $\{E_i\}$, we have

$$-Scal = \sum_{i,j} r(E_i,E_j,E_i,E_j) = n(n-1)k.$$

Let us see the first equality. Put $r' := k(g \owedge g)$. We remind that

$$k = K = \frac{r(x,y,x,y)}{g(x,x)g(y,y) - g(x,y)^2} = \frac{r(x,y,x,y)}{(g \owedge g)(x,y,x,y)}$$

So we have that r and r' are two curvature tensors such that $r'(x,y,x,y) = r(x,y,x,y)$ for every $x,y \in TM$. We shall see that this implies $r = r'$. Indeed, we have that, being t any of r or r',

$$t(x+z,y,x+z,y) = t(x,y,x,y) + 2t(x,y,z,y) + t(z,y,z,y).$$

As we know that $r(x+z,y,x+z,y) = r'(x+z,y,x+z,y)$, it follows that

$$r(x,y,z,y) = r'(x,y,z,y). \tag{43}$$

With t as before we have

$$t(x,y+u,z,y+u) = t(x,y,z,y) + t(x,y,z,u) + t(x,u,z,y) + t(x,u,z,u).$$

so, by (43) this implies that $r(x,y,z,u)+r(x,u,z,y) = r'(x,y,z,u)+r'(x,u,z,y)$ which is equivalent to

$$r(x,y,z,u) - r'(x,y,z,u) = r'(x,u,z,y) - r(x,u,z,y) = r(y,z,x,u) - r'(y,z,x,u).$$

If we define $s(x,y,z,u) := r(x,y,z,u) - r'(x,y,z,u)$, then we just saw that $s(x,y,z,u) = s(y,z,x,u)$ so s is invariant under cyclic permutations of the first three entries. But then

$$3(r-r')(x,y,z,t) = 3s(x,y,z,t) = s(x,y,z,t) + s(y,z,x,t) + s(z,x,y,t) = 0$$

the last equality because r and r' satisfy the Bianchi identity. This proves that $r = r'$. □

Corollary 5. If (M,g) is a Riemannian manifold of constant sectional curvature $K \equiv k$ with $g \in C^3$ then (M,g) is locally conformally flat.

Proof. By the lema above, $r = k(g \owedge g) = w + s \owedge g$ and from the uniqueness of the descomposition we have that $w = 0$ and $s = kg$. Then $\nabla s = 0$, so the Cotton tensor vanishes and we can apply the Weyl-Schouten theorem. □

Example 2. Consider a Riemannian surface (M,g) and let $M' := (M \times I, g' = g \times ds^2)$. The product metric acts as follows. Every $x' \in T_{(p,s)}M'$ is uniquely descomposed as $x' = x + a(x)\frac{\partial}{\partial s}$ where $x = \pi_1(x')$ is the proyection onto T_pM and $a(x) = \pi_2(x')$ onto I. Then $g \times ds^2(x',y') = g(x,y) + a(x)a(y)$. We compute in a product chart (x_1, x_2, s). For simplicity, put $x_3 := s$. It is clear that in this chart the metric g' is, in matrix notation

$$\begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and note that g_{ij} do not depend on x_3. From this and the formula for the Christoffel symbols Γ^k_{ij} of g' we have

$$\Gamma^3_{ij} = \frac{1}{2}g^{3l}[\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}] = \frac{1}{2}[\partial_i g_{3j} + \partial_j g_{3i} - \partial_3 g_{ij}] = 0$$
$$\Gamma^k_{3j} = \frac{1}{2}g^{kl}[\partial_3 g_{lj} + \partial_j g_{l3} - \partial_l g_{3j}] = 0$$

and the Christoffel symbols of g and g' coincide if $i,j,k \in \{1,2\}$. Then, $\nabla'_{\partial_i}\partial_j = \nabla_{\partial_i}\partial_j$ if $j,k \in \{1,2\}$ and $\nabla'_{\partial_i}\partial_3 = \nabla'_{\partial_3}\partial_j = 0$ for all i,j.

This implies that the $(3,1)$ curvature tensors of g' and g, respectively R' and R, coincide when evaluated in the set $\{\partial_1, \partial_2\}$, and $R'(\partial_3, \partial_j)\partial_k = R'(\partial_i, \partial_j)\partial_3 = 0$ for every i,j,k.

The same occurs for r and r', i.e, $r' = r$ in the plane spanned by $\{\partial_1, \partial_2\}$ and $r' = 0$ in $L\{\partial_3\}$ and by tensoriality $r'(x',y',z',t') = r(x,y,z,t)$. This is commonly expressed by saying that $r' = \pi_1^* r$ is the pullback of r by $\pi_1 : M \times I \to M$. This notation allows us to make calculations easier. For instance, note that $g' = \pi_1^* g + \pi_2^* ds^2$.

On the other hand, as M is a surface, its sectional curvature K becomes a function and we have $r = K(g \owedge g)$. Note too that $(\pi_2^* ds^2 \owedge \pi_2^* ds^2)(x,y,z,t) = 2a(x)a(z)a(y)a(t) - 2a(x)a(t)a(y)a(z) = 0$.

Combining these facts we have

$$r' = \pi_1^* r = \pi_1^* K(g \owedge g) = (\pi_1^* K)(\pi_1^* g - \pi_2^* ds^2) \owedge (\pi_1^* g + \pi_2^* ds^2) = (\pi_1^* K)(\pi_1^* g - \pi_2^* ds^2) \owedge g'$$

so the Weyl and Schouten tensors are $w' = 0$ (as we know since $dim(M') = 3$) and $s' = (\pi_1^* K)(\pi_1^* g - \pi_2^* ds^2)$. Let us compute the Cotton tensor.

$$(\nabla'_{\partial_i}\pi_1^* g)(\partial_j, \partial_k) = \partial_i[g(\pi_1(\partial_j), \pi_1(\partial_k))] - g(\pi_1(\nabla'_{\partial_i}\partial_j), \pi_1(\partial_k)) - g(\pi_1(\partial_j), \pi_1(\nabla'_{\partial_i}\partial_k)) = 0$$

because if $i, j, k \in \{1, 2\}$ then $(\nabla'_{\partial_i}\pi_1^* g) = (\nabla_{\partial_i} g) = 0$, and the case when any of i, j, k is 3 is obvious. We have too that

$$(\nabla'_{\partial_i}\pi_2^* ds^2)(\partial_j, \partial_k) = \partial_i[ds^2(\pi_2(\partial_j), \pi_2(\partial_k))] - ds^2(\pi_2(\nabla'_{\partial_i}\partial_j), \pi_2(\partial_k)) - ds^2(\pi_2(\partial_j), \pi_2(\nabla'_{\partial_i}\partial_k)) = 0$$

so by tensoriality it follows that $\nabla\pi_1^* g = \nabla\pi_2^* ds^2 = 0$.

On the other hand, as $\pi_1^* K$ does not depend on x_3, for any vector field $X' = X + a(X')\partial_3 \in TM'$ we have $X'(\pi_1^* K) = dK(X)$. We finally conclude that

$$\nabla'_{X'} s' = dK(X)(\pi_1^* g - \pi_2^* ds^2)$$

and the Cotton tensor is given by

$$c(\partial_i, \partial_j, \partial_k) = pi_1^* dK(\partial_i)(\pi_1^* g(\partial_j, \partial_k) - \pi_2^* ds^2(\partial_j, \partial_k)) - pi_1^* dK(\partial_j)(\pi_1^* g(\partial_i, \partial_k) - \pi_2^* ds^2(\partial_i, \partial_k))$$

in particular for $j \in \{1, 2\}$

$$c(\partial_3, \partial_j, \partial_3) = dK(\partial_j)$$

and from this it is obvious than $C \equiv 0$ if and only if K is constant, i.e, $(M \times I, g \times ds^2)$ is locally conformally flat if and only if M has constant sectional curvature. Thus we can construct a lot of three manifolds which are not locally conformally flat by taking a surface with not constant sectional curvature and making the product with an interval.

§5. Analytic Tool-Box.

Now we adress the cuestion of characterize when a Riemannian manifold (M, g) is locally conformally flat when g is not regular enough to directly apply the Weyl-Schouten theorem, i.e, when the metric is less regular than C^3.

We could make sence of the Weyl tensor w for C^1 metrics for example by considering distributional derivatives of the metric and then we can wonder if $w = 0$ implies that (M, g) is locally conformally flat.

Our strategy here will be to find suitable coordinates around any point in such a way that, when expressed in these coordinates, the equation $w(g) = 0$ of the Weyl tensor being zero as a distribution, viewed as a PDE which g satisfies, is elliptic. This will allow us to apply regularity results and conclude that in these coordinates the metric is in fact more regular than expected. Then we can apply the previous results about local conformal flatness for C^3 metrics.With this aim at mind we shall use n-harmonic coordinates, which are coordinates that satisfy a Laplacian type equation. These coordinates are optimal when dealing with regularity results, as we shall see.

First of all we collect some results of elliptic regularity. These results will provide us some of the technical machinery we will need.

5.1. Differential Operators, Holder Spaces and Elliptic Regularity

From now on we will use the *multiindex notation* to express iterated partial derivatives of functions, so for a multiindex $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we denote $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. Also we denote $D_j := -i\partial_j$ and put $D^\alpha := D_1^{\alpha_1} \dots D_n^{\alpha_n}$. Sometimes, when we are only interested in the ordr of differentiation, we will write ∂^l and D^l to mean respectively ∂^α and D^α for some arbitrary multiindex $\alpha \in \mathbb{N}^n$ with $|\alpha| = l$.

Definition 19. Let $\Omega \subset \mathbb{R}^n$ be an open set. Consider an operator L of the form

$$p(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha \tag{44}$$

acting on functions $u \in C^\infty(\Omega)$ as follows

$$p(x,\partial)u(x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u(x) \tag{45}$$

where a_α are functions on Ω not necessarily regular. Suppose that there exists α with $|\alpha| = m$ such that a_α is not the zero function. Then we say that L is a *differential operator of order* m .

The principal part of $p(x,\partial)$, which we shall denote $p_m(x,\partial)$, is defined to be its higher order terms, i.e,

$$p_m(x,\partial) = \sum_{|\alpha|=m} a_\alpha(x)\partial^\alpha$$

When dealing with differential operators the letter α will usually refer to a multiindex ,i.e, an element of $\mathbb{N}^n$.

Definition 20. Let $p(x,\partial)$ be a differential operator of order m. For $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ denote

$$\xi^\alpha = \xi_1^{\alpha_1}\dots\xi_n^{\alpha_n}$$

With this notation, we define the symbol of $p(x,\partial)$ as

$$p(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \tag{46}$$

and the principal symbol as

$$p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \tag{47}$$

Furthermore, we say that $p(x,\partial)$ is *elliptic* if for every $\xi \neq 0$ the principal symbol $p_m(x,\xi)$ never vanishes as a function on $x \in \Omega$.

Finally, we say that $p(x,\partial)$ is *uniformly elliptic* if $|p_m(x,\xi)| \ge C|\xi|^m$ for some $C > 0$.

Recall that if $p(x,\xi)$ is elliptic in $\overline{\Omega} \times \mathbb{R}^n$, with $\overline{\Omega}$ compact, then we can take the constant as $C = \sup\{|p(x,\xi)| : x \in \overline{\Omega}, |\xi| = 1\} > 0$, so $p(x,\xi)$ is actually uniformly elliptic in $\overline{\Omega} \times \mathbb{R}^n$.

The typical example of uniformly elliptic operator of order 2 is the usual Laplacian on $\mathbb{R}^2$, $p(x,\partial) = \Delta$ whose principal symbol is $p_2(x,\partial) = |\xi|^2$.

We define now a class of functions well behaved for elliptic regularity.

Definition 21. Let $0 \le \alpha \le 1$, $k,m,n \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^n$ an open set. We define the holder space $C^{k,\alpha}(\Omega)$, or simply $C^{k+\alpha}(\Omega)$, as the functions $u = (u_1,\dots,u_m) : \Omega \to \mathbb{R}^m$ with continuous partial derivatives until order k such that the following two conditions hold:

$$\|u\|_{C^k(\Omega)} := \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{L^\infty(\Omega)} < \infty$$

$$[u]_{C^\alpha(\Omega)} := \sum_{|\alpha|=k} \sup_{x,y\in\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\alpha} < \infty$$

and we define $\|u\|_{C^{k,\alpha}(\Omega)} := \|u\|_{C^k(\Omega)} + [u]_{C^\alpha(\Omega)}$. From the definition it is straighforward that $u \in C^{k,\alpha}(\Omega)$ if and only if all its components $u_i \in C^{k,\alpha}(\Omega)$.

Some remarks about notation. When $\Omega = \mathbb{R}^n$ we may denote $C^{k+\alpha}$ for the Holder spaces $C^{k,\alpha}(\Omega)$. Also, if $s \in \mathbb{R}$, $s \ge 0$, we denote C^s for the Holder space $C^{k+\delta}$ where $s = k+\delta$ for $k \in \mathbb{N}$, $0 \le \delta < 1$. Note that when $s = k \in \mathbb{N}$ there could be confussion between functions with k continuous derivatives (not necessarily bounded), and functions in the Holder space C^k, because both sets are denoted by the symbol C^k. That's why we will try to write C^{k+0} for the holder space, though we will probably forget, and hope the context will clarify the ambiguity. Also we may denote C^{k+1} for the Holder space $C^{k,1}$. Finally, if $k = \infty$ we denote $C^{\infty+0}$ the space of smooth functions with derivatives bounded in $\mathbb{R}^n$. Note that we allow the bound to become arbitrarily large as the order of the derivative increases, so in this space we do not have the norm $\|\cdot\|_{C^{\infty+0}}$.

Now we estate the main theorem we will need.

Theorem 7. (Elliptic regularity in Holder spaces) Let L be an elliptic differential operator of order m with coefficients $a_\alpha(x)$ defined is some open set $\Omega \subset \mathbb{R}^n$. Let $l \geq 0$ be an integer, and let $0 < \delta < 1$. Assume that $u \in W^{m,p}(\Omega)$ satisfies the equation $Lu(x) = f(x)$ for a.e. $x \in \Omega$, with $f, a_\alpha \in C^{l,\delta}(\Omega)$. Then in fact $u \in C^{l+m,\delta}(\Omega)$.

The statement of this Theorem can be found in [5], Appendix J. The proof is spread over various references cited there. The case in which m is even (which covers the cases we need here) is done in [16], Appendix 5. Such proof makes use of the fact that every linear elliptic equation with *constant* coeficcients a_α admits a fundamental solution $K(y)$ which is regular outside the origin $0 \in \mathbb{R}^n$ and has a pole in 0 of certain order. This fact is well known for the Laplacian, and the construction of the fundamental solution for the general case can be found in [17], Chapther III.

The strategy to deal with variable coefficients $a_\alpha(x)$ is to freeze these coefficients at one fixed point x_0, so one has a fundamental solution $K_{x_0}(y) = K(x_0, y)$. Besides, one can check from the explicit formula of $K(x_0, y)$ that it depends on x_0 with the same regularity than $a_\alpha(x_0)$. Finally, some manipulations allow to write our solution $u(x) \in W^{m,p}(\Omega)$ as a convolution of $K(x, y)$ with the function $f(x) = Lu(x)$, plus lower order terms that can be controlled. From this expression of u and from the $C^{l,\delta}$ dependence of $K(x, y)$ in the x-variable, we can deduce, via Sobolev Embeding Theorems, the regularity desired for u.

This theorem shows that things work properly when treating elliptic equations in Holder spaces with Holder exponent $\delta > 0$, i.e, solutions gain all the regularity that naturally allow the coeficcients a_α and f. This nice behaviour of elliptic equations is no longer true if we work with continuous functions (i.e, with Holder exponent $\delta = 0$), as the following example shows.

Example 3. In this example we show that

$$u(x, y) = (x^2 - y^2)(-log|(x, y)|)^{\frac{1}{2}}$$

is a function such that $u \in W^{2,p}(B_1)$ for all $1 \leq p < \infty$, Δu is continuous but $u \notin C^2(B_1)$. First note that u is defined in some neighborhood of $(0, 0)$, for example B_1, after setting $u(0, 0) = 0$. A tedious computation shows that

$$u_x(x, y) = 2x(-log|(x, y)|)^{\frac{1}{2}} + (-log|(x, y)|)^{-\frac{1}{2}} \frac{-x(x^2 - y^2)}{2(x^2 + y^2)}$$

$$u_{xx}(x, y) = 2(-log|(x, y)|)^{\frac{1}{2}} - \frac{2x^2}{x^2 + y^2}(-log|(x, y)|)^{-\frac{1}{2}}$$

$$-\frac{x^2(x^2 - y^2)}{4(x^2 + y^2)^2}(-log|(x, y)|)^{-\frac{3}{2}} + \frac{1}{2}\frac{(x^2 - y^2)^2}{(x^2 + y^2)^2}(-log|(x, y)|)^{-\frac{1}{2}}$$

$$u_{xy}(x, y) = \frac{xy(x^2 - y^2)}{4(x^2 + y^2)^2}[(-log|(x, y)|)^{-\frac{1}{2}} - 4(-log|(x, y)|)^{\frac{1}{2}}]$$

Now, using $2ab \leq a^2 + b^2$ and triangle inequality, we see that the terms $\frac{2x^2}{x^2+y^2}$, $\frac{x^2|x^2-y^2|}{(x^2+y^2)^2} = \frac{x^2}{x^2+y^2}\frac{|x^2-y^2|}{x^2+y^2}$, $\frac{xy(x^2-y^2)}{4(x^2+y^2)^2}$ and $\frac{(x^2-y^2)^2}{(x^2+y^2)^2}$ are bounded as $(x, y) \to (0, 0)$, so they can be substituted by constants near

$(0,0)$ to compute that both $u_{xy}(x,y)$ and $u_{xx}(x,y)$ tend to ∞ as $(x,y) \to (0,0)$. Nevertheless the function u has symmetries which, when computing its laplacian, cancel out the unbounded terms.

Indeed, $u(x,y) = -u(y,x)$, so the first and second variables play an antisymmetric role. Hence, $(\partial_1 u)(x,y) = -(\partial_2 u)(y,x)$ and $(\partial_{11} u)(x,y) = -(\partial_{22} u)(y,x)$. Then we have

$$\Delta u(x,y) = u_{xx}(x,y) - u_{xx}(y,x) = 2\frac{y^2-x^2}{x^2+y^2}(-log|(x,y)|)^{-\frac{1}{2}} + \frac{y^4-x^4}{4(x^2+y^2)^2}(-log|(x,y)|)^{-\frac{3}{2}}$$

$$= \frac{y^2-x^2}{4(x^2+y^2)}[8(-log|(x,y)|)^{-\frac{1}{2}} + (-log|(x,y)|)^{-\frac{3}{2}}]$$

so $\Delta u(x,y) \to 0$ if $(x,y) \to (0,0)$, and this shows that, defining $f(x,y) := \Delta u(x,y)$ for $(x,y) \neq (0,0)$ and $f(0,0) := 0$, then $\Delta u = f$ almost everywhere, with f continuous, but $u \notin C^2(B_1)$.

However we claim that $u \in W^{2,p}$ for each $1 \leq p < \infty$. To see this, note that u and the first derivatives of u are continuous since in polar coordinates (r,θ) they are controlled near $(0,0)$ by $r(-log(r))^{\frac{1}{2}}$ which tends to 0 as $r \to 0$. For the second derivatives of u, in polar coordinates we see that, near $(0,0)$, they are controlled by $(-log(r))^{\frac{1}{2}}$ which is the typical example of a function in $L^p(0,1)$ for every $1 \leq p < \infty$ but not in $L^\infty(0,1)$.

Note then that theorem 7 tell us that this can only happen because $f := \Delta u$ is not $C^\alpha(B_1)$ for any $\alpha > 0$, i.e, near $(0,0)$ f oscillates too quickly.

5.2. Fourier Analysis, Pseudodifferential Operators and Elliptic Regularity

We will need for some purposes sharper elliptic regularity results, stated in other spaces of functions. For this we make a quick review of the basic results from Fourier analysis and distributions that we will need.

Definition 22. Let $\Omega \subset \mathbb{R}^n$. We will denote $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ for the smooth functions with compact support in Ω equipped with the topology of test functions. We say that a secuence $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$ if there exists a compact set $K \subset \Omega$ such that $supp(\varphi_n) \subset K$ for all $n \in \mathbb{N}$, and for each multiindex $\alpha \in \mathbb{N}^n$ we have $\|\partial^\alpha \varphi_n - \partial^\alpha \varphi\|_{L^\infty(\Omega)} \to 0$ as $n \to \infty$. This convergence comes from a certain topology, but this topology is neither straightforward nor very useful, so we will stick to the notion of convergence. Besides it can be proved that in this topology a function is continuous if and only if it is sequentially continuous, and this is all we need.

We say that a linear functional $u : \mathcal{D}(\Omega) \to \mathbb{C}$ is a *distribution* if it is continuous with respect to this notion of convergence. We will denote $\mathcal{D}'(\Omega)$ for the space of distributions in Ω. It can be seen that $u \in \mathcal{D}'(\Omega)$ if and only if for each compect $K \subset \Omega$ there are a constant C_K and a natural number n_K such that $|u(\varphi)| \leq C_K \sum_{|\alpha| \leq n_K} \|\partial^\alpha \varphi\|_{L^\infty}$ for every $\varphi \in \mathcal{D}(\Omega)$ with support in K.

Remark 17. Every function f in $L^1_{loc}(\mathbb{R}^n)$ can be identified with the distribution given by $(I_f, \varphi) := \int f\varphi dx$. We mean by this that the map $f \to I_f$ is inyective. Indeed, if the set $E := \{f > 0\}$ has positive measure, we consider the convolutions $\phi_\varepsilon := \rho_\varepsilon \star \chi_E$, being $\rho_\varepsilon \subset C_c^\infty(\mathbb{R}^n)$ a standard aproximattion of the dirac δ. As $(I_f, \chi_E) > 0$ it is easy to prove that for $\varepsilon > 0$ small we have also $(I_f, \phi_\varepsilon) > 0$, and we are done.

Definition 23. Given $U \subset \mathbb{R}^n$ two open sets with $U \subset \mathbb{R}^n$, note that $\mathcal{D}(U) \subset \mathcal{D}(\mathbb{R}^n)$. Then, given $u \in \mathcal{D}'(\mathbb{R}^n)$ we define the restriction of u to U as $u|_U := u|_{\mathcal{D}(U)}$.

Remark 18.
(1) Obviously if ϕ is a function defined in $\mathbb{R}^n$, we have that $I_\phi|_U = I_{\phi|_U}$.
(2) Also we have $(D^\alpha u)|_U = D^\alpha [u|_U]$.

Definition 24. A function $\varphi \in C^\infty(\mathbb{R}^n)$ is a Schwartz function, if for every multiindex $\beta \in \mathbb{N}^n$ we have that $\partial^\beta \varphi(x)$ is a *rapidly decreasing function* , which means that for every multiindex $\alpha \in \mathbb{N}^n$

$$[\varphi]_{\alpha,\beta} := sup\{|x^\alpha \partial^\beta \varphi(x) : x \in \mathbb{R}^n\} < \infty.$$

We denote the space of Schwartz functions as $\mathcal{S}(\mathbb{R}^n)$, or simply $\mathcal{S}$.

In the space $\mathcal{S}$ we define convergence as follows. A sequence $\varphi_j \to \varphi$ in $\mathcal{S}$ if for every α, $\beta \in \mathbb{N}^n$ we have that $[\varphi_j - \varphi]_{\alpha,\beta} \to 0$ as $j \to \infty$. It can be proved that this topology is metrizable, and in fact it is given by the metric $d : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$ such that

$$d(\varphi, \phi) := \sum_{\alpha,\beta \in \mathbb{N}^n} 2^{-|\alpha|-|\beta|} \frac{[\varphi - \phi]_{\alpha,\beta}}{1 + [\varphi - \phi]_{\alpha,\beta}}.$$

It is easy to prove that with this topology a linear form $u : \mathcal{S} \to \mathbb{C}$ is continuous if and only if there are constants C and N such that for every $\varphi \in \mathcal{S}$ we have

$$|(u, \varphi)| \leq C \sum_{|\alpha|+|\beta| \leq N} [\varphi]_{\alpha,\beta}.$$

If a linear form u satisfies the estimate above, then $u \in \mathcal{S}'$ and we say that u is a *tempered distribution* .

Remark 19. The fourier transform satisfies $\mathcal{F} : \mathcal{S} \to \mathcal{S}$. Then, its transpose map $\mathcal{F}' : \mathcal{S}' \to \mathcal{S}'$ acts on $\mathcal{S}'$ by $(\mathcal{F}'(u), \varphi) := (u, \mathcal{F}(\varphi))$ for all $u \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$. As the fourier inversion formula holds in $\mathcal{S}$, by this definition it also holds on $\mathcal{S}'$. However, this abstract definition of $\mathcal{F}'$ has more meaning than one would expect. As we know, $\mathcal{S}$ can be regarded as a subset of $\mathcal{S}'$ via the embeeding

$$\iota : \mathcal{S} \to \mathcal{S}' : \varphi \mapsto I_\varphi$$
$$(I_\varphi, \phi) := \int \varphi \phi dx$$

Then, one then might wonder if $\mathcal{F}'(I_\varphi) = I_{\mathcal{F}[\varphi]}$, and this is indeed true by Plancharel theorem: for each $\phi \in \mathcal{S}$ we have

$$(\mathcal{F}'(I_\varphi), \phi) = \int \varphi \hat{\phi} dx = \int \hat{\varphi} \phi dx = (I_{\mathcal{F}[\varphi]}, \phi).$$

Therefore $\mathcal{F}' : \mathcal{S}' \to \mathcal{S}'$ is actually an extension of $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ to all $\mathcal{S}'$. This is why $\mathcal{F}'$ is written as $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$.

This allows us to define the Fourier transform on a class of functions much larger than $\mathcal{S}$. For example, the polinomials are embeded in $\mathcal{S}'$ via the same mapping than embeds $\mathcal{S}$, so although polinomials are far from being integrable, we can extend the classical definition of Fourier transform to polinomials, and more generally to functions that growth at infinity like a polinomial.

In general given a map $T : \mathcal{S} \to \mathcal{S}$ we can always define $T' : \mathcal{S}' \to \mathcal{S}'$ by putting $(T'u, \varphi) := (u, T\varphi)$. But if we have the additional fact that T' extends T in the sense that $T' = T$ when restricted to $\mathcal{S}$, then we can directly write $T : \mathcal{S}' \to \mathcal{S}'$ as an extension of T.

For example if p is a polinomial or a function with the same asymptotic growth, we can take $T(\varphi) := p\varphi$, the multiplication operator by p, and then we have $T' = T : \mathcal{S}' \to \mathcal{S}'$, since T' obviously extends T to all $\mathcal{S}'$.

Another important example follows by considering $T(\varphi) = D^\alpha \phi$ for some $\alpha \in \mathbb{N}^n$. Integrating by parts we see that $(-1)^{|\alpha|} T' = T : \mathcal{S}' \to \mathcal{S}'$ extends T to $\mathcal{S}'$. Therefore we can differentiate tempered distributions. It is straighforward to check that all the classic formulas relating derivation and Fourier transforms are true in the space of tempered distributions. It is enough to note that they are true in $\mathcal{S}$ and we have defined this operators in $\mathcal{S}'$ via his actions on $\mathcal{S}$.

Recall also that a similar discussion can be done substituting the space $\mathcal{S}(\mathbb{R}^n)$ by $\mathcal{D}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ an open set and $\mathcal{S}'(\mathbb{R}^n)$ by $\mathcal{D}'(\Omega)$, so we can differentiate distributions and multiply distributions by smooth functions. Therefore given a differential operator $p(x, D) = \sum a_\alpha(x) D^\alpha$, we can make sense of $p(x, D)u$ if $u \in \mathcal{D}'(\Omega)$ and the $a_\alpha(x)$ are smooth. However in general we cannot define the Fourier transform of a distribution even when we impose the obvious technichal assumption $\Omega = \mathbb{R}^n$, since $\mathcal{F} : \mathcal{D}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}(\mathcal{D}(\mathbb{R}^n))$ is not contained in $\mathcal{D}(\mathbb{R}^n)$.

Definition 25. A function $a \in C^\infty(\mathbb{R}^n)$ is said to be slowly increasing if for all multiindex $\alpha \in \mathbb{N}^n$ there exists $k_\alpha \in \mathbb{N}$ and $C_\alpha > 0$ such that

$$|\partial^\alpha a(x)| \leq C_\alpha (1 + |x|)^{k_\alpha}.$$

The vectorial space of slowly incresing functions is denoted by $\mathcal{O}(\mathbb{R}^n)$. The typical examples are polinomials.

If $a \in \mathcal{O}(\mathbb{R}^n)$, and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we define the operator $a(D) : \mathcal{S} \to \mathcal{S}$ as

$$a(D)\varphi(x) := \mathcal{F}^{-1}[a(\xi)\mathcal{F}(\varphi)(\xi)](x).$$

Note that when $a(\xi) = \xi^\alpha$ we have the well known result $\mathcal{F}[D^\alpha u](\xi) = \xi^\alpha \mathcal{F}(u)(\xi)$ so this definition generalizes this fact.

Also note that $a(D)' = a(D) : \mathcal{S}' \to \mathcal{S}'$, i.e, $a(D)$ can be extended to $\mathcal{S}'$ by duality. To see this, given $\varphi, \phi \in \mathcal{S}$ we compute

$$\begin{aligned}
&(a(D)' I_\varphi, \phi) = (I_\varphi, a(D)\phi) = \int \varphi(x) \mathcal{F}^{-1}[a(\xi)\mathcal{F}(\phi)(\xi)](x) dx \\
&= \int a(\xi) \mathcal{F}^{-1}[\varphi](\xi) \mathcal{F}[\phi](\xi) d\xi = \int \phi(x) \mathcal{F}[a(\xi)\mathcal{F}^{-1}[\varphi](\xi)](x) dx \\
&= \int \phi(x) \mathcal{F}[[\mathcal{F} \circ \mathcal{F}^{-1}][a(\xi)\mathcal{F}^{-1}[\mathcal{F}^{-1} \circ \mathcal{F}\varphi](\xi)](x) dx \\
&= \int \phi(x)(-[\mathcal{F}^{-1}][a(\xi)[-\mathcal{F}\varphi](\xi)](x)) dx = (I_{a(D)\varphi}, \phi)
\end{aligned}$$

Notation is awful, but we used that the Fourier inversion formula holds for the functions $a(\xi)\mathcal{F}^{-1}[\varphi](\xi) \in \mathcal{S}$, $\varphi(x) \in \mathcal{S}$, and the fact that $\mathcal{F} \circ \mathcal{F} = -Id = \mathcal{F}^{-1} \circ \mathcal{F}^{-1}$ in $\mathcal{S}$.

Now we introduce more spaces of functions, the Zigmund Spaces C^s_*. These spaces are an slighter generalization of the Holder spaces for positive exponent s. However they are also defined for negative exponent. It is more convenient to work with them rather than Holder spaces because they have nicer properties respect to the Fourier transform. For example, we will prove that the distributional derivative of a C^s_* function lies in the space C^{s-1}_* for every $s \in \mathbb{R}$, so Zigmund spaces accomodate well to the world of distributions, while Holder spaces do not.

Definition 26. Take any family of functions $\varphi_j \in C_c(\mathbb{R}^n)$ for $j \geq 0$ such that:

(1)φ_0 is a radial function, $Supp(\varphi_0) \subset \{|\xi| \leq 1\}$ and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ for $j \geq 1$. Note that, in particular, $Supp(\varphi_j) \subset \{2^{j-2} \leq \xi \leq 2^j\}$.

(2) For all $|\xi| \in \mathbb{R}^n$ we have $\sum_{j \geq 0} \varphi_j(\xi)^2 = 1$. Any family $\{\varphi_j\}$ which satisfies these properties is called a *Littlewood-Paley partition of unity* . The existence of such a family of functions is classical and easy to prove.

Definition 27. For $s \in \mathbb{R}$ we define the Zigmund spaces $C_*^s(\mathbb{R}^n)$, or simply C_*^s, as follows. Consider as in 26 a Littlewood-Pailey partition of unity φ_j, $j \geq 0$. A tempered distribution $u \in \mathcal{S}'$ is said to be in the Zigmund space C_*^s if we have that

$$\|u\|_{C_*^s(\mathbb{R}^n)} := \sup_{k \in \mathbb{N}} 2^{ks} \|\varphi_k(D)u\|_{L^\infty} < \infty$$

One can check that $\|\cdot\|_{C_*^s}$ is indeed a norm.

One can also define these spaces for functions $u = (u_1, \ldots, u_m) : \mathbb{R}^n \to \mathbb{R}^m$. Note that in this case we have, integrating on each component, that $\varphi_k(D)u = (\varphi_k(D)u_1, \ldots, \varphi_k(D)u_m)$ is an $\mathbb{R}^m$ valued function, so the definition is analogous.

Remark 20. (1) By the definition of the norm $\|\cdot\|_{C_*^r}$ in 27, it follows that for $s < r$ we have $C_*^r \subset C_*^s$ with continuous inclusion.

(2) As we shall see later, for $s > 0$, (and only in this case), these spaces can be defined also in terms of difference quotients in a similar way than Holder spaces.

(3) The definition we give above in 27 coincides with the definition of the Besov spaces $B_{\infty,\infty}^s$, which are a particular case of a two parameter scale of function spaces $B_{p.q}^s$ for $p, q \geq 1$ and $s \in \mathbb{R}$. See [19] for more on these spaces.

We have a nice characterization of the Zigmund spaces for $s > 0$, which we state below.

Theorem 8. Let $s > 0$, $s \notin \mathbb{Z}$, and let $u \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution. Descompose $s = k + \delta$ with $k \in \mathbb{N}$ and $0 < \delta < 1$. Then $u \in C_*^s(\mathbb{R}^n)$ if and only if $u \in C^s(\mathbb{R}^n) = C^{k,\delta}(\mathbb{R}^n)$. Moreover the norms $\|\cdot\|_{C_*^s}$ and $\|\cdot\|_{C^s}$ are equivalent.

Proof. A very straightforward and readable proof of this can be found in the lecture notes [22], page 55. □

When $s \in \mathbb{N}$, the Zigmund spaces are larger than Holder spaces, but still they can be characterized in terms of (more elaborated) difference quotients.

Theorem 9. Let $s = k \in \mathbb{N}$. Then $C^{k,0}(\mathbb{R}^n) \subset C_*^k(\mathbb{R}^n)$. Moreover, for a tempered distribution $u \in S'(\mathbb{R}^n)$ we have that $u \in C_*^k(\mathbb{R}^n)$ if and only if $u \in C^{k-1,0}(\mathbb{R}^n)$ and

$$[u]_{C_*^k(\mathbb{R}^n)} := \sum_{|\alpha|=k-1} sup \left\{ \frac{|\partial^\alpha u(x+h)| - 2\partial^\alpha u(x) + \partial^\alpha u(x-h)}{|h|} : x, h \in \mathbb{R}^n, h \neq 0 \right\} < \infty$$

Moreover, the norm $\|\cdot\|_{C_*^k(\mathbb{R}^n)}$ is equivalent to the norm $\|\cdot\|'_{C_*^k(\mathbb{R}^n)} := \|\cdot\|_{C^{k-1,0}(\mathbb{R}^n)} + [\cdot]_{C_*^k(\mathbb{R}^n)}$

Proof. Again this result can be found in [22], page 57. □

Remark 21. In virtue of these characterizations for the Zigmund spaces $C_*^s(\mathbb{R}^n)$ for $s > 0$, we will think of these spaces just as Holder spaces when $s > 0$ is not integer. Also if $1 \leq s = k \in \mathbb{N}$ we will think of these spaces equipped with the norm $\|\cdot\|'_{C_*^k(\mathbb{R}^n)}$ defined in Theorem 9 above.

Corollary 6. If $k \in \mathbb{N}$, the inclusions $C^{k,0}(\mathbb{R}^n) \subset C^{k-1,1}(\mathbb{R}^n) \subset C_*^k(\mathbb{R}^n)$ are continuous, i.e, there exist constants C_1, C_2 such that $\|\cdot\|_{C_*^k(\mathbb{R}^n)} \leq C_1 \|\cdot\|_{C^{k-1,1}(\mathbb{R}^n)} \leq C_2 \|\cdot\|_{C^{k,0}(\mathbb{R}^n)}$.

Proof. This is straighforward from Theorem 9. Indeed, let $u \in C^{k,0}(\mathbb{R}^n)$. Let $\alpha \in \mathbb{N}^n$ with $|\alpha| = k-1$. By the mean value theorem, for any $h, x \in \mathbb{R}^n$ we have

$$|\partial^\alpha u(x+h) - \partial^\alpha u(x)| \leq |h| C(\sum_{|\beta|=k} \|\partial^\beta u\|_{L^\infty(\mathbb{R}^n)})$$

and taking supremum on h this yields that $\|u\|_{C^k(\mathbb{R}^n)} \geq C_1\|u\|_{C^{k-1,1}(\mathbb{R}^n)}$. For the second inclusion, note that by the triangle inequality we have $[u]_{C^k_*(\mathbb{R}^n)} \leq 2[u]_{C^{k-1,1}(\mathbb{R}^n)}$, and therefore $\|\cdot\|'_{C^k_*(\mathbb{R}^n)} \leq C\|\cdot\|_{C^{k,0}(\mathbb{R}^n)}$. Note that $\|\cdot\|'_{C^k_*(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{C^k_*(\mathbb{R}^n)}$, and this gives the claim. □

The inclusions from Corollary 6 above are strict. First let us see the intuitive reason why it is logical to expect that the space $C^1_*(\mathbb{R}^n)$ is bigger than the space $C^{0,1}(\mathbb{R}^n)$ of Lipschitz continuous functions. Suppose $f \in C^1(\mathbb{R})$. Then by Taylor's theorem

$$f(x+h) = f(x) + hf'(x) + o(h) \quad ; \quad f(x-h) = f(x) - hf'(x) + o(h) \quad \text{as } h \to 0$$

If we sum, we see that $f(x+h) - 2f(x) + f(x+h) = o(h)$ as $h \to 0$, while we only have $f(x+h) - f(x) = O(h)$ as $h \to 0$. So if h is small the difference quotient considered to define C^1_* is smaller than the one considered to define $C^{0,1}$. For h large we estimate both difference quotients by $2\|f\|_{L^\infty}$.

We conclude from this that, at least for C^1 functions, the difference quotient of $C^1_*(\mathbb{R}^n)$ is much smaller as $h \to 0$ than the one of $C^{0,1}(\mathbb{R}^n)$, so it is reasonable that the first difference quotients are bounded for a larger class of functions. In fact, it can be proved that the function $f(x) = \sum_{k\geq 0} 2^{-k}e^{i2^k x}$ is not Lipschitz continuous, but $f \in C^1_*(\mathbb{R})$.

The characterization given in Theorems 8 and 9 of the Zigmund spaces for positive exponent s allow us to define them in an arbitrary open set, this time in terms of difference quotients (note that the classical Fourier transform does not make sense for functions whose domain is an open set, so the definition given in $\mathbb{R}^n$ in terms of the Fourier transform cannot be generalized here in a straightforward manner).

Definition 28. Let $\Omega \subset \mathbb{R}^n$ be open. Let $k \in \mathbb{N}$ and $0 < \delta < 1$ and put $s = k + \delta$. We define the Zigmund space $C^k_*(\Omega)$ as the functions $u \in C^{k-1,0}(\Omega)$ such that

$$[u]_{C^k_*(\Omega)} := \sum_{|\alpha|=k-1} sup \left\{ \frac{|\partial^\alpha u(x+h)| - 2\partial^\alpha u(x) + \partial^\alpha u(x-h)}{|h|} : x, x+h, x-h \in \Omega \right\} < \infty$$

and the space $C^k_*(\Omega)$ is equipped with the norm $\|u\|_{C^k_*(\Omega)} := \|u\|_{C^{k-1,0}(\Omega)} + [u]_{C^k_*(\Omega)}$.

We define the Zigmund space $C^s_*(\Omega)$ as the Holder space $C^s(\Omega) = C^{k,\delta}(\Omega)$.

We need to define also the Zigmund spaces in open sets for $s \leq 0$. In fact, the following definition makes sense for all $s \in \mathbb{R}$ and generalizes the definition given above for positive exponent.

Definition 29. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $s \in \mathbb{R}$. We define the Zigmund spaces $C^s_*(\Omega)$ as the distributions $u \in \mathcal{D}'(\Omega)$ such that there exists an $v \in C^s_*(\mathbb{R}^n)$ with $v|_\Omega = u$ as distributions. We define the norm

$$\|u\|_{C^s_*(\Omega)} := \inf\{\|v\|_{C^s_*(\mathbb{R}^n)} : v \in C^s_*(\mathbb{R}^n), v|_\Omega = u\}$$

Remark 22. We have defined twice the Zigmund spaces $C^s_*(\Omega)$ for $s > 0$. It can be proved that both definitions 28 and 29 are equivalent for $s > 0$ (i.e, both norms are equivalent). This fact can be found in [19], Section 3, page 90.

The next Theorem asserts basically that all the results true for the Zigmund spaces $C^s_*(\mathbb{R}^n)$ are also true for the spaces $C^s_*(\Omega)$ if Ω has a regular boundary. The reason is that if $f \in C^s_*(\Omega)$ we can extend f to some function $Ef \in C^s_*(\mathbb{R}^n)$, and the norm $\|Ef\|_{C^s_*(\mathbb{R}^n)}$ is equivalent to the norm $\|f\|_{C^s_*(\Omega)}$.

Theorem 10. Let $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary. Then there exists a bounded and linear operator

$$E : C^s_*(\Omega) \to C^s_*(\mathbb{R}^n) : f \mapsto Ef$$

such that $Ef|_\Omega = f$ as distributions for all $f \in C^s_*(\Omega)$. We call E an extension operator. Recall that we have $\|f\|_{C^s_*(\Omega)} \leq \|Ef\|_{C^s_*(\mathbb{R}^n)} \leq \|E\|\|f\|_{C^s_*(\Omega)}$

Proof. See [19], Section 3.3.4, page 200. □

We shall see now some properties of Zigmund and Holder spaces that will prove to be very useful. We will try to give the simpler proofs here (which usually apply for positive and non-integer exponent). For negative exponent (and in some cases for integer exponent also) we will just state the properties we will need and refer to [19] for the proofs.

Lemma 14. Let $0 \leq s, r \leq 1$. Suppose $f \in C^{0,s}(\Omega)$ and $g \in C^{0,r}(\Omega)$. Then $fg \in C^{0,t}(\Omega)$ being $t = min\{s, r\}$, and

$$\|fg\|_{C^{0,t}(\Omega)} \leq 2\|f\|_{C^{0,s}(\Omega)}\|g\|_{C^{0,r}(\Omega)}$$

Proof. We use the standard trick to control products. Suppose first $|h| \leq 1$

$$|f(x)g(x) - f(x+h)g(x+h)| = |f(x)[g(x) - g(x+h)] + g(x+h)[f(x) - f(x+h)]|$$
$$\leq \|f\|_{C^{0,s}(\Omega)}\|g\|_{C^{0,r}(\Omega)}[|h|^r + |h|^s] \leq 2\|f\|_{C^{0,s}(\Omega)}\|g\|_{C^{0,r}(\Omega)}|h|^t$$

If $|h| \geq 1$, proceeding as above we see that

$$|f(x)g(x) - f(x+h)g(x+h)| \leq 2\|f\|_{C^{0,s}(\Omega)}\|g\|_{C^{0,r}(\Omega)} \leq 2\|f\|_{C^{0,s}(\Omega)}\|g\|_{C^{0,r}(\Omega)}|h|^t$$

and this proves the lemma. □

Remark 23. Note that in the Lemma 14 above we could have taken $s = r$ and prove that $fg \in C^s(\Omega)$, since for $s \leq r$ we clearly have $C^s(\Omega) \subset C^r(\Omega)$ with continuous inclusion.

Corollary 7. Let $k \in \mathbb{N}$, $0 \leq s \leq 1$. Suppose $f, g \in C^{k,s}(\Omega)$. Then $fg \in C^{k,s}(\Omega)$

Proof. If $k = 0$ it is an inmediante consequence of the Lemma 14 above. So suppose $k \geq 1$, so f and g are Lipschitz. Firstly, it is clear that $fg \in C^k(\Omega)$. Besides if $\alpha \in \mathbb{N}^n$ is such that $|\alpha| = k$, we have

$$D^\alpha[fg] = \sum_{|\eta|+|\beta|=k} D^\eta f D^\beta g$$

In the above sum all the terms $D^\beta g D^\eta f$ are $C^{1,0}$ except the terms $gD^\alpha f$, which is at least $C^{0,s}$ by Lemma 14, and the term $fD^\alpha g$, which is at least $C^{0,s}$ for the same reason. We see that $D^\alpha[fg]$ is $C^{0,s}$, and this proves the Corollary. □

Now we want to see under what conditions the product of two functions in some Zigmund space lies in other Zigmund space. First we need to define what we mean by the pointwise product of a C^r_* and a C^s_* function for general $r, s \in \mathbb{R}$. If $r, s > 0$ we know how to define the pointwise product since we deal with continuous functions. But for negative exponents we deal with tempered distributions, so we need to be more careful. In fact it is known that there is no way to extend the usual product of functions to the space of all distributions. For example, the pointwise product of two L^1 functions could not be in L^1_{loc}. So we have to live with it and define the product whenever it makes sense. Before defining this product we need some results.

Definition 30. Let $u \in D'(\mathbb{R}^n)$. We define its support and denote $supp(u)$ for the complement of the biggest open set $U \subset \mathbb{R}^n$ such that $u|_U = 0$. It is straightforward that if u is a continuous function this definition coincides with the traditional definition of support, i.e, in this case we have $supp(u) = \overline{\{u \neq 0\}}$.

Remark 24. If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ then $supp(\varphi u) \subset supp(\varphi)$. To see this note that if $\varphi|_V = 0$ in some open set $V \subset \mathbb{R}^n$, then also $(u\varphi)|_V = 0$ since $\varphi\mathcal{D}(V) = 0$ in all $\mathbb{R}^n$.

Definition 31. Let $\lambda \in \mathbb{R}$. We define the *dilation operator* d_λ acting on a function $f : \mathbb{R}^n \to \mathbb{C}$ by $d_\lambda[f](x) := f(\lambda x)$. It is straigtforward that $d_\lambda : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ and $d_\lambda : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.

Proposition 6. The Zigmund Spaces $C^r_*(\mathbb{R}^n)$ with $r \in \mathbb{R}$ are Banach spaces. Moreover, the convergence in the $C^r_*(\mathbb{R}^n)$ norm implies weak convergence in $\mathcal{S}'$.

Proof. The proof can be found in [19], 2.3.3, page 48. □

Corollary 8. If $\Omega \subset \mathbb{R}^n$ is open, the Zigmund Spaces $C^r_*(\Omega)$ with $r \in \mathbb{R}$ are Banach spaces. Moreover, the convergence in the $C^r_*(\Omega)$ norm implies weak convergence in $\mathcal{D}'(\Omega)$.

Proof. In this proof we make the typical use of the extension operator E. If $u_n \subset C^r_*(\Omega)$ is a Cauchy sequence, then by linearity and boundness of E, we have also that Eu_n is a Cauchy sequence in $C^r_*(\mathbb{R}^n)$ so $Eu_n \to v \in C^r_*(\mathbb{R}^n)$ by Proposition 6 above. Now, by definition of the norm $C^r_*(\Omega)$ we have

$$\|u_n - v|_\Omega\|_{C^r_*(\Omega)} \leq \|Eu_n - v\|_{C^r_*(\mathbb{R}^n)} \to 0$$

so we see that $u_n \to v|_\Omega$ in $C^r_*(\Omega)$, as desired. Finally, as $Eu_n \to v$ weekly in $\mathcal{S}'$, then $u_n \to v|_\Omega$ weekly in $\mathcal{D}'(\Omega)$ and this proves the Corollary. □

Definition 32. (Product in $\mathcal{S}'$)

(1) Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\varphi = 1$ in $B(0,1) = \{x \in \mathbb{R}^n : |x| = 1\}$ and $\varphi = 0$ outside $B(0,2)$. Given $g \in \mathcal{S}'$ we consider the family of functions given by $g_j := \mathcal{F}^{-1} d_{2^{-j}}[\varphi]\mathcal{F}g$. Note that $d_{2^{-j}}[\varphi]\mathcal{F}g$ is a distribution with compact support. Recall here the Payley-Wiener-Schwartz Theorem, which says that the Fourier transform of a distribution with compact support is an slowly increasing C^∞ function, so is in $\mathcal{O}(\mathbb{R}^n)$.

Therefore $g_j \in \mathcal{O}(\mathbb{R}^n)$, and the product $g_j u$ makes sense for any $u \in \mathcal{S}'$. That said, we define the product $gu := \lim_j g_j u$ whenever this limit exists in $\mathcal{S}'$ (in the weak topology). Note that with this definition it could happen that $\lim_j g_j u$ converged in some Zigmund space or not, so even though u and g are in some Zigmund spaces, its product does not have to belong necessarily to any Zigmund space.

(2) Let $g \in C^r_*(\Omega)$ and $u \in C^s_*(\Omega)$. Consider extensions $\hat{g} \in C^r_*(\mathbb{R}^n)$ and $\hat{u} \in C^s_*(\mathbb{R}^n)$. We define the product gu as $gu := (\hat{g}\hat{u})|_\Omega \in \mathcal{D}'(\Omega)$ whenever the product $\hat{g}\hat{u}$ is well defined in $\mathcal{S}'$. In the next remark we prove this product does not depend on the extensions.

Remark 25. In (1) of 32, it can be checked that the product gu does not depend on the particular choice of $\varphi \in \mathcal{D}(\mathbb{R}^n)$, that it is conmutative (so it can be defined considering u_j instead of g_j) and that it is associative respect to the usual sum of distributions. Moreover, when the product ug is well defined in a different sense (for example if $u \in \mathcal{O}(\mathbb{R}^n)$, or if both u, g are continuous or L^2 functions), then both definitions of ug coincide. See [19], 2.6.1, for details.

Also, in (2) it can be checked that this definition does not depend on the choice of the extensions $\hat{u}$ and $\hat{g}$. As we did not find this in the references, let us see why. Let $\varphi \in \mathcal{D}(\Omega)$. Then

$$(\hat{u}\hat{g}, \varphi) = \lim_j (\hat{u}, g_j\varphi) = \lim_j (u, g_j\varphi) \quad \text{does not depend on } \hat{u}$$

$$(\hat{u}\hat{g}, \varphi) = \lim_j (\hat{g}, u_j\varphi) = \lim_j (g, u_j\varphi) \quad \text{does not depend on } \hat{g}$$

Note that although u_j and g_j do depend on the extensions, we see from here that $(\hat{u}\hat{g}, \varphi)$ does not.

Lemma 15. Let $s \in \mathbb{R}$ and let $r > |s|$. Then $C^r_*(\mathbb{R}^n)C^s_*(\mathbb{R}^n) \subset C^s_*(\mathbb{R}^n)$.

This means that for every $g \in C^r_*(\mathbb{R}^n)$ and $u \in C^s_*(\mathbb{R}^n)$ the product ug (pointwise product of tempered distributions as defined in 32 above) belongs to $C^s_*(\mathbb{R}^n)$. Moreover we have

$$\|ug\|_{C^s_*(\mathbb{R}^n)} \leq C\|g\|_{C^r_*(\mathbb{R}^n)}\|u\|_{C^s_*(\mathbb{R}^n)}$$

for some constant C.

Proof. See [20], 2.6.1, page 128. □

We can change $\mathbb{R}^n$ by an arbitrary open set Ω in the Lemma 15 above and get the same result.

Corollary 9. Let $s \in \mathbb{R}$ and let $r > |s|$. Then $C^r_*(\Omega)C^s_*(\Omega) \subset C^s_*(\Omega)$.

So for every $g \in C^r_*(\Omega)$ and $u \in C^s_*(\Omega)$ the product ug belongs to $C^s_*(\Omega)$ and we have

$$\|ug\|_{C^s_*(\mathbb{R}^n)} \leq C\|g\|_{C^r_*(\mathbb{R}^n)}\|u\|_{C^s_*(\mathbb{R}^n)}$$

for some constant C.

Proof. Consider extensions $\hat{g} \in C^r_*(\mathbb{R}^n)$ and $\hat{u} \in C^s_*(\mathbb{R}^n)$. Then, by definition of ug and 15 we have that

$$\hat{u}\hat{g} \in C^s_*(\mathbb{R}^n) \text{ and } \|ug\|_{C^s_*(\Omega)} \leq \|\hat{u}\hat{g}\|_{C^s_*(\mathbb{R}^n)} \leq C\|\hat{g}\|_{C^r_*(\mathbb{R}^n)}\|\hat{u}\|_{C^s_*(\mathbb{R}^n)}$$

Now if we take the infimum over the extensions we get

$$\|ug\|_{C^s_*(\Omega)} \leq C\|g\|_{C^r_*(\Omega)}\|u\|_{C^s_*(\Omega)}$$

as desired. □

The Lemma 15 above can be sharpened when the exponets satisfy $r = s > 0$. Note that in this case we deal with continuous functions so the pointwise product is well defined in a classic sense and coincides with the product of tempered distributions.

Lemma 16. Let $r > 0$ and $\Omega \subset \mathbb{R}^n$ be an open set. Suppose $u, g \in C^r_*(\Omega)$. Then $ug \in C^r_*(\Omega)$. So $ug \in C^r_*(\Omega)$ is an algebra under pointwise multiplication.

Proof. For non integer r we already know this by Corollary 7 above. It remains to see the case $r \in \mathbb{N}$. The proof can be found in [20], 2.6.2, page 133. □

Lemma 17. Let $k \in \mathbb{N}$ and $0 \leq \alpha \leq 1$. Suppose $|f(x)| \geq c > 0$ for all $x \in \Omega$.

(1) If $f \in C^{k,\alpha}(\Omega)$ then $f^{-1} = \frac{1}{f} \in C^\alpha(\Omega)$.

(2) If $f \in C^{k+2}_*(\Omega)$ then $f^{-1} = \frac{1}{f} \in C^{k+2}_*(\Omega)$

Proof. Let us prove (1). First suppose $k = 0$. We compute

$$|\frac{1}{f(x)} - \frac{1}{f(y)}| = \frac{|f(y) - f(x)|}{|f(y)f(x)|} \leq \frac{1}{c^2}|x-y|^\alpha$$

and this proves the lemma for $k = 0$. Now suppose $k \geq 1$. We have

$$D^1(f^{-1}) = -f^{-2}D^1f \quad \text{ if } k = 1$$
$$D^2(f^{-1}) = 2f^{-3}D^1fD^1f + f^{-4}D^2f \quad \text{ if } k = 2$$
$$D^k(f^{-1}) = \ \{\text{sum of products of derivatives of } f \text{ of order} \leq k-1\} \ + (-1)^{k-1}f^{-2k}D^kf$$

If $k = 1$ we see from the formula above that $D^1(f^{-1})$ is C^α since f^{-2} is C^1 and $D^1 f$ is C^α. If $k = 2$, we know that f^{-3}, f^{-4} and $D^1 f$ are C^1 and from the formula above it follows that $D^2(f^{-1})$ is C^α. For general k follows applying induction.

Now let us see (2). If $k + 2 = 2$, then $f \in C^2_* \subset C^1$, so the formula for $D^1(f^{-1})$ above is valid. As f^{-2} is C^1 and $D^1 f$ is C^1_*, its product also is C^1_* by Lemma 16, so $D^1(f^{-1}) \in C^1_*$ and we are done.

If $k + 2 = 3$, then f^{-1} is C^2 and the formula above for $D^2(f^{-1})$ holds. The first summand in that formula is C^1. For the second summand, as $D^2 f$ is C^1_* and f^{-4} is C^2 , again by Lemma 16 we conclude that $D^2(f^{-1})$ is C^1_*, so $f \in C^3_*$ and we are done.

For general k it follows by induction. □

Lemma 18. Let $\Omega \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be two open sets. Let $f : \Omega \to V$ be Holder of exponent α, with $0 < \alpha \leq 1$, and let $g : V \to \mathbb{R}^k$ be Holder of exponent β, with $0 < \beta \leq 1$. Then $g \circ f : \Omega \to \mathbb{R}^k$ is Holder of exponent $\alpha\beta$.

Proof. Recall that for every $t, s \in V$ we have by definition that $|g(t) - g(s)| \leq [g]_\beta |t - s|^\beta$, so

$$|g(f(x)) - g(f(x + h))| \leq [g]_\beta |f(x) - f(x + h)|^\beta \leq [g]_\beta [f]^\beta_\alpha |h|^{\alpha\beta}$$

and this gives that $[g \circ f]_{\alpha\beta} \leq [g]_\beta [f]^\beta_\alpha$, so the lemma is proved. □

Note that this result is optimal, since taking $f(x) = x^\alpha, g(x) = x^\beta$, both defined in $[0, 1]$, we see that $g \circ f(x) = x^{\alpha\beta}$, and this function is not C^γ for any $\gamma > \alpha\beta$, since $x^{\alpha\beta} x^{-\gamma} \to \infty$ as $x \to 0$.

Lemma 19. Let $U \subset \mathbb{R}^m$, $\Omega \subset \mathbb{R}^n$ be open sets. Let $k \in \mathbb{N}$. Let $\alpha \in [0, 1]$. The Holder spaces are closed for composition. The Zigmund spaces of exponent $s > 1$ are also closed for composition. More precisely:

(1) If $k \geq 1$, $f \in C^{k,\alpha}(U)$, $g \in C^{k,\alpha}(\Omega; U)$ then $f \circ g \in C^{k,\alpha}(\Omega)$ (false if $k = 0$)

(2) If $r > 1$, $f \in C^r_*(U)$, $g \in C^r_*(\Omega; U)$ then $f \circ g \in C^r_*(\Omega)$

Proof. First we prove (1). By the chain rule it is clear that $f \circ g \in C^k(\Omega)$. Then

$$\begin{aligned} &D^1(f \circ g) = (D^1 f \circ g) D^1 g \in C^\alpha \quad \text{ if } k = 1 \\ &D^2(f \circ g) = (D^2 f \circ g) D^1 g D^1 g + (D^1 f \circ g) D^2 g \quad \text{ if } k = 2 \\ &D^k(f \circ g) = (D^k f \circ g) D^1 g \dots D^1 g + (D^{k-1} f \circ g) D^2 g D^1 g \dots D^1 g + \dots + (D^1 f \circ g) D^k g \end{aligned} \tag{48}$$

For $k = 1$ we claim that $D^1(f \circ g) \in C^\alpha(U)$. To see this note that $D^1 f \circ g$ is a composition of a Lipschitz function g and a C^α function $D^1 f$ so by Lemma 18 we have $D^1 f \circ g \in C^\alpha$.

For $k = 2$, looking at the formula above for $D^2(f \circ g)$ we see that both summands are C^α since they are a product of a C^α and a C^1 function (note that $D^2 f \circ g$ is C^α again by Lemma 18).

For general k, the only summands we have to pay attention in the formula for $D^k(f \circ g)$ are the first and the last, because the others are C^1. These summands are C^α by the same argument as in the case $k = 2$. Thus $f \circ g$ is $C^{k,\alpha}$, and the result is proved.

The proof of (2) is considerably more difficult. The case when g is real valued can be found in the relatively recent paper [23]. For the case where g is vector valued we do not have any reference yet, but according to an expert on Besov spaces the result is true. Recall that the hipothesis of $r > 1$ is crucial. In [23] it is proved that $C^1 \circ C^1_*$ is not contained in C^1_*. □

Now we shall introduce a class of operators which generalize differential operators. They are called pseudodifferential operators. As usual, by expanding the class of operators we get some nice properties such as closeness under certain operations (composition, multiplication, and more). Working with pseudodifferential operators we can also define ellipticity, and we can improve some results of elliptic regularity.

Let $p(x,\xi) = \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$ be the symbol of the differential operator of order m given by $p(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$. Note that here we use D instead of ∂ beacause we are going to use Fourier analysis, so it is more confortable to work with D to avoid constants in the formulas. By the inversion Fourier formula we have that for $f \in \mathcal{S}$, denoting $\hat{f}(\xi) := \mathcal{F}(f)(\xi)$, we have

$$f(x) = \int \hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi. \tag{49}$$

And if we apply $P(x,D)$ we obtain that

$$P(x,D)f(x) = \int p(x,\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi. \tag{50}$$

So a differential operator can be recovered from its symbol $p(x,\xi)$, which is a polinomial in ξ. Imagine that we want to find a composition inverse $q(x,D)$ of $p(x,D)$. Then we could think of an operator $q(x,D)$ which acts on functions as in (49) but changing $p(x,\xi)$ for $q(x,\xi) := p(x,\xi)^{-1}$, the multiplicative inverse of $p(x,\xi)$. Whenever this can be done, $q(x,D)$ is an inverse of $p(x,D)$. But note that the operator $q(x,D)$ defined this way is not differential, since its symbol $q(x,\xi)$ is not a polinomial in ξ. That's why we introduce the following.

Definition 33. Given $m \in \mathbb{R}$, $\rho,\delta \in [0,1]$, we define the class of symbols $S^m_{\rho,\delta}$ as

$$S^m_{\rho,\delta} := \{p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : |D^\beta_x D^\alpha_\xi p(x,\xi)| \leq C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\}$$

for every $x \in \mathbb{R}^n$ and multiindexes $\alpha,\beta \in \mathbb{N}^n$, where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$.

Any element $p(x,\xi) \in S^m_{\rho,\delta}$ is called a symbol, and it defines an operator

$$\begin{aligned} &p(x,D) : \mathcal{S} \to \mathcal{S} : \varphi \mapsto p(x,D)(\varphi) \\ &p(x,D)\varphi(x) = \int p(x,\xi)\hat{\varphi}(\xi)e^{2\pi i x\cdot\xi}d\xi. \end{aligned} \tag{51}$$

We will denote $OPS^m_{\rho,\delta}$, for this class of operators $p(x,D)$.

To see that $p(x,D)\varphi \in \mathcal{S}$, recall that for each multiindex $\beta \in \mathbb{N}^n$ we can differentiate under the integral sign in (51) because the derivatives D^β_x of both $p(x,\xi)$ and $e^{2\pi i x\cdot\xi}$ have a polinomial growth in ξ, so we have absolute convergence. Doing so and integrating by parts we get

$$\begin{aligned} &|x^\alpha D^\beta_x[p(x,D)\varphi(x)]| \leq |\int D^\beta_x p(x,\xi)\hat{\varphi}(\xi)x^\alpha e^{2\pi i x\cdot\xi} + C_\beta p(x,\xi)\hat{\varphi}(\xi)\xi^\beta x^\alpha e^{2\pi i x\cdot\xi}d\xi| \\ &= |\int [D^\beta_x p(x,\xi) + C_\beta p(x,\xi)\xi^\beta]\hat{\varphi}(\xi)D^\alpha_\xi[e^{2\pi i x\cdot\xi}]d\xi| \\ &= |\int e^{2\pi i x\cdot\xi}D^\alpha_\xi[\{D^\beta_x p(x,\xi) + C_\beta p(x,\xi)\xi^\beta\}\hat{\varphi}(\xi)]d\xi| \\ &\leq \sum_{|\gamma|+|\lambda|\leq|\alpha|}\int |D^\beta_x D^\gamma_\xi p(x,\xi) + C_\beta D^\gamma_\xi[p(x,\xi)\xi^\beta]||D^\lambda_\xi\hat{\varphi}(\xi)|d\xi \\ &\leq C_{\alpha,\beta}\langle\xi\rangle^{m+|\beta|+\delta|\beta|}\sum_{|\lambda|\leq|\alpha|}\int |D^\lambda\hat{\varphi}(\xi)|d\xi \leq C_{\alpha,\beta} \end{aligned} \tag{52}$$

and this proves that $p(x,D)\varphi(x) \in \mathcal{S}$.

Now we can ask whether $p(x,D)$ can be extended to $\mathcal{S}'$ in such a way that $p(x,D) = p(x,D)'$ when restricted to $\mathcal{S}$. This time it is not possible in general. If we look at the proof that $a(D) = a(D)'$ in (25), it was crucial that the symbol $a(\xi)$ did not depend on x. In this case we can still define $p(x,D)' : \mathcal{S}' \to \mathcal{S}'$, but if $u, \varphi \in \mathcal{S}$ we have

$$\begin{aligned}(p(x,D)'u,\varphi) &= (u, p(x,D)\varphi) = \int\int u(x)p(x,\xi)\hat{\varphi}(\xi)e^{2\pi i x\cdot\xi}d\xi dx \\ &= [\text{ Fubini }] = \int \hat{\varphi}(\xi)\overline{u}(\xi)d\xi = \int \varphi(x)\mathcal{F}[\overline{u}](x)dx\end{aligned} \tag{53}$$

The last step is not justified, because we are not sure wheter $\overline{u}(\xi)$ has a Fourier transform, being

$$\overline{u}(\xi) := \int p(y,\xi)u(y)e^{2\pi i y\cdot\xi}dy$$

In any case, even when the last step is valid, it is easy to check that, as $p(y,\xi)$ depends on y, in general $\mathcal{F}[\overline{u}](x) \neq p(x,D)u(x)$. In the next proposition we analize the cases when the last step is valid, and then we can extend $p(x,D)$ to $\mathcal{S}'$. But note that possibly this extension does not coincide with the transpose map $p(x,D)'$.

This fact is related with one important property of pseudodifferential operators: the L^2-adjoint of a pseudodifferential operator is pseudodifferential, but it is not easy to see and of course the symbol of the adjoint is not the conjugate of the symbol.

Proposition 7. If $p(x,D) \in OPS^m_{\rho,\delta}$ for $m \in \mathbb{R}$, $\rho \in [0,1]$ and $0 \le \delta < 1$, then we have $p(x,D) : \mathcal{S}' \to \mathcal{S}'$.

Proof. As in the calculation (53) , we see that if $\varphi, \phi \in \mathcal{S}$, considering $\varphi \in \mathcal{S}'$, we have

$$(p(x,D)I_\varphi, \phi) := (P(x,D)\varphi, \phi) = (\hat{\varphi}, \overline{\phi})$$

So $p(x,D)$ can be extended to $\mathcal{S}'$ if and only if for any $u \in \mathcal{S}'$ we can make sense of $(\hat{u}, \overline{\phi})$, which happens if and only if $\overline{\phi} \in \mathcal{S}$, being, as before,

$$\overline{\phi}(\xi) := \int p(y,\xi)\phi(y)e^{2\pi i y\cdot\xi}dy \tag{54}$$

Take $\alpha, \beta \in \mathbb{N}^n$. We claim that if we apply D^α_ξ to the integrand of (54) and multiply it by ξ^β, the result is integrable, so the formal derivation is true. Indeed, formal differentiation yields

$$\begin{aligned}&|\xi^\beta D^\alpha_\xi \overline{\phi}(\xi)| = |\int C_\beta D^\alpha_\xi p(y,\xi)\phi(y)D^\beta_y e^{2\pi i y\cdot\xi} + C_{\alpha,\beta}p(y,\xi)\phi(y)y^\alpha D^\beta_y e^{2\pi i y\cdot\xi}dy| \\ &= |\int C_\beta D^\beta_y[D^\alpha_\xi p(y,\xi)\phi(y)]e^{2\pi i y\cdot\xi} + C_{\alpha,\beta}D^\beta_y[p(y,\xi)\phi(y)y^\alpha]e^{2\pi i y\cdot\xi}dy| \\ &\le C_{\alpha,\beta}\{\langle\xi\rangle^{m+|\beta|\delta-|\alpha|\rho}\int|\phi(y)|dy + \langle\xi\rangle^{m+|\beta|\delta}\int|y^\alpha\phi(y)|dy\} \le C_{\alpha,\beta}\langle\xi\rangle^{m+|\beta|\delta}\end{aligned}$$

we conclude that for every monomial ξ^γ with $\gamma \in \mathbb{N}^n$, $|\gamma| \le |\beta|$ we have $|\xi^\gamma D^\alpha_\xi\overline{\phi}(\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m+|\beta|\delta}$. As $\langle\xi\rangle^\beta$ has an equivalent growth that a certain sum of monomials ξ^γ with $\gamma \in \mathbb{N}^n$, $|\gamma| \le |\beta|$, we conclude that $|\langle\xi\rangle^\beta D^\alpha_\xi\overline{\phi}(\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m+|\beta|\delta}$, and this yields

$$|D^\alpha_\xi\overline{\phi}(\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m+|\beta|(\delta-1)}$$

Since $\delta < 1$, this trick shows that we can take $|\beta|$ as large as we want to obtain an arbitraryly rapid decay in ξ of $D^\alpha_\xi\overline{\phi}(\xi)$, and this shows that $\overline{\phi}(\xi) \in \mathcal{S}$, as we wanted. $\square$

In general when we speak of a pseudodifferential operator we mean an operator $p(x, D)$ associated to some function $p(x, \xi)$ that acts as in (51). If $p(x, \xi)$ is not a symbol in $S^m_{\rho,\delta}$, the convergenge of the integral defining $p(x, D)$ is not clear, and that is why in any case we will require some kind of decay of $p(x, \xi)$ in the ξ variable to assure the convergence.

Remark 26. Let $p(x, D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ a differential operator of order $m \in \mathbb{N}$, and let $p(x, \xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$ be its symbol. If $a_\alpha(x) \in C^{\infty+0}(\mathbb{R}^n)$, (remind that this means all the derivatives of a_α are bounded in $\mathbb{R}^n$, as said in 21), then it is obvious that $p(x, \xi) \in S^m_{1,0}$. This is why pseudodifferential operators generalize differential operators in presence of $C^{\infty+0}$ regularity of the coeficients $a_\alpha(x)$.

However, we are interested in the case where $a_\alpha \in C^s$ for $s \in \mathbb{R}$, $s \ge 0$, so the class of symbols $S^m_{\rho,\delta}$ does not fullfill our requirements. To this end, we define now a more general class of symbols that generalize less regular differential operators.

Definition 34. Let $r \in (0, \infty)$, $m \in \mathbb{R}$, and let $\rho, \delta \in [0, 1]$. We say that the function $p(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ belongs to the space of symbols $C^r_* S^m_{\rho,\delta}(\mathbb{R}^n)$ provided:

1) The maps $\xi \mapsto p(x, \xi)$ are smooth for every $\xi \in \mathbb{R}^n$, and moreover for each multiindex $\alpha \in \mathbb{N}^n$ there exists a constant C_α such that $\|D^\alpha_\xi p(\cdot, \xi)\|_{L^\infty} \le C_\alpha \langle\xi\rangle^{m-\rho|\alpha|}$.

2) The maps $x \mapsto D^\alpha_\xi p(x, \xi)$ are $C^r_*(\mathbb{R}^n)$ for each $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$, and moreover we have the estimate $\|D^\alpha_\xi p(\cdot, \xi)\|_{C^r_*(\mathbb{R}^n)} \le C_\alpha \langle\xi\rangle^{m-\rho|\alpha|+\delta r}$ for some constant C_α.

Remark 27. Note that the class of symbols $C^r_* S^m_{\rho,\delta}(\mathbb{R}^n)$ is defined in the same spirit that the class $S^m_{\rho,\delta}$, only that in this case we have less derivatives in the x variable. To see this, note that for $r \notin \mathbb{N}$ the norm C^r_* is equivalent to the norm C^r, so taking the norm $\|\cdot\|_{C^r_*}$ is like 'taking r derivatives' on the variable x. For $r = k \in \mathbb{N}$ the norm $\|\cdot\|_{C^k_*}$ is like taking $k-1$ derivatives and a little more.

Also, if $p(x, D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha$ is a differential operator of order m, and $a_\alpha(x) \in C^r_*(\mathbb{R}^n)$ for $r > 0$, it is obvious that $p(x, \xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \in C^r_* S^m_{1,0}(\mathbb{R}^n)$.

Let us now prove an elementary result on how $p(x, D)$ acts when its symbol has lower regularity. This results is not optimal at all, but it is easy to prove, and at least tell us that the operator $p(x, D)$ is well defined acting on $\mathcal{S}$.

Proposition 8. Let $r > 0$, $r \notin \mathbb{N}$, and suppose $p(x, \xi) \in C^r_* S^m_{\rho,\delta}(\mathbb{R}^n)$ for $m \in \mathbb{R}$, $\rho, \delta \in [0, 1]$. Then we have $p(x, D) : \mathcal{S} \to C^r_*(\mathbb{R}^n)$. Moreover, if $\beta \in \mathbb{N}^n$ satisfies $|\beta| \le r$, then the function $D^\beta_x p(x, D)\varphi$ is rapidly decreasing for every $\varphi \in \mathcal{S}$.

Proof. First, since $r \notin \mathbb{N}$ we have $C^r_* = C^r$ and both spaces have equivalent norms, so we will work with the space C^r. Let $r = k + s$ with $k \ge 0$, $k \in \mathbb{N}$ and $0 < s < 1$. Let $\beta \in \mathbb{N}^n$ with $|\beta| \le k$. By hipothesis, for every $x \in \mathbb{R}^n$ we have $|D^\beta_x p(x, \xi)| \le C_\beta \langle\xi\rangle^{m+\delta r}$. Also, if $\gamma \in \mathbb{N}^n$ and $|\gamma| = k$ we have $[D^\gamma_x p(\cdot, \xi)]_{C^s} \le C_\gamma \langle\xi\rangle^{m+\delta r}$. That said, given $\varphi \in \mathcal{S}$, we derive formally under the integral sign to obtain

$$|D^\beta_x p(x, D)\varphi(x)| = |\int \hat{\varphi}(\xi) e^{2\pi i x\cdot\xi} [D^\beta_x p(x, \xi) + C_\beta p(x, \xi)\xi^\beta] d\xi|$$

$$\le C \int \langle\xi\rangle^{m+\delta r+|\beta|} |\hat{\varphi}(\xi)| d\xi \le C$$

We conclude that $p(x, D)\varphi(x) \in C^{k+0}$. Now we shall see that $D^\beta_x p(x, D)\varphi(x)$ is a rapidly decreasing function. Indeed, given $\alpha \in \mathbb{N}^n$, integration by parts yields

$$
\begin{aligned}
&|x^\alpha D_x^\beta p(x,D)\varphi(x)| = |\int \hat{\varphi}(\xi) C_\alpha D_\xi^\alpha [e^{2\pi i x\cdot\xi}]\{D_x^\beta p(x,\xi) + C_\beta p(x,\xi)\xi^\beta\} d\xi| \\
&= |\int C_\alpha e^{2\pi i x\cdot\xi} D_\xi^\alpha \{\hat{\varphi}(\xi)[D_x^\beta p(x,\xi) + C_\beta p(x,\xi)\xi^\beta] d\xi| \\
&\le \sum_{|\lambda|\le|\alpha|} \int C_\alpha \langle\xi\rangle^{m+\delta r+|\beta|} |D_\xi^\lambda \hat{\varphi}(\xi)| d\xi \le C_\alpha
\end{aligned}
$$

Now let $|\gamma| = k$ and we have that

$$
\begin{aligned}
&|D_x^\gamma p(x+h,D)\varphi(x+h) - D_x^\gamma p(x,D)\varphi(x)| \\
&= |\int \hat{\varphi}(\xi)[\Delta_h\{D_x^\gamma p(x,\xi)e^{2\pi i x\cdot\xi}\} + C_\gamma \xi^\gamma \Delta_h\{p(x,\xi)e^{2\pi i x\cdot\xi}\}] d\xi|
\end{aligned} \tag{55}
$$

where

$$
\begin{aligned}
\Delta_h\{D_x^\gamma p(x,\xi)e^{2\pi i x\cdot\xi}\} &:= D_x^\gamma p(x+h,\xi)e^{2\pi i(x+h)\cdot\xi} - D_x^\gamma p(x,\xi)e^{2\pi i x\cdot\xi} \\
\Delta_h\{p(x,\xi)e^{2\pi i x\cdot\xi}\} &:= p(x+h,\xi)e^{2\pi i(x+h)\cdot\xi} - p(x,\xi)e^{2\pi i x\cdot\xi}
\end{aligned}
$$

Denote $f(x,\xi)$ for any of $p(x,\xi)$, $D_x^\gamma p(x,\xi)$. We claim that $[f(\cdot,\xi)]_{C^s} \le C\langle\xi\rangle^{m+\delta r}$. Indeed, if $k=0$ then $p(x,\xi) = D_x^\gamma p(x,\xi)$ is the result is known. If $k \ge 1$, as the claim is known for $D_x^\gamma p(x,\xi)$, suppose $f(\cdot,\xi) = p(\cdot,\xi) \in C^{1+0}$. Then, by distinction of cases $|h| > 1$ and $|h| < 1$ it is easy to see that $[f(\cdot,\xi)]_{C^s} \le 2\|f(\cdot,\xi)\|_{C^1} \le \langle\xi\rangle^{m+\delta r}$ and this gives the claim.

Moreover, denoting $g(x,\xi) := e^{2\pi i x\cdot\xi}$, by the same argument we have $[g(\cdot,\xi)]_{C^s} \le 2\|g(\cdot,\xi)\|_{C^1} \le 2\langle\xi\rangle$.

Let us see that also $[f(\cdot,\xi)g(\cdot,\xi)]_{C^s} \le C\langle\xi\rangle^{m+\delta r+1}$. To see this, we use the standard trick to deal with a product

$$
\begin{aligned}
&|\Delta_h\{f(x,\xi)g(x,\xi)| = |f(x+h,\xi)g(h+x,\xi) - f(x,\xi)g(x,\xi)| \\
&= |[f(x+h,\xi) - f(x,\xi)]g(x+h,\xi) + f(x,\xi)[g(x+h,\xi) - g(x,\xi)]| \\
&\le \|g(\cdot,\xi)\|_{L^\infty}[f]_{C^s}|h|^s + \|f(\cdot,\xi)\|_{L^\infty}\|g(\cdot,\xi)\|_{C^s}|h|^s \\
&\le C\langle\xi\rangle^{m+\delta r}|h|^s + \langle\xi\rangle^{m+1}|h|^s \le C|h|^s\langle\xi\rangle^{m+1+\delta r}.
\end{aligned}
$$

Now, coming back to (55) we have that

$$
\begin{aligned}
&|D_x^\gamma p(x+h,D)\varphi(x+h) - D_x^\gamma p(x,D)\varphi(x)| \\
&\le \int |\hat{\varphi}(\xi)|[\xi^{m+\delta r+1}|h|^s + C\xi^{m+|\beta|+\delta r+1}|h|^s] d\xi \le C|h|^s
\end{aligned}
$$

and this concludes that $p(x,D)\varphi \in C^r = C^r_*$. □

This proof above could possibly be adapted for $r \in \mathbb{N}$, $r > 0$. However, as the difference quotients defining C^r_* for $r \in \mathbb{N}$ are more complicated, we will not worry about this. In any case, we have the following corollary for free.

Corollary 10. Let $r > 0$, $m \in \mathbb{R}$, and $\rho,\delta \in [0,1]$. Given $p(x,\xi) \in C^r_* S^m_{\rho,\delta}(\mathbb{R}^n)$, it follows that $p(x,D) : \mathcal{S} \to C^s_*(\mathbb{R}^n)$ for every $s < r$. In addition, if $\beta \in \mathbb{N}^n$ is such that $\beta \le r-1$, then $D_x^\beta p(x,D)\varphi$ is rapidly decreasing for any $\varphi \in \mathcal{S}$.

Proof. We have $\|\cdot\|_{C_*^s} \leq \|\cdot\|_{C_*^r}$. So, if $p(x,\xi) \in C_*^r S_{\rho,\delta}^m$, then $p(x,\xi) \in C_*^s S_{\rho,\delta}^m$, and now the proposition 8 gives the claim. □

In fact, with a considerable amount of work, the following result can be proved, which is far better than Corollary 10 above. The important case for us will be to take $\delta = 0$ in the Theorem below.

Theorem 11. Let $r > 0$, $\delta \in [0,1)$, $m \in \mathbb{R}$ and $p(x,\xi) \in C_*^r S_{1,\delta}^m(\mathbb{R}^n)$. Suppose that $s \in \mathbb{R}$ satisfies $-(1-\delta)r < s < r$. Then we have

$$p(x,D) : C_*^{m+s}(\mathbb{R}^n) \to C_*^s(\mathbb{R}^n)$$

This always means unless explicit mention that $p(x,D)$ is continuous, i.e, there exists a constant C so that for every $u \in C_*^{s+m}(\mathbb{R}^n)$ we have $\|p(x,D)u\|_{C_*^s(\mathbb{R}^n)} \leq C\|u\|_{C_*^{m+s}(\mathbb{R}^n)}$.

Proof. See [8], Chapter 13, Proposition 9.10. The proof is highly technical and requires some background. Note that m, s could be negative, so the proof has to work with the Fourier transform. □

Remark 28. (1) Let us think about what the Theorem 11 above tell us in the particular case of $p(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ being a differential operator of order m with coefficients $a_\alpha \in C_*^r(\mathbb{R}^n)$. We know that in this case the symbol $p(x,\xi) \in C_*^r S_{1,0}^m(\mathbb{R}^n)$. Given $u \in \mathcal{S}'$ one is tempted to make sense of $p(x,D)u = \sum a_\alpha(x)D^\alpha u$ as an element of $\mathcal{S}'$ in the usual sense, i.e, by putting for $\varphi \in \mathcal{S}$

$$(a_\alpha(x)D^\alpha u, \varphi) = (D^\alpha u, a_\alpha\varphi) = (-1)^{|\alpha|}(u, D^\alpha[a_\alpha\varphi]) \tag{56}$$

The problem here is that a_α, though is slowly increasing, is not smooth, so for general $u \in \mathcal{S}'$ the expression (56) will not make sense. The Theorem above tell us that if we require more than simply $u \in \mathcal{S}'$, i.e, if we require that $u \in C_*^{m+s}(\mathbb{R}^n)$ for some $s \in (-r,r)$, then we can make sense of $p(x,D)u$ as a tempered distribution, so the expression (56) holds in some sense. Moreover the Theorem says that $p(x,D)u \in C_*^s(\mathbb{R}^n)$.

(2) A trivial example of a situation in which the expression (56) makes sense would be to suppose that $u \in C_*^{m+s}$ for some $s > 0$. Then $D^\alpha u$ is a bounded function, so in (56) we can make sense of $(D^\alpha u, a_\alpha\varphi)$. Note however that the Theorem allow s to be negative also, so certainly is more powerful than this naive approach.

(3) Consider the special case of $p(x,D)u = a(x)u$ a differential operator of order 0, for $a \in C_*^r(\mathbb{R}^n)$, then the Theorem says that the operator given by multiplication by a maps $C_*^s(\mathbb{R}^n)$ to $C_*^s(\mathbb{R}^n)$ if $|s| < r$. This is just what Lemma 15 says.

Now we shall define the notion of ellipticity for pseudodifferential operators.

Definition 35. Let $r > 0$, and let $p(x,\xi) \in C_*^r S_{\rho,\delta}^m(\mathbb{R}^n)$. Consider the associated operator $p(x,D)$. We say that $p(x,D)$ is elliptic if the following conditions are satisfied:
(a) There exits $R > 0$ such that $p(x,\xi) \neq 0$ for any $|\xi| \geq R$, $x \in \mathbb{R}^n$
(b) We have $|p(x,\xi)^{-1}| \leq C\langle\xi\rangle^{-m}$ if $x \in \mathbb{R}^n$, $|\xi| \geq R$

Remark 29. This definition is somewhat different than the definition given for ellipticity of differential operators, but we have the following.

Given $p(x,D)$ a linear differential operator with continuous and bounded coeficcients a_α, if $p(x,D)$ is elliptic in the sense of differential operators, then it is also elliptic regarded as a pseudodifferential operator. Indeed, let $p_k(x,\xi) := \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha$ for $k = 0, \ldots, m$. By definition of uniform ellipticity, we know that for some $C_m > 0$ we have $|p_m(x,\xi)| \geq C_m|\xi|^m$. Therefore, as the a_α are bounded, this yields

$$|p(x,\xi)| = |\sum_{k=0}^{m} |\xi|^k p_k(x,\xi|\xi|^{-1})| \geq C_m|\xi|^m - C(|\xi|^{m-1} + \cdots + 1)$$

So if $|\xi|$ is large, $|p(x,\xi)| \neq 0$ for all $x \in \mathbb{R}^n$. Besides we see that $|p(x,\xi)| \geq C\langle\xi\rangle^m$ for ξ large, so $|p(x,\xi)^{-1}| \leq C\langle\xi\rangle^{-m}$ for ξ large, and this shows that $p(x,\xi)$ is elliptic as a pseudodifferential operator.

Now we state, without proof, a powerful result concerning elliptic regularity in the world of pseudodifferential operators in $\mathbb{R}^n$.

Theorem 12. Let $s > 0$, $m \in \mathbb{R}$, and assume $p(x,\xi) \in C^s_* S^m_{1,0}(\mathbb{R}^n)$ is elliptic. Suppose that $u \in C^{m-s+\varepsilon}_*(\mathbb{R}^n)$ for some $0 < \varepsilon < 2s$, and that $f \in C^r_*(\mathbb{R}^n)$ for some $-s < r \leq s$.

Under this hipothesis, if u is a solution (in the distributional sense) of the equation

$$p(x,D)u = f$$

then actually $u \in C^{m+r}_*(\mathbb{R}^n)$.

Remark 30. Recall that under the hipothesis of Theorem 12 above, it obviously holds that $-s < -s+\varepsilon < s$, so by Proposition 11 it follows that

$$p(x,D) : C^{m-s+\varepsilon}_*(\mathbb{R}^n) \to C^{-s+\varepsilon}_*(\mathbb{R}^n).$$

From this we see that we can make sense of $p(x,D)u$ as an element of $C^{-s+\varepsilon}_*(\mathbb{R}^n)$.

Proof. See [8], Chapter 14, Theorems 4.2 and 4.3. □

Remark 31. Under the hipothesis and notations of the Theorem 12 above, note that when $r = s$ this result is similar to the classic elliptic regularity result given in Theorem 7. Indeed, suppose that $p(x,D)$ is a differential operator with coefficients a_α, and suppose $a_\alpha, f \in C^s_*(\mathbb{R}^n)$. Then, if s is not integer, this Theorem ends up with the same conclusion than the classic elliptic regularity in Theorem 7. The singular cases in Theorem 7 arise, as we saw, when a_α and f are not $C^{k,\alpha}$ for $k \in \mathbb{N}$ and $0 < \alpha \leq 1$. This Theorem shows that when a_α and f are C^k_* then u is C^{m+k}_*, so this cases are also singular here because for integer exponent the Zigmund spaces are not Holder spaces anymore.

In the future we will need to use elliptic regularity working in coordinates on a Riemannian manifold M, so we shall encounter differential operators with coefficients defined not in all $\mathbb{R}^n$ but in some open set Ω. This is why it is interesting for us to deduce a local version of 12, as we do in the next propositions. First we prove an analogous result of Theorem 11 but changing $\mathbb{R}^n$ by an open set Ω. Although it can be done, here we do not need to define the concept of symbols defined in Ω. We will simply work with linear differential operators with coefficients defined in Ω.

Proposition 9. Let $\Omega \subset \mathbb{R}^n$ be an open set and $r > 0$. Suppose we have a linear differential operator

$$p(x,D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

with coefficients a_α in $C^r_*(\Omega)$. Let $s \in \mathbb{R}$ satisfy $-r < s < r$. Then we have the mapping property

$$p(x,D) : C^{m+s}_*(\Omega) \to C^s_*(\Omega)$$

This means that there exists a constant C so that for every $u \in C^{m+s}_*(\Omega)$ we have

$$\|p(x,D)u\|_{C^s_*(\Omega)} \leq C\|u\|_{C^{m+s}_*(\Omega)}.$$

Proof. Let $u \in C_*^{s+m}(\Omega)$, and consider any extension $\hat{u} \in C_*^{s+m}(\mathbb{R}^n)$. Consider also extensions $\hat{a}_\alpha \in C_*^r(\mathbb{R}^n)$ of a_α and let $\hat{p}(x,D) := \sum_{|\alpha|\leq m} \hat{a}_\alpha(x) D^\alpha$ be the corresponding extension of $p(x,D)$. As discussed above, we have that $\hat{p}(x,\xi) \in C_*^r S_{1,0}^m(\mathbb{R}^n)$. Therefore we apply Theorem 11 to see that

$$\|\hat{p}(x,D)\hat{u}\|_{C_*^s(\mathbb{R}^n)} \leq C_{\hat{p}} \|\hat{u}\|_{C_*^{m+s}(\mathbb{R}^n)} \tag{57}$$

In particular we see that $p(x,D)u$ has an extension $\hat{p}(x,D)\hat{u} \in C_*^s(\mathbb{R}^n)$, so by definition $p(x,D)u \in C_*^s(\Omega)$, and from (57) we see that $\|p(x,D)u\|_{C_*^s(\Omega)} \leq C_{\hat{p}}\|\hat{u}\|_{C_*^{m+s}(\mathbb{R}^n)}$. Now, as $C_{\hat{p}}$ does not depend on $\hat{u}$, we can take the infimum over the extensions $\hat{u}$ of u to conclude that $\|p(x,D)u\|_{C_*^s(\Omega)} \leq C_{\hat{p}}\|u\|_{C_*^{m+s}(\Omega)}$, and this gives the claim. ◻

Remark 32. Note that we can take in the Proposition 9 above $p(x,D) = D^\alpha$ for any multiindex $\alpha \in \mathbb{N}^n$ whose symbol $p(x,\xi) = \xi^\alpha \in S_{1,0}^{|\alpha|} C^{\infty+0}(\mathbb{R}^n)$ and therefore for any $s \in \mathbb{R}$ and for any open set $\Omega \subset \mathbb{R}^n$ we have

$$D^\alpha : C_*^s(\Omega) \to C_*^{s-|\alpha|}(\Omega) \tag{58}$$

This result seems simple if one think in the exponent s as the order of differentiability of the space C_*^s, but we recall that the exponent just expresses how quikly the Fourier multipliers $\varphi_j(D)u$ decay, being φ_j a Littlewood-Payley partition of unity as in Definition 27.

Let us see that the naive approach does not work to see (58). For example take $\Omega = \mathbb{R}^n$ and $u \in C_*^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. Differentiating under the integral sign and integrating by parts, it is easy to check that

$$\varphi_j(D)\partial_{x_i} u = \partial_{x_i}[\varphi_j(D)u]$$

as distributions, so for $\phi \in \mathcal{S}$ we have

$$|(\varphi_j(D)\partial_{x_i}u, \phi)| = |(\varphi_j(D)u, \partial_{x_i}\phi)| \leq \|\varphi_j(D)u\|_{L^\infty(\mathbb{R}^n)}\|\partial_{x_i}\phi\|_{L^1(\mathbb{R}^n)} \leq \|u\|_{C_*^s(\mathbb{R}^n)} 2^{-js}\|\partial_{x_i}\phi\|_{L^1(\mathbb{R}^n)}$$

If we take the supremum on $\phi \in \mathcal{S}$ with $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$ in the expresion above we obtain the trivial estimate $\|\varphi_j(D)\partial_{x_i}u\|_{L^\infty(\mathbb{R}^n)} \leq \infty$, since the differential operators are unbounded in $L^1(\mathbb{R}^n)$.

This shows that the estimate we did above does not work at all to get the assymptotic decay for $\|\varphi_j(D)\partial_{x_i}u\|_{L^\infty(\mathbb{R}^n)}$ of type $2^{-j(s-1)}$ we are looking for. So it is necessary to make much sharper estimates even for this simple result.

Proposition 10. Let $B \subset \mathbb{R}^n$ be a ball and let $\Omega \subset \mathbb{R}^n$ be an open set such that $B \subset\subset \Omega$. Suppose we have a linear elliptic differential operator

$$p(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$$

with coefficients a_α in $C_*^r(\Omega)$, where $r > 0$. Suppose in addition that $p(x,D)u = f$ in $\mathcal{D}'(\Omega)$, where $u \in C_*^{m-r+\varepsilon}(\Omega)$ for some $0 < \varepsilon < 2r$ and $f \in C_*^\mu(\Omega)$ for some $-r < \mu \leq r$. Then $u \in C_*^{m+\mu}(B)$.

Remark 33. By the Proposition 9 above, as $-r < -r+\varepsilon < r$, we know that

$$p(x,D) : C_*^{m-r+\varepsilon}(\Omega) \to C_*^{-r+\varepsilon}(\Omega)$$

so in particular $p(x,D)u \in C_*^{-r+\varepsilon}(\Omega)$ and the equation $p(x,D)u = f$ in $D'(\Omega)$ makes sense.

Proof. For simplicity we assume that B is centered at 0. Let B' another ball centered at 0 such that $B \subset\subset B' \subset\subset \Omega$, and let $\chi \in C_c^\infty(B')$ such that $\chi = 1$ in a neighborhood of $\overline{B}$. Write $v := \chi u$. Then we have

$$\begin{aligned} p(x,D)v &= \chi \sum_{|\alpha|\le m} p_\alpha(x) D^\alpha u(x) + \sum_{|\alpha|\le m} \sum_{\substack{|\gamma|+|\eta|=|\alpha| \\ |\eta|\le|\alpha|-1}} p_\alpha(x) D^\gamma \chi D^\eta u = \chi f + \hat{f} \\ \hat{f} &:= \sum_{|\alpha|\le m} \sum_{\substack{|\gamma|+|\eta|=|\alpha| \\ |\eta|\le|\alpha|-1}} p_\alpha(x) D^\gamma \chi D^\eta u \end{aligned} \tag{59}$$

Note that in the sum above $|\eta| \le |\alpha| - 1 \le m-1$, and $\hat{f}$ is a sum of products of a $C_*^r(\Omega)$ function p_α, a $C_c^\infty(\Omega)$ function $D^\gamma\chi$, and a $C_*^{1-r+\varepsilon}(\Omega)$ function $D^\eta u$. We claim that $\hat{f} \in C_*^\sigma(\mathbb{R}^n)$ for $\sigma := min\{r, 1-r+\varepsilon\}$.

Indeed, if $1-r+\varepsilon > 0$, by Proposition 16 we see that $\hat{f} \in C_*^\sigma(\mathbb{R}^n)$. On the other hand, if $1-r+\varepsilon < 0$, we have that $r > |1-r+\varepsilon| = r-1-\varepsilon$, so this time we apply Lemma 15 to conclude that $\hat{f} \in C_*^\sigma(\mathbb{R}^n)$.

In order to apply Theorem 12 we need to extend the coefficients $p_\alpha \in C_*^r(\Omega)$ to all $\mathbb{R}^n$ in such a way that the extensions $\hat{p}_\alpha$ are $C_*^r(\mathbb{R}^n)$ and besides the extended symbol $\hat{p}(x,\xi) = \hat{p}_\alpha(x)\xi^\alpha$ is elliptic (in the sense of pseudodifferential operators). To do this, consider K the inversion of the ball B'. If r is the radious of B', K has the formula

$$K(x) = r^2 \frac{x}{|x|^2}. \tag{60}$$

Note that from this formula we see that K and all its derivatives are uniformly bounded away from zero, so $K \in C^{\infty+0}(\mathbb{R}^n \backslash B)$. Define now

$$q_\alpha(x) = \begin{cases} p_\alpha(x) & x \in B' \\ p_\alpha(K(x)) & \text{otherwise} \end{cases}$$

As the principal symbol $p_m(x,\xi)$ of $p(x,D)$ is non zero for $x \in \Omega, \xi \neq 0$, it is clear that $q_m(x,\xi) := \sum_{|\alpha|=m} q_\alpha(x)\xi^\alpha$ satisfies that $q_m(x,\xi) \neq 0$ for $x \in \mathbb{R}^n$, $\xi \neq 0$. However, the functions q_α could have a corner in $\partial B'$, so we need to smooth them near $\partial B'$ with minimal changes elsewhere. For this, take $\hat{\chi} \in C_c^\infty(B')$ with $\hat{\chi} = 1$ in a neighborhood of $supp(\chi)$, and define

$$\hat{p_\alpha} := \rho_\delta \star ((1-\hat{\chi})q_\alpha) + \hat{\chi} q_\alpha$$

where $\rho_\delta := \delta^{-n}\rho(x/\delta)$, for some $\rho \in C_c^\infty(B(0,1))$ such that $\int \rho = 1$, are a family of standard mollifiers.

We claim that $\hat{p_\alpha}$ converges uniformly to q_α in $\mathbb{R}^n$ as $\delta \to 0$.

To see this, we shall see first that q_α is bounded and uniformly continuous in $\mathbb{R}^n$. Indeed take the compact $C = \overline{B' + B(0,1)}$. Then q_α is uniformly continuous in C, as it is a continuous function defined in a compact. On the other hand, it is straighforward looking at (60) that the differential of the function K is uniformly bounded $\mathbb{R}^n \backslash B'$, so K is Lipschitz in $\mathbb{R}^n \backslash B'$, and then $p_\alpha \circ K$ is uniformly continuous there. Now, given $x, y \in \mathbb{R}^n$, with $|x-y| < 1$, either both x, y are in C or in $\mathbb{R}^n \backslash B'$, and this gives that q_α is uniformly continuous. That q_α is bounded is obvious.

So, we see that $(1-\hat{\chi})q_\alpha$ is bounded and uniformly continuous in $\mathbb{R}^n$, and therefore we have that $\rho_\delta \star ((1-\hat{\chi})q_\alpha)$ converges to $(1-\hat{\chi})q_\alpha$ uniformly in all $\mathbb{R}^n$ as $\delta \to 0$. This can be seen by the classic computation that shows that for any function f it holds

$$|(\rho_\delta \star f)(x) - f(x)| = \left| \int_{B_\delta} \rho_\delta(x-y)[f(y)-f(x)]dy \right| \le \sup_{x,y\in\mathbb{R}^n} \{|f(x)-f(y)| : |x-y| \le \delta\}$$

Therefore, also $\hat{p_\alpha} = \rho_\delta \star ((1-\hat{\chi})q_\alpha) + \hat{\chi} q_\alpha$ coverges to $q_\alpha = (1-\hat{\chi})q_\alpha + \hat{\chi} q_\alpha$ uniformly in all $\mathbb{R}^n$ as $\delta \to 0$, as we wanted to see.

Now we shall see that $\hat{p_\alpha} \in C^r_*(\mathbb{R}^n)$ if δ is small. Of course $\hat{p_\alpha}$ is smooth, but we need to see the boundness of the derivatives up to order k for any integer $k \leq r$ ($\leq r-1$ if r is integer) , and the boundness of the difference quotients.

Since $\hat{\chi} q_\alpha \in C^r_*(\mathbb{R}^n)$, it suffices to see that $\rho_\delta \star ((1-\hat{\chi})q_\alpha) \in C^r_*(\mathbb{R}^n)$. To see this, take a compact F such that $\Omega \subset F$, and note that obviously $\rho_\delta \star ((1-\hat{\chi})q_\alpha) \in C^r_*(F)$ since it is smooth in $\mathbb{R}^n$.

That said, note that $(1-\hat{\chi})q_\alpha(x) = q_\alpha(x) = p_\alpha(K(x))$ for x outside Ω. As K is $C^{\infty+0}$ outside Ω and p_α is C^r_* in B, we have that

$$(1-\hat{\chi})q_\alpha \in C^r_*(V) \text{ for some neighborhood } V \text{ of } \mathbb{R}^n \backslash F$$

We will need this neighborhood V to mollify well later.

We claim that this implies that $\rho_\delta \star ((1-\hat{\chi})q_\alpha)$ is $C^r_*(\mathbb{R}^n \backslash F)$. Indeed, for each integer k with $k < r-1$ and every $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k$, we know that $D^\alpha[(1-\hat{\chi})q_\alpha])$ exists on V. Then, for δ so small that $\mathbb{R}^n \backslash F + B(0,\delta) \subset V$, and for $x \in \mathbb{R}^n \backslash F$, we have

$$\begin{aligned}
&|D^\alpha[\rho_\delta \star ((1-\hat{\chi})q_\alpha)](x)| = |(\rho_\delta \star (D^\alpha[(1-\hat{\chi})q_\alpha]))(x)| \\
&\leq \|D^\alpha[(1-\hat{\chi})q_\alpha]\|_{L^\infty(V)} \|\rho_\delta\|_{L^1} = \|D^\alpha[(1-\hat{\chi})q_\alpha]\|_{L^\infty(V)}
\end{aligned}$$

This shows that $D^\alpha[\rho_\delta \star ((1-\hat{\chi})q_\alpha)]$ is bounded in $\mathbb{R}^n \backslash F$. If r is not integer, this applies also for $k < r$. So the derivatives are bounded, and it remains to see that the difference quotients also are.

Let us see that the difference quotients of the higher derivatives are well bahaved. This follows by linearity. Indeed, suppose $r \notin \mathbb{N}$ and let $D^\alpha[(1-\hat{\chi})q_\alpha]$ be any of this higher derivatives, i.e, with $|\alpha|$ equal to the biggest integer smaller that r. Then we have, for $x, y \in \mathbb{R}^n \backslash F$

$$\begin{aligned}
&|D^\alpha[\rho_\delta \star ((1-\hat{\chi})q_\alpha)](x) - D^\alpha[\rho_\delta \star ((1-\hat{\chi})q_\alpha)](y)| \\
&= |(\rho_\delta \star (D^\alpha[(1-\hat{\chi})q_\alpha]))(x) - (\rho_\delta \star (D^\alpha[(1-\hat{\chi})q_\alpha]))(y)| \\
&\leq \int \rho_\delta(\sigma) |D^\alpha[(1-\hat{\chi})q_\alpha](x-\sigma) - D^\alpha[(1-\hat{\chi})q_\alpha](y-\sigma)| d\sigma \leq [D^\alpha[(1-\hat{\chi})q_\alpha]]_{C^r_*(\mathbb{R}^n \backslash F)}
\end{aligned}$$

If r is an ineteger, we make an analogous computation, and we estimate also the second order differece quotients in $\mathbb{R}^n \backslash F$ of $D^\alpha[(1-\hat{\chi})q_\alpha]$, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| = r-1$.

This shows that $\rho_\delta \star ((1-\hat{\chi})q_\alpha) \in C^r_*(\mathbb{R}^n \backslash F)$. But recall that we also saw before that $\rho_\delta \star ((1-\hat{\chi})q_\alpha) \in C^r_*(F)$. As $\rho_\delta \star ((1-\hat{\chi})q_\alpha)$ is smooth in a neighborhood of ∂F, this finally shows that $\rho_\delta \star ((1-\hat{\chi})q_\alpha) \in C^r_*(\mathbb{R}^n)$.

Summarising, we have seen that $\hat{p_\alpha} \in C^r_*(\mathbb{R}^n)$ if δ is small, and that $\hat{p_\alpha}$ converges uniformly to q_α in $\mathbb{R}^n$ as $\delta \to 0$, being

$$\hat{p_\alpha} := \rho_\delta \star ((1-\hat{\chi})q_\alpha) + \hat{\chi} q_\alpha$$

Consider now $\hat{p}_m(x,\xi) = \sum_{|\alpha|=m} \hat{p_\alpha}(x)\xi^\alpha$. We want to see that this symbol satisfies the ellipticity conditions.

By hipothesis, we know that $p_m(x,\xi) : \Omega \times \{|\xi| = 1\} \to \mathbb{R}$ never vanishes, so it has a minumun $\gamma > 0$ in $B' \times \{|\xi| = 1\}$. This γ is also a minumun of $q_m(x,\xi) = \sum_{|\alpha|=m} q_\alpha(x)\xi^\alpha$ in $\mathbb{R}^n \times \{|\xi| = 1\}$.

Now, as $\hat{p_\alpha}$ converges uniformly to q_α in $\mathbb{R}^n$ as $\delta \to 0$, it is clear that also $\hat{p}_m(x,\xi)$ converges uniformly to $q_m(x,\xi)$ in $\mathbb{R}^n \times \{|\xi| = 1\}$ as $\delta \to 0$, so for δ small we know that $\hat{p}_m(x,\xi) \geq \frac{1}{2}\gamma$ if $(x,\xi) \in \mathbb{R}^n \times \{|\xi| = 1\}$. By homogenety in ξ, this shows that

$$\hat{p}_m(x,\xi) \geq \frac{1}{2}\gamma|\xi|^m \text{ for all } x, \xi \in \mathbb{R}^n$$

and therefore $\hat{p}_m(x,\xi)$ satisfies the elliticity conditions, so the differential operator $\hat{p}(x,D)$ is elliptic (in the sense of pseudodifferential operators) with principal symbol $\hat{p}_m(x,\xi) \in C^r_* S^m_{1,0}(\mathbb{R}^n)$.

Now, as $\hat{\chi} = 1$ in a neighborhood of $supp(\chi)$, we have that, if δ is so small that $supp(\chi) + B(0,2\delta) \subset \{\hat{\chi} = 1\}$, then $p_\alpha = \hat{p_\alpha}$ in a neighboorhood of $supp(\chi)$. Then, being as before $v = \chi u$, we remind that, by the computations in (59), we have

$$\hat{p}(x,D)v = p(x,D)v = \chi f + \hat{f} := g$$

with $\hat{f} \in C^\sigma_*(\mathbb{R}^n)$, being $\sigma = min\{1-r+\varepsilon, r\}$, $v \in C^{m-r+\varepsilon}_*(\mathbb{R}^n)$, and $\chi f \in C^\mu_*(\mathbb{R}^n)$, $-r < \mu \leq r$. Now we distinguish cases.

If $0 < r \leq \frac{1}{2}(1+\varepsilon)$, then $\hat{f} \in C^r_*(\mathbb{R}^n)$, and therefore $g = \chi f + \hat{f} \in C^\mu_*(\mathbb{R}^n)$. Now we can use Theorem 12 to conclude that $v \in C^{m+\mu}_*(\mathbb{R}^n)$, so, as $\chi = 1$ in a neighborhood of B, we see that $u \in C^{m+\mu}_*(B)$, and the proposition is proved.

If $r > \frac{1}{2}(1+\varepsilon)$, then $\hat{f} \in C^{1-r+\varepsilon}_*(\mathbb{R}^n)$, so $g \in C^{min\{1-r+\varepsilon,\mu\}}$. As $-r < min\{1-r+\varepsilon,\mu\} < r$, we can use Theorem 12 and conclude that $v \in C^{m+min\{\mu,1-r+\varepsilon\}}_*(\mathbb{R}^n)$. If $\mu \leq 1-r+\varepsilon$ we have $v \in C^{m+\mu}_*(\mathbb{R}^n)$, so $u \in C^{m+\mu}_*(B)$ and we are done. Otherwise we keep going. If $1-r+\varepsilon < \mu$, then $v \in C^{m+1-r+\varepsilon}_*(\mathbb{R}^n)$, so $u \in C^{m+1-r+\varepsilon}_*(B)$. Thus, u has gained 1 derivative, so we have to repeat the argument, using that u has gained 1 derivative.

Recalling that $\hat{p_\alpha} = p_\alpha$ in a neighborhood of $supp(\chi)$, from (59) we see that

$$\hat{f} := \sum_{|\alpha|\leq m} \sum_{\substack{|\gamma|+|\eta|=|\alpha| \\ |\eta|\leq|\alpha|-1}} \hat{p_\alpha}(x) D^\gamma \chi D^\eta u$$

As $\hat{p_\alpha} \in C^r_*(\mathbb{R}^n)$ and $D^\gamma \chi D^\eta u \in C^{2-r+\varepsilon}_*(\mathbb{R}^n)$, we conclude that $\hat{f} \in C^{min\{2-r+\varepsilon,r\}}_*(\mathbb{R}^n)$ by the same argument we gave before, i.e, if $2-r+\varepsilon > 0$ we use Proposition 16 and if $2-r+\varepsilon \leq 0$ we use Lemma 15 since $|2-r+\varepsilon| = r-2-\varepsilon < r$.

Now, if $0 < r \leq \frac{1}{2}(2+\varepsilon)$, then $\hat{f} \in C^r_*(\mathbb{R}^n)$, so $g = \chi f + \hat{f} \in C^\mu_*(\mathbb{R}^n)$ and therefore by Theorem 12, $v \in C^{m+\mu}_*(\mathbb{R}^n)$, so $u \in C^{m+\mu}_*(B)$.

If $r > \frac{1}{2}(2+\varepsilon)$, then $\hat{f} \in C^{2-r+\varepsilon}_*(\mathbb{R}^n)$, so $g \in C^{min\{2-r+\varepsilon,\mu\}}$. As $-r < min\{2-r+\varepsilon,\mu\} < r$, we use Theorem 12 to conclude that $v \in C^{m+min\{\mu,2-r+\varepsilon\}}_*(\mathbb{R}^n)$. If $\mu \leq 2-r+\varepsilon$ we have $v \in C^{m+\mu}_*(\mathbb{R}^n)$, so $u \in C^{m+\mu}_*(B)$ and we are done. Otherwise we keep going. If $2-r+\varepsilon < \mu$, then $v \in C^{m+2-r+\varepsilon}_*(\mathbb{R}^n)$, so $u \in C^{m+2-r+\varepsilon}_*(B)$. Thus, u has gained 1 more derivative, and we have to repeat the argument.

After repeating the argument $j \in \mathbb{N}$ times, we must get that either $0 < r \leq \frac{1}{2}(j+\varepsilon)$ or $r > \frac{1}{2}(j+\varepsilon)$ and $\mu \leq j-r+\varepsilon$. In any case we conclude $u \in C^{m+\mu}_*(B)$. This proves the proposition. □

In the future we will need to deal with regularity problems not only for a PDE alone, but also for systems of PDE'S. To tackle this problem, we need to define differential operators with matrix coefficients and the corresponding pseudodifferential operators with matrix-valued symbols. The notion of ellipticity is this setting follows the same idea, i.e, a differential operator $p(x,D)$ with matrix coefficients will be elliptic if we are able to find some kind of inverse $a(x,D)$ so that $a(x,D)p(x,D)$ is similar to the identity. Therefore we will require the symbol of $p(x,D)$ to be a left invertible matrix (i.e, inyective).

Let us see heuristically the idea. Suppose we have the system of PDE'S given by $p(x,D)u = f$. If we apply $a(x,D)$, we see that $u = a(x,D)f$. Therefore, if $a(x,D)f$ is smoother than f was (i.e, if $a(x,D)$ is an smoother operator), then we see that u gains regularity. The notion of smoother operator is related with the decay of its symbol at infinity in the ξ variable. Therefore, as in the case of scalar symbols, we shall require that the left inverse of the symbol of $p(x,D)$ has a good decay at infinity in the ξ variable.

All the definitions are totally analogous for matrix coefficients. We illustrate this by an example. We will try to use capital letters for matrix valued differential and pseudodifferential operators and for its associated symbols.

Example 4. Consider the system of PDE'S given by the Cauchy-Riemannn equations. For $f(x_1,x_2) = (u(x_1,x_2), v(x_1,x_2)) : \mathbb{R}^2 \to \mathbb{R}^2$ we consider the system

$$\begin{aligned}&\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}; \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1} \quad \text{equivalent to}\\ &P_1\left[(u_{x_1}, v_{x_1})\right]^t + P_2\left[(u_{x_2}, v_{x_2})\right]^t = (0,0)^t \qquad (61)\\ &P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; P_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

So the differential operator associated to this system in multiindex notation is given by $P(x,\partial) = P_1\partial^{(1,0)} + P_2\partial^{(0,1)}$, whose symbol is $P(x,\xi) = P_1\xi_1 + P_2\xi_2$, and therefore

$$P(x,\xi) = \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix} \quad ; \quad [P(x,\xi)]^{-1} = |\xi|^{-2}\begin{pmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{pmatrix}$$

Note that the determinant of $P(x,\xi)$ is $|\xi|^2$ so it is invertible and the formula above holds. Besides it is straightforward that $|P(x,\xi)^{-1}| \leq |\xi|^{-1}$, where for any $n \times m$ matrix A we denote $|A|$ for the norm of A regarded as an element of $\mathbb{R}^{nm}$.

We conclude that the Cauchy-Riemann equations can be regarded as an elliptic differential operator or oder 1. Indeed, it is well known from complex analysis that any C^1 function (u,v) satisfying these equations is smooth (in fact analytic), so the elliptic regularity results are satisfied.

Now we briefly define the notions for matrix psedodifferential and differential operators, recalling that thery are analogous to the scalar case. We shall explicitly give here only the definitions that we will use later

Definition 36. Let $\Omega \subset \mathbb{R}^n$ be open. A matrix-coefficient differential operator of order m in Ω is an expression $P(x,D)$ of the form

$$P(x,D) = \sum_{|\alpha|\leq m} P_\alpha(x)D^\alpha$$

where $\alpha \in \mathbb{N}^n$ and $P_\alpha(x) : \Omega \to \mathbb{C}^{kl}$ are $k \times l$ complex matrix valued functions of the form

$$P_\alpha(x) = \begin{pmatrix} p_{11}(x) & p_{12}(x) & \dots & p_{1l}(x) \\ p_{21}(x) & p_{22}(x) & \dots & p_{2l}(x) \\ \vdots & \vdots & \vdots & \vdots \\ p_{k1}(x) & p_{k2}(x) & \dots & p_{kl}(x) \end{pmatrix}$$

The differential operator $P(x,D)$ acts on vectorial functions $u = (u_1, \dots, u_l)$ with values in $\mathbb{C}^l$ by the usual matrix multiplication, so $P(x,D)u = P_\alpha(x)D^\alpha u$ is a function with values in $\mathbb{C}^k$.

The symbol $P(x,\xi)$ of P and its homogeneous part $P_l(x,\xi)$ of dedree l are the matrix coefficient polinomials given by

$$P(x,\xi) = \sum_{|\alpha|\leq m} P_\alpha(x)\xi^\alpha \quad ; \quad P_l(x,\xi) = \sum_{|\alpha|=l} P_\alpha(x)\xi^\alpha$$

The principal part of $P(x,\xi)$ is $P_m(x,\xi)$, i.e, its homogeneous part of higher degree.

Moreover we have analogous concepts of ellipticity.

(1) If $k=l$ then $P_m(x,\xi)$ is an square matrix and we say that the operator $P(x,D)$ is *elliptic* provided that $P_m(x,\xi)$ is an invertible matrix for $\xi\neq 0$.

(2) If $l\leq k$ we say that $P(x,D)$ is *overdetermined elliptic* if the symbol $P_m(x,\xi)$ has l linearly independent columns (is inyective) for $\xi\neq 0$.

Definition 37. For $m\in\mathbb{R}$, $\rho,\delta\in[0,1]$, and $r>0$ we define the class of matrix symbols $C^r_*S^m_{\rho,\delta}(\mathbb{R}^n;\mathbb{C}^{k\times l})$ as the $k\times l$ complex matrix valued functions

$$P(x,\xi)=(p_{ij}(x,\xi))_{i,j=1,1}^{k,l}$$

such that every of its components $p_{ij}(x,\xi)$ belongs to the scalar class of $C^r_*S^m_{\rho,\delta}(\mathbb{R}^n)$ symbols. This is equivalent to the conditions

(1) $\|D^\alpha_\xi P(\cdot,\xi)\|_{L^\infty(\mathbb{C}^{kl})}\leq C_\alpha\langle\xi\rangle^{m-\rho|\alpha|}$ for some constant $C_\alpha>0$.

(2) $\|D^\alpha_\xi P(\cdot,\xi)\|_{C^r_*(\mathbb{R}^n;\mathbb{C}^{kl})}\leq C_\alpha\langle\xi\rangle^{m-\rho|\alpha|+\delta r}$.

Recall that the norms in (1) and (2) are the corresponding matrix norms. One can take any matrix norm since all of them are equivalent, but we could just think on vector valued functions $P(x,\xi)$ instead of matrix-valued functions and take the Euclidean norm for the matrixes, considered as elements of $\mathbb{C}^{kl}$.

The pseudodifferential operator $P(x,D)$ associated to a symbol $P(x,\xi)$ acting on a $\mathbb{C}^l$ valued function $u=(u_1,\ldots,u_l)$ is given by

$$P(x,D)u(x)=\int P(x,\xi)\hat{u}(\xi)e^{2\pi i x\cdot\xi}d\xi=\left(\sum_{a=1}^{l}p_{1,a}(x,D)u_a(x),\ldots,\sum_{a=1}^{l}p_{k,a}(x,D)u_a(x)\right) \tag{62}$$

Note that the integration is done component by component, so $\hat{u}=(\hat{u_1},\ldots,\hat{u_l})$ is also an $\mathbb{C}^l$ valued function, and $P(x,\xi)\hat{u}(\xi)\in\mathbb{C}^k$. Therefore also $P(x,\xi)u(x)$ is a $\mathbb{C}^k$ valued function in the x-variable. Let us define the notions of ellipticity.

(1) If $k=l$ then $P(x,\xi)$ is an square matrix, we say that the operator $P(x,D)$ is *elliptic* provided there exists $R>0$ big enough so that for $|\xi|\geq R$ we have that $P(x,\xi)$ is an invertible matrix and besides $|P(x,\xi)^{-1}|\leq C|\xi|^{-m}$.

(2) If $l\leq k$ we say that $P(x,D)$ is *overdetermined elliptic* if the symbol $P(x,\xi)$ has l linearly independent columns for $|\xi|\geq R$, (i.e, if $P(x,\xi)$ is inyective for $|\xi|\geq R$) and we demand also that, if for $|\xi|>R$ we call $A(x,\xi)$ to the left inverse of $P(x,\xi)$ such that $A(x,\xi)P(x,\xi)=Id$, then we have the estimate $|A(x,\xi)|\leq C|\xi|^{-m}$.

(3) If $l\leq k$ we say that $P(x,D)$ is *uniformly overdetermined elliptic* if there exists $R>0$ such that for $|\xi|\geq R$ we have that $P(x,\xi)$ is inyective and besides

$$(\zeta,P(x,\xi)^*P(x,\xi)\zeta):=\zeta^*\cdot P(x,\xi)^*P(x,\xi)\zeta\geq C|\zeta|^2|\xi|^{2m} \tag{63}$$

for $\xi\in\mathbb{R}^n$, $|\xi|\geq R$ and $\zeta\in\mathbb{C}^l$. Here $(\cdot,\cdot)$ denotes the hermitian product in $\mathbb{C}^n$, and $P(x,\xi)^*:=\overline{P(x,\xi)}^t$, $\zeta^*:=\overline{\zeta}^t$ are the conjugate transpose.

Remark 34. With notation as in Definition 37 above, suppose that $P(x,\xi)$ is uniformly overdetermined elliptic. Note that $P(x,\xi)^*P(x,\xi)$ is a $l\times l$ matrix, and it as an autoadjoint operator defined on $\mathbb{C}^l$ with its Hermitian product, so its norm is easy to compute. Therefore, taking into account (63) and the formula to compute the norm of an autoadjoint operator, we see that

$$|P(x,\xi)^*P(x,\xi)|=\sup_{|\zeta|=1}(\zeta,P(x,\xi)^*P(x,\xi)\zeta)\geq C|\xi|^{2m}$$

On the other hand, it is well known that $|P(x,\xi)^*P(x,\xi)| \approx |P(x,\xi)|^2$ (this means that they are equivalent up to universal constants, and can be seen by considering the spectral norm for example), so $|P(x,\xi)| \geq C|\xi|^m$. Now consider, for $|\xi| \geq R$, the left inverse $A(x,\xi)$ of $P(x,\xi)$. Then we have

$$C = |Id| = |A(x,\xi)P(x,\xi)| \leq C_1|A(x,\xi)||P(x,\xi)|$$

so we see that $|A(x,\xi)| \leq C|P(x,\xi)|^{-1} \leq C_1|\xi|^{-m}$, so as expected $P(x,\xi)$ is elliptic.

Remark 35. Suppose we have a linear $k \times l$ matrix valued differential operator $P(x,D)$ of order m with symbol $P(x,\xi)$ given by

$$P(x,D) = \sum_{|\alpha|\leq m} A_\alpha(x)D^\alpha \quad ; \quad P(x,\xi) = \sum_{|\alpha|\leq m} A_\alpha(x)\xi^\alpha$$

and with bounded coefficients $A_\alpha(x)$. Suppose that $P(x,D)$ is overdetermined elliptic in the sense of differential operators. We claim that also $P(x,D)$ is uniformly overdetermined elliptic in the pseudodifferential sense. To see this note that as $P_m(x,\xi)$ is inyective for $|\xi| \geq R$, by homogeneity in ξ we see that in fact $P_m(x,\xi)$ is inyective for all $\xi \neq 0$.

That said, we claim that for $\xi \neq 0$, $P_m(x,\xi)^*P_m(x,\xi)$ is an invertible $l \times l$ matrix. Indeed, if $P_m(x,\xi)^*P_m(x,\xi)\zeta = 0$, for some $\zeta \in \mathbb{C}^l$, then we have

$$0 = \zeta^* \cdot P_m(x,\xi)^*P_m(x,\xi)\zeta = \zeta^*P_m(x,\xi)^* \cdot P_m(x,\xi)\zeta = |P_m(x,\xi)\zeta|^2$$

so, as $P_m(x,\xi)$ is inyective, we see that $\zeta = 0$. Therefore $P_m(x,\xi)^*P_m(x,\xi)$ is inyective, and hence invertible. Moreover we can take

$$C := \inf\{\zeta^* \cdot P_m(x,\xi)^*P_m(x,\xi)\zeta : |\xi| = |\zeta| = 1\} = \inf\{|P(x,\xi)\zeta|^2 : |\xi| = |\zeta| = 1\} > 0$$

As $\zeta^* \cdot P_m(x,\xi)^*P_m(x,\xi)\zeta$ is homogeneous of degree $2m$ in ξ and degree 2 in ζ, we see that $P_m(x,\xi)$ satisfies estimate (63) for every $\xi \neq 0$. Moreover, $|\xi|$ is big enough, $P_m(x,\xi)$ dominates the behaviour of the entire symbol $P(x,\xi)$, so using that $A_\alpha(x)$ are bounded, the same calculation done for scalar symbols show that $P(x,\xi)$ also satisfies estimate (63), so $P(x,D)$ is uniformly overdetermined elliptic.

Now we shall check that the mapping properties of scalar pseudodifferential operator have analogous here.

Theorem 13. Let $r > 0$, $\delta \in [0,1)$, $m \in \mathbb{R}$ and $P(x,\xi) \in C^r_*S^m_{1,\delta}(\mathbb{R}^n;\mathbb{C}^{k\times l})$ be an $k \times l$ complex matrix valued symbol. Suppose that $s \in \mathbb{R}$ satisfies $-(1-\delta)r < s < r$. Then we have

$$P(x,D) : C^{m+s}_*(\mathbb{R}^n;\mathbb{C}^l) \to C^s_*(\mathbb{R}^n;\mathbb{C}^k)$$

Also, there exists a constant C so that for every $\mathbb{C}^l$ valued function u so that $u \in C^{s+m}_*(\mathbb{R}^n;\mathbb{C}^l)$ we have

$$\|P(x,D)u\|_{C^s_*(\mathbb{R}^n;\mathbb{C}^k)} \leq C\|u\|_{C^{m+s}_*(\mathbb{R}^n;\mathbb{C}^l)}.$$

Proof. Recall that $P(x,D)$ acts by multipliying $P(x,\xi)$ and $\hat{u}$, and then each component of $P(x,D)u$ is a sum of scalar psedodifferential operators acting on the the components of u, as noted in (62). Therefore this result is an inmediate consecuence for the scalar case given in Theorem 11. □

We have the corresponding version for linear matrix valued pseudodifferential operators defined in an open set, taqking $\delta = 1$ in the prevous Theorem 13.

Proposition 11. Let $\Omega \subset \mathbb{R}^n$ be an open set and $r > 0$. Suppose we have a linear $k \times l$ matrix valued differential operator

$$P(x, D) = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha$$

with coefficients A_α in $C_*^r(\Omega; \mathbb{C}^{k\times l})$. Let $s \in \mathbb{R}$ satisfy $-r < s < r$. Then we have the mapping property

$$P(x, D) : C_*^{m+s}(\Omega; \mathbb{C}^l) \to C_*^s(\Omega; \mathbb{C}^k)$$

So there exists a constant C so that for every $u \in C_*^{m+s}(\Omega; \mathbb{C}^l)$ we have

$$\|P(x, D)u\|_{C_*^s(\Omega;\mathbb{C}^k)} \leq C\|u\|_{C_*^{m+s}(\Omega;\mathbb{C}^l)}.$$

Proof. This result is also an inmediate consequence of the scalar case given in Proposition 9, taking into account how $P(x, D)$ acts on u. □

Now we give a generalization of Theorem 12 for uniformly overdetermined elliptic operators.

Theorem 14. Let $s > 0$, $m \in \mathbb{R}$ and let $P(x, \xi) \in C_*^s S_{1,0}^m(\mathbb{R}^n; \mathbb{C}^{k\times l})$ be a $k \times l$ complex matrix valued symbol. Suppose $P(x, \xi)$ is uniformly overdetermined elliptic as in (63). Suppose that $u \in C_*^{m-s+\varepsilon}(\mathbb{R}^n; \mathbb{C}^l)$ for some $0 < \varepsilon < 2s$, and suppose that $f \in C_*^r(\mathbb{R}^n; \mathbb{C}^k)$ for some $-s < r \leq s$.

Under these hipothesis, if u is a solution (in the distributional sense) of the equation

$$P(x, D)u = f$$

then actually $u \in C_*^{m+r}(\mathbb{R}^n; \mathbb{C}^l)$.

Proof. The proof of this Theorem is not an inmediate consequence of the scalar case since in this case the notion of ellipticity involves how the matrix of $P(x, \xi)$ is in global terms, and not how its components are. It must be checked that the proof of Theorem 12 can be adapted for these matrix symbols, which is not difficult because the hard work is done to prove the scalar case (actually the notion of ellipticity for matrix symbols is defined so that the proof for the scalar case can be adapted for matrixes). The details are given in [15], Appendix A, Proposition A.2. □

Finally we arrive to the result in which we are really interested. It is an easy modification of Proposition 10.

Proposition 12. Let $B \subset \mathbb{R}^n$ be a ball and let $\Omega \subset \mathbb{R}^n$ be an open set such that $B \subset\subset \Omega$. Suppose we have a linear $k \times l$ matrix valued differential operator

$$P(x, D) = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha$$

with coefficients A_α in $C_*^r(\Omega; \mathbb{C}^{k\times l})$, where $r > 0$. Suppose that $P(x, D)$ is overdetermined elliptic in the sense of differential operators (i.e that its principal symbol $P_m(x, \xi)$ is inyective for $\xi \neq 0$).

Suppose also that $P(x, D)u = f$ in $\mathcal{D}'(\Omega)$, where $u \in C_*^{m-r+\varepsilon}(\Omega; \mathbb{C}^l)$ for some $0 < \varepsilon < 2r$ and $f \in C_*^\mu(\Omega; \mathbb{C}^k)$ for some $-r < \mu \leq r$. Then $u \in C_*^{m+\mu}(B; \mathbb{C}^l)$.

Remark 36. By the Proposition 11 above, as $-r < -r + \varepsilon < r$, we know that

$$P(x, D) : C_*^{m-r+\varepsilon}(\Omega; \mathbb{C}^l) \to C_*^{-r+\varepsilon}(\Omega; \mathbb{C}^k)$$

so in particular $P(x, D)u \in C_*^{-r+\varepsilon}(\Omega; \mathbb{C}^k)$ and the equation $P(x, D)u = f$ in $D'(\Omega)$ makes sense.

Proof. We shall see that the proof of the scalar case given in Proposition 10 can be adapted here without a problem. Assume that B is centered at 0. Let B' another ball centered at 0 such that $B \subset\subset B' \subset\subset \Omega$, and let $\chi \in C_c^\infty(B')$ such that $\chi = 1$ in a neighborhood of $\overline{B}$. Write $v := \chi u$ as in the scalar case. We have

$$P(x,D)v = \chi f + \hat{f}$$
$$\hat{f} := \sum_{|\alpha|\leq m} \sum_{\substack{|\gamma|+|\eta|=|\alpha| \\ |\eta|\leq|\alpha|-1}} P_\alpha(x) D^\gamma \chi D^\eta u \tag{64}$$

Note that in the sum above $\hat{f}$ is a sum of products of a $C_*^r(\Omega;\mathbb{C}^{k\times l})$ function P_α, a $C_c^\infty(\Omega)$ function $D^\gamma\chi$, and a $C_*^{1-r+\varepsilon}(\Omega;\mathbb{C}^l)$ function $D^\eta u$. In the scalar case (when $k = l = 1$) we saw that $\hat{f} \in C_*^\sigma(\mathbb{R}^n)$ for $\sigma := min\{r, 1-r+\varepsilon\}$ and obviously it is also true for matrixes.

Now we shall extend the coefficients $P_\alpha \in C_*^r(\Omega;\mathbb{C}^{k\times l})$ to $\hat{P}_\alpha \in C_*^r(\mathbb{R}^n;\mathbb{C}^{k\times l})$ so that the extended symbol $\hat{P}(x,\xi) = \hat{P}_\alpha(x)\xi^\alpha$ is uniformly overdetermined elliptic. We will follow the same procedure as in the scalar case. Denote K the inversion of the ball B', and define

$$Q_\alpha(x) = \begin{cases} P_\alpha(x) & x \in B' \\ P_\alpha(K(x)) & \text{otherwise} \end{cases}$$

Let ρ_δ a standard approximation of the dirac delta and set $\hat{P}_\alpha := \rho_\delta \star ((1-\hat{\chi})Q_\alpha) + \hat{\chi}Q_\alpha$, where the convolution is done component by component. As done in Proposition 10 we see that for $\delta \to 0$ we have that $\hat{P}_\alpha$ converges to Q_α uniformly in $\mathbb{R}^n$, and besides $\hat{P}_\alpha \in C_*^r(\mathbb{R}^n;\mathbb{C}^{k\times l})$.

Consider now the symbol given by

$$\hat{P}(x,\xi) = \sum_{|\alpha|\leq m} \hat{P}_\alpha(x)\xi^\alpha$$

We want to see that this symbol is uniformly overdetermined elliptic, i.e, that satisfies estimate (63). As mentioned before, it is enough to see that the principal symbol $\hat{P}_m(x,\xi)$ is inyective for $\xi \neq 0$.

By hipothesis, $P_m(x,\xi)$ is injective for $(x,\xi) \in \Omega \times \{|\xi| = 1\} \to \mathbb{C}^{k\times l}$. As being inyective is locally expressed as having an invertible submatrix of range $l \times l$, we see that there exists $\gamma > 0$ and there exists a finite open cover A_i so that

$$A_i = U_i \times V_i \subset\subset \Omega \times \{|\xi| = 1\} \quad \text{and} \quad B' \times \{|\xi| = 1\} \subset \bigcup_i A_i$$

and such that in every A_i we can find a submatrix $M_i(x,\xi)$ of $P_m(x,\xi)$ whose order is $l \times l$ and with determinant $D_i := Det(M_i)(x,\xi) \geq \gamma$ in A_i.

Denote $\tilde{K} := Id\chi_{B'} + K\chi_{\mathbb{R}^n\setminus B'}$ so that $Q_\alpha = P_\alpha \circ \tilde{K}$.

By construction, it is clear that $Q_m(x,\xi) = P_m(\tilde{K}(x),\xi)$ is injective for every $(x,\xi) \in \mathbb{R}^n \times \{|\xi| = 1\}$. Besides if we consider the open cover

$$\tilde{A}_i = \tilde{K}^{-1}(U_i) \times V_i \quad \text{of} \quad \mathbb{R}^n \times \{|\xi| = 1\}$$

it is clear that in every $\tilde{A}_i$ the submatrix of $Q_m(x,\xi)$ given by $\tilde{M}_i(x,\xi) := M_i(\tilde{K}(x),\xi)$ has order $l \times l$ and determinant $\tilde{D}_i(x,\xi) := Det(\tilde{M}_i)(x,\xi) \geq \gamma$ in $\tilde{A}_i$.

As $\hat{P}_m(x,\xi)$ converges uniformly to $Q_m(x,\xi)$ in $\mathbb{R}^n \times \{|\xi| = 1\}$ as $\delta \to 0$, and as the determinat is an uniformly continuous function, we see that taking δ small enough we can achieve that, in $\tilde{A}_i$, the submatrix $\hat{M}_i$ of $\hat{P}_m(x,\xi)$ with the same entries as $\tilde{M}_i$ has determinant $\hat{D}_i := Det(\hat{M}_i) \geq \frac{1}{2}\gamma$ in $\tilde{A}_i$. In particular the principal symbol $\hat{P}_m(x,\xi)$ is injective in the set $\mathbb{R}^n \times \{|\xi| = 1\}$, and by homogeneity in all $\mathbb{R}^n \times \mathbb{R}^n$.

Therefore the differential operator $\hat{P}(x,\xi)$ is overdetermined elliptic as a differential operator, and by Remark 35 we see that also $\hat{P}(x,\xi)$ is uniformly overdetermined eliptic and besides $\hat{P}_\alpha$ coincides with P_α in a neighborhood of $supp(\chi)$, so, being $v=\chi u$ as before, we have $\hat{P}(x,D)v = P(x,D)u = \chi f + \hat{f}$.

Now we are under the hipothesis of Theorem 14. The rest of the proof is the same as the proof of Proposition 10, with a boothstrap argument this time using Theorem 14 repeteadly. □

§6. Construction of p-Harmonic Coordinates.

Now we focuss on constructing n-harmonic coordinates, which will be highly important to solve the problem of conformal fatness for low regular metrics. In fact, we prove the existence of p-harmonic coordinates por $1<p<\infty$, since the case $p=n$ does not make easier the problem. First we shall set the problem and make some computations needed later. The case $p=n$ plays an special role in conformal geometry, since the n-harmonic equation is conformally invariant. This means that a solution of the n-harmonic equation for some metric g (this equation as we shall see depends on the metric) is also a solution of the n-harmonic equation for any metric in the conformal class of g.

6.1. First Step. Weak Solution of p-Laplace Type Equations.

The first step to construct p-harmonic coordinates will be to find weak solutions to the p-Laplace equation on a Riemannian manifold. We are only interested in local results near a point x_0 in our Riemannian manifold M, so via coordinates it will be enough to work in some open set $\Omega\subset\mathbb{R}^n$. We need to see first how the p-Laplace equation is intrinsically defined on M. We shall see that the equation on M has much analogies that the ususal p-Laplace equation in $\mathbb{R}^n$.

Definition 38. Given a Riemannian manifold (M,g) and two functions $f,g\in L^2(M)$ we define its L^2 inner product as

$$(f,g) := \int_M fg dV$$

where dV is the volume form of M, whose expression in coordinates is $dV = |g|^{\frac{1}{2}} dx^1\wedge\cdots\wedge dx^n$. In a similar matter, for 1-forms $\alpha,\beta\in L^2(M)$ we define its L^2 inner product as

$$(\alpha,\beta) := \int_M g(\alpha,\beta) dV$$

where we remind that $g(\alpha,\beta) := g(\alpha^{\#},\beta^{\#})$ by definition.

In $\mathbb{R}^n$ the p-Laplace equation is defined as $div(|grad(u)|^{p-2}grad(u)) = 0$, and this equation can be set on a Riemannian manifold with no changes. In remark 13 we saw that, in local coordinates, if $X = b^j\partial_j$ is a vector field, then

$$\begin{aligned} div(X) &= \partial_j b^j + b^k\Gamma^j_{kj} = \partial_j b^j + b^k\frac{1}{2}g^{ja}[\partial_k g_{ja} + \partial_j g_{ka} - \partial_a g_{kj}] \\ &= \partial_j b^j + \frac{1}{2}b^k g^{ja}\partial_k g_{ja} \end{aligned} \tag{65}$$

The last inequality for simmetry of the last two summands in the indices j,a. However, this coordinate expression is unconfortable for many things. In order to obtain a better coordinate expression we need a cupple of lemmas.

Lemma 20. Let $A\in GL(n) :=$ {invertible square matrixes of order n} and $B\in\mathcal{M}_n :=$ {matrices of ordern n}. Then $det(A+tB) = det(A) + t det(A) Tr(A^{-1}B) + O(t^2)$

Proof. First we shall see that $det(I+tB) = 1+tTr(B)+O(t^2)$. We proceed by induction on n. The case $n=1$ is trivial. Suppose the result is true for $(n-1)$-order matrices. Let $B=(b_{ij})$, and denote B^{ij} the $(n-1)$ order matrix obtained from B by supressing its i-th row and j-th column. We have

$$\begin{aligned} det(I+tB) &= (1+tb_{11})det(I+tB)^{11} + \sum_{i=2}^{n}(-1)^i tb_{i1}det(I+tB)^{i1} \\ &= (1+tb_{11})(1+tTr(B^{11}))) + O(t^2) = 1+tTr(B)+O(t^2) \end{aligned}$$

where we used that for $i>1$ the polinomial $det(I+tB)^{i1}$ is divisible by t because its i-th column is multiplied by t. Once this is proved, we use the multiplicative property of the determinant to get

$$det(A+tB) = det(A\circ I + A\circ tA^{-1}\circ B) = det(A)(1+tTr(A^{-1}\circ B)))$$

□

Remark 37. Note that $Gl(n)$ is lie group whose tangent space at every matrix $A\in GL(n)$ is $\mathcal{M}_n$ because $GL(n)$ is an open subset of $\mathbb{R}^{n^2}$. We can consider the function in $C^\infty(GL(n))$ gicen by

$$det : GL(n) \to \mathbb{R} : A \mapsto det(A)$$

Witn notations as lemma above, $B\in T_A GL(n)$, and for t small, the curve $\alpha(t) := A+tB$ is in $GL(n)$ and $\alpha'(0)=B$, $\alpha(0)=A$. We conclude that the differential at A of the determinant function is given by $(d_A det)(B) = det(A)Tr(A^{-1}B)$

Lemma 21. Let $A:(a,b)\to GL(n) : t\mapsto A(t)$ be a differentiable matrix function depending on t, where $GL(n)$. Then $\partial_t[det(A(t)] = det(A(t))Tr(A(t)^{-1}\circ A'(t))$, where $\circ$ denotes product of matrices.

Proof. By the chain rule, $\partial_t[det(A(t)] = (d_{A(t)}det)(A'(t)) = det(A(t))Tr(A(t)^{-1}A'(t))$ by the remark above. □

Lemma 22. In local coordinates, $div(X) = |g|^{-\frac{1}{2}}\partial_j[b^j|g|^{\frac{1}{2}}]$

Proof. Note first that, if $G=(g_{ij})$, then $Tr(G^{-1}\circ\partial_j G) = g^{ab}\partial_j g^{ab}$. Now we apply the chain rule and the formula to differentiate the determinant to obtain

$$\begin{aligned} |g|^{-\frac{1}{2}}\partial_j[b^j|g|^{\frac{1}{2}}] &= \partial_j b^j + |g|^{-\frac{1}{2}}b^j\frac{1}{2}|g|^{-\frac{1}{2}}|g|g^{ab}\partial_j g_{ab} \\ &= \partial_j b^j + \frac{1}{2}b^j g^{ab}\partial_j g_{ab} = div(X) \end{aligned} \tag{66}$$

where the last equality follows according to equation (65). □

Let us see how one would think that the divergence has the expresion showed in the lemma in coordinates. For this, we shall see that the divergence is the adjoint of the gradient operator $grad : C^\infty \to \Gamma(TM)$, in a sense we state now.

Proposition 13. Let u be a compactly interior supported function, i.e, u is zero outside a compact set of $int(M)$. We will note for this $u\in C_c^\infty(int(M))$. Then we have that

$$\int_M g(grad(u),X)dV = -\int_M div(X)udV$$

Proof. Ordinary calculus show us that this is true in $\mathbb{R}^n$, and it in natural to expect it on a manifold. In fact, the divergence can be defined as the operator that satisfies the above formula. To see it, let us first consider u with support contained in a coordinate domain U. Then

$$\int_M g(grad(u), X)dV = \int_U g_{jk}g^{ak}(\partial_b u)b^j|g|^{\frac{1}{2}}dx = \int_U (\partial_j u)b^j|g|^{\frac{1}{2}}dx$$
$$= [\text{boundary terms are zero}] = -\int_U u\partial_j[b^j|g|^{\frac{1}{2}}]|g|^{-\frac{1}{2}}dV$$
$$= -\int_M u\partial_j[b^j|g|^{\frac{1}{2}}]|g|^{-\frac{1}{2}}dV = -\int_M div(X)u$$

For general $u \in C_c^\infty(int(M))$ we consider a partition of unity φ_i subordinate to an open cover of coordinate patches U_i such that if $supp(u) \cap U_i \neq \emptyset$ then $U_i \subset int(M)$ (this way we make sure that boundary terms are zero). Then

$$\int_M g(grad(u), X)dV = \int_M g(grad(\sum_i \varphi_i u), X)dV = \sum_i \int_M g(grad(\varphi_i u), X)dV$$
$$= [\text{by the previous case}] = \sum_i -\int_M \varphi_i u div(X)dV = -\int_M div(X)udV$$

□

Corollary 11. In local coordinates, for a C^2 function u, we have

$$\Delta u = div(grad(u)) = |g|^{-\frac{1}{2}}\partial_i[|g|^{\frac{1}{2}}g^{ij}\partial_j u] = g^{ij}\partial_{ij}u + |g|^{-\frac{1}{2}}\partial_i[|g|^{\frac{1}{2}}g^{ij}]\partial_j u \tag{67}$$

Now we consider the operator $d : C^\infty(M) \to \Omega^1(M)$. This operator and the gradient are dual, so the adjoint of d for the L^2 escalar product will be dual to the divergence operator. This motivates the following.

Definition 39. We define the codifferential as

$$\delta : \Omega^1(M) \to C^\infty(M) : \alpha \mapsto \delta(\alpha) := div(\alpha^\#)$$

Note that, defined this way, δ is the adjoint for d. Indeed, given $\alpha \in \Omega^1(M)$ and $u \in C_c^\infty(int(M))$ we have

$$\int_M g(\alpha, du)dV = \int_M g(\alpha^\#, grad(u))dV = -\int_M div(\alpha^\#)udV = -\int_M \delta(\alpha)u.$$

By lemma 22, we obtain the expression in coordinates for δ. Given $\alpha = a_i dx^i$, we know that $\alpha^\# = g^{ij}a_i\partial_j$, so

$$\delta(\alpha) = div(\alpha^\#) = |g|^{-\frac{1}{2}}\partial_j[g^{ij}a_i|g|^{\frac{1}{2}}].$$

We have set the background to define the $p-$Laplace equation.

Definition 40. A function $f \in C^2(M)$ is p-harmonic if $\delta(|du|^{p-2}du) = 0$. Note that, by definition, $\delta(|du|^{p-2}du) = div(|grad(u)|^{p-2}grad(u))$ so this equation generalize the p-harmonic equation in $\mathbb{R}^n$. In local coordinates u is p-harmonic if and only if

$$\delta(|du|^{p-2}du) = |g|^{-\frac{1}{2}}\partial_j[|g|^{\frac{1}{2}}g^{ij}(g^{ab}\partial_a u\partial_b u)^{\frac{p-2}{2}}\partial_i u] = 0. \tag{68}$$

If we differentiate this expression applying leibnitz's rule we obtain

$$\begin{aligned}\delta(|du|^{p-2}du) &= \frac{1}{2}g^{ab}\partial_j g_{ab}g^{ij}|du|^{p-2}\partial_i u\\ &+\partial_j g^{ij}|du|^{p-2}\partial_i u + g^{ij}\partial_i u\partial_j[|du|^{p-2}] + g^{ij}|du|^{p-2}\partial_{ij}u = 0\end{aligned} \tag{69}$$

Proposition 14. Fix a coordinate system $(x^1, \dots, x_n)$ in a coordinate path $U \subset M$. Define $\Gamma^l = g^{ij}\Gamma^l_{ij}$. Then the following identities are true

$$\Gamma^l = -\Delta x^l = -|g|^{-\frac{1}{2}}\partial_i[|g|^{\frac{1}{2}}g^{il}] = -\frac{1}{2}g^{ij}g^{lk}\partial_k g_{ij} - \partial_i g^{li} \tag{70}$$

Proof. We know the second identity is true by (67). The third identity follows by the formula to differentiate the determinant. Now, to see the first, we compute

$$\begin{aligned}
\Gamma^l &= g^{ij}\Gamma^k_{ij} = g^{ij}\frac{1}{2}g^{lk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \\
&= -\frac{1}{2}g^{ij}g^{lk}\partial_k g_{ij} - \frac{1}{2}g^{lk}g_{jk}\partial_i g^{ij} - \frac{1}{2}g^{lk}g_{ik}\partial_j g^{ij} \\
&= -\tfrac{1}{2}g^{ij}g^{lk}\partial_k g_{ij} - \tfrac{1}{2}\partial_i g^{il} - \tfrac{1}{2}\partial_j g^{lj} = -\tfrac{1}{2}g^{ij}g^{lk}\partial_k g_{ij} - \partial_i g^{il}
\end{aligned}$$

where we used that $g^{ij}\partial_i g_{jk} = -\partial_i g^{ij} g_{jk}$. This proves the claim. □

The next proposition provides a useful characterization of when a coordinate function x^k is p-harmonic.

Proposition 15. Fix a coordinate system $(x^1, \dots, x^n)$ in a coordinate path $U \subset M$. Then x^l is $p-$harmonic on U if and only if

$$\Gamma^l = \frac{1}{2}(p-2)g^{jl}\partial_j[log(g^{ll})] = (2-p)\frac{g^{jl}\Gamma^l_{ij}g^{il}}{g^{ll}}$$

Proof. If we substitute $u = x^l$ in equation (69) we get, noting $|x^l|^2 = g^{ll}$, that

$$\begin{aligned}
0 &= \frac{1}{2}g^{ab}\partial_j g_{ab}g^{jk}\delta^l_k(g^{ll})^{\frac{p-2}{2}} + \partial_j g^{jk}\delta^l_k(g^{ll})^{\frac{p-2}{2}} + g^{jk}\delta^l_k\partial_j[(g^{ll})^{\frac{p-2}{2}}] \\
&= \frac{1}{2}g^{ab}\partial_j g_{ab}g^{jl}(g^{ll})^{\frac{p-2}{2}} + \partial_j g^{jl}(g^{ll})^{\frac{p-2}{2}} + g^{jl}\frac{p-2}{2}(g^{ll})^{\frac{p-4}{2}}\partial_j g^{ll} \\
&= -(g^{ll})^{\frac{p-2}{2}}[-\frac{1}{2}g^{ab}\partial_j g_{ab}g^{jl} - \partial_j g^{jl} - \frac{p-2}{2}g^{jl}\frac{\partial_j g^{ll}}{g^{ll}}] \\
&= -(g^{ll})^{\frac{p-2}{2}}\{\Gamma^l - \frac{1}{2}(p-2)g^{jl}\partial_j[log(g^{ll})]\}
\end{aligned}$$

and this gives the first equality. To see the second, we compute

$$\begin{aligned}
\Gamma^l_{ij}g^{il} &= \frac{1}{2}g^{al}g^{il}[\partial_j g_{ia} + \partial_i g_{ja} - \partial_a g_{ij}] \\
&= \frac{1}{2}[-g_{ia}g^{il}\partial_j g^{al} - g_{ja}g^{il}\partial_i g^{al} + g^{al}g^{ij}\partial_a g^{il}] \\
&= -\frac{1}{2}\partial_j g^{ll} - \frac{1}{2}g_{ja}g^{il}\partial_i g^{al} + \frac{1}{2}g_{ij}g^{al}\partial_a g^{il} = -\frac{1}{2}\partial_j g^{ll}
\end{aligned}$$

where we used that $g^{al}\partial_j g_{ia} = -g_{ia}\partial_j g^{al}$. This shows that the second equality is in fact an identity. □

Once the basic calculations are done, we focus on the problem of proving the local existence of solutions of the $p-$laplacian equation. Since we deal with a local question, we fix a local coordinate chart $(x^1, \dots, x^n)$ from now on, and we work in a fixed coordinte patch $\Omega \subset \mathbb{R}^n$. Moreover, we can suppose that Ω has C^∞ boundary (in fact we can suppose that Ω is a ball, though this will not be necessary). From the expression in local coordinates of the $p-$laplace equation given by equation (68) we see that a function u is $p-$harmonic if and only if

$$\partial_j[|g|^{\frac{1}{2}}g^{ij}(g^{ab}\partial_a u\partial_b u)^{\frac{p-2}{2}}\partial_i u] = 0.$$

If we define the function $A = (A^1, \ldots, A^n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with components

$$A^j(x,q) := |g|^{\frac{1}{2}}(x)g^{ij}(x)(g^{ab}(x)q_a q_b)^{\frac{p-2}{2}}q_i \tag{71}$$

then u is $p-$harmonic if and only if it satisfies

$$div(A(x,\nabla u(x))) = 0 \tag{72}$$

Here ∇u is the usual gradient of u as a function defined on $\mathbb{R}^n$, and $div(A(x,\nabla u(x)))$ is the usual divergence of the vector field defined on $\mathbb{R}^n$ by $x \mapsto A(x,\nabla u(x))$. This type of equations arises in many contexts, and we have a quite general theory for them. First of all we aim to find a weak solution of (72). In order to do this, we need to set some definitions and basic results first. All of this can be found in many references, for example in [6].

Lemma 23. Let $\Omega \subset \mathbb{R}^n$ be an open set. The normed spaces $W^{1,p}(\Omega) := \{u \in L^p(\Omega) :$ exists ∇u in the distributio $L^p(\Omega;\mathbb{R}^n)\}$ are reflexive Banach spaces.

Lemma 24. Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^1$. Then there exits the trace map $Tr : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, a bounded linear map such that $Tr(f) = f|_{\partial\Omega}$ for every $f \in C^1(\Omega)$.

Definition 41. Let $\Omega \subset \mathbb{R}^n$ be an open set. We define the subespace $W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$.

Lemma 25. Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\partial\Omega \in C^1$, and $f \in W^{1,p}(\Omega)$. Then $f \in W_0^{1,p}$ if and only if $Tr(f) = 0$.

Definition 42. A function $u \in W^{1,p}$ is a weak solution of (72) if

$$\int A(x,\nabla u(x))\nabla v(x)dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$.

To prove the existence of weak solutions we will need some results of functional analysis.

Definition 43. Let X be a reflexive Banach space with dual space X'. Denote (u,ϕ) the pairing between $u \in X$ and $\phi \in X'$. Let $K \subset X$ be a closed convex subset, and let $F : K \to X'$. We say that F is *monotone* if $(u-v, F(u)-F(v)) \geq 0$ for every $u,v \in X$.

We say that F is *coercive* if there exists a $h \in K$ such that for every secuence $u_j \in K$ such that $\|u_j\| \to \infty$ we have

$$\frac{(F(u_j)-F(h), u_j - h)}{\|u_j - h\|} \to \infty \qquad \text{as } j \to \infty$$

We say that F is *weakly continuous* on a point $u \in K$ if for every sequence u_j converging to u strongly, we have that $F(u_j)$ converges to $F(u)$ weakly. Remark that as X is reflexive the weak and weak* topologies coincide in X'.

Lemma 26. Let K be a convex and closed subset of X, and let $F : K \to X'$ be monotone and weakly continuous. Fix $u \in K$. Then the following are equivalent:

(1) For all $v \in K$ we have $(v-u, F(u)) \geq 0$
(2) For all $v \in K$ we have $(v-u, F(v)) \geq 0$

Proof. Suppose (1) is true. Then by monoticity

$$0 \leq (v-u, F(v)-F(u)) = (v-u, F(v)) - (v-u, F(u))$$

so $(v-u, F(v)) \geq (v-u, F(u)) \geq 0$ and this is (2).

Now suppose (2) is true. Let $w \in K$ and for $t \in [0,1]$ consider $v_t := u + t(w-u) \in K$ by convexity. Then $(v_t - u, F(v_t)) = t(w-u, F(v_t)) \geq 0$, so for $t > 0$ we have $(w-u, F(v_t)) \geq 0$. As $v_t \to u$ as $t \to 0$ strongly in X, by weak continuity we see that $F(v_t) \to F(u)$ weakly in X'. Then $0 \leq (w-u, F(v_t)) \to (w-u, F(u))$ as $t \to 0^+$ so we conclude that $(w-u, F(u)) \geq 0$ and this is (1). □

Lemma 27. Let $K \subset \mathbb{R}^N$ compact and convex, $G : K \to K$ a continuous function. Then G admits a fixed point.

Proof. As K is compact, there exists a closed ball B such that $K \subset B$. Let $Pr_K : \mathbb{R}^N \to K$ be the projection, which is continuous. Then the map $G \circ Pr_K|_B : B \to K \subset B$ is continuous. By the Brouwer fixed point theorem, there is a fixed point $x \in B$ such that $x = G \circ Pr_K|_B(x) \in K$ so in fact $x \in K$ and $x = G(x)$ is a fixed point for G. □

Proposition 16. Let $K \neq \emptyset$ be a bounded, closed and convex subset of X. Let $F : K \to X'$ be monotone and weakly continuous. Then there exists $u \in K$ such that $(v-u, F(u)) \geq 0$ for every $v \in K$.

Proof. Step one: Finite dimensional case. If X is finite dimensional, we can suppose that $X = \mathbb{R}^N$ since X is linearly isomorpfhic to $\mathbb{R}^N$ and linear transformations conserve convex sets. Note that in finite dimension the weak and strong topologies are the same. Then $F : K \to (\mathbb{R}^N)' \equiv \mathbb{R}^N$ is continuous. As K is compact and convex, there exists a projection over K which we denote $Pr_K : \mathbb{R}^N \to K$. This projection is characterized by the property $|x - Pr_K(x)| = min\{|x-y| : y \in K\}$.

Let us first see an important property of the projection. Let $\eta \in K$, and $x \in \mathbb{R}^N$. By convexity, for all $t \in [0,1]$ we have $(1-t)Pr_K(x) + t\eta = Pr_K(x) + t(\eta - Pr_K(x)) \in K$ so the function defined for $t \in [0,1]$ by

$$\begin{aligned}\phi(t) &:= |x - Pr_K(x) - t(\eta - Pr_K(x))|^2 \\ &= |x - Pr_K(x)|^2 - 2t(x - Pr_K(x), \eta - Pr_K(x)) + t^2|\eta - Pr_K(x))|^2\end{aligned}$$

attain its minimum at $t = 0$ so $\phi'(0) = -2(x - Pr_K(x), \eta - Pr_K(x)) \geq 0$. So we conclude that for any $x \in \mathbb{R}^N$ and $\eta \in K$

$$(x, \eta - Pr_K(x)) \leq (Pr_K(x), \eta - Pr_K(x)). \tag{73}$$

Note by $\pi : (\mathbb{R}^N)' \to \mathbb{R}^N$ the canonical identification. Then we consider the continuous map

$$Pr_K \circ (Id_K - \pi \circ F) : K \to K$$

As K is compact this map has a fixed point $x^* \in K$ such that if we note $x := x^* - \pi(F(x^*)) \in \mathbb{R}^N$ then $x^* = Pr_K(x)$. From (73) we conclude that

$$(x^*, \eta - x^*) = (Pr_K(x), \eta - Pr_K(x)) \geq (x, \eta - Pr_K(x)) = (x^* - \pi(F(x^*)), \eta - x^*)$$

and this yields $(\pi(F(x^*)), \eta - x^*) = (\eta - x^*, F(x^*)) \geq 0$, as we wanted. So step one is proved.

Step two: General case. By a translation we can suppose that $0 \in K$. Let $M \subset X$ a finite dimensional subespace, and note $K_M := K \cap M$. Let $j : M \to X$ be the inclusion and $j' : X' \to M'$

its transpose map such that $j'(f) = f \circ j$. Then the map $j'F|_{K_M} : K_M \to M'$ is under the hipothesis of step one, en hence there exists $u_M \in K_M$ such that for every $v \in K_M$ we have

$$0 \leq (v - u_M, j'F|_{K_M}(u_M)) = (v - u_M, F(u_M))$$

by lemma 26 this is equivalent to $(v - u_M, F(v)) \geq 0$ for all $v \in K_M$.

Now consider an arbitrary $v \in K$ an define $S(v) := \{u \in K : (v - u, F(v)) \geq 0\}$, which is weakly closed by definition of the weak topology. Now we use the Banach-Alaoglu theorem to see that, as K is bounded, its weakly-closure is weakly-compact. But, as K is convex and norm-closed, by the geometric version of the Hanh-Banach theorem, it is weakly-closed. Combining both facts we see that K is weakly compact. Then the elements of the family $\mathcal{A} := \{S(v)\}_{v\in K}$ are weakly-closed subsets of a weakly-compact set K. Let us see that any finite subfamily has non-empty intersection.

Indeed, given $v_1, \dots, v_m \in K$, let $M := Span\{v_1, \dots, v_m\}$, so putting $K_M := M \cap K$ as before, we see that there exists $u_M \in K_M$ such that $(v - u_M, F(v)) \geq 0$ for all $v \in K_M$, and since $v_i \in K_M$ for $i = 1, \dots, m$, then $u \in S(v_1) \cap \dots \cap S(v_m)$.

Then, by the finite intersection property, we conclude that there exists $u \in \bigcup_{v\in K} S(v)$ so $(v - u, F(v)) \geq 0$ for all $v \in K$. We use again lemma 26 to get that $(v - u, F(u)) \geq 0$ for all $v \in K$, and this proves the proposition.

□

Using the coercivity, we can prove the same result as above for K unbounded.

Proposition 17. Let $K \neq \emptyset$ be a closed and convex subset of X. Let $F : K \to X'$ be monotone, coercive and weakly continuous. Then there exists $u \in K$ such that $(v - u, F(u)) \geq 0$ for every $v \in K$.

Proof. As F is coercive, there exists $h \in K$ such that for every secuence $u_j \in K$ such that $\|u_j\| \to \infty$ we have

$$\frac{(F(u_j) - F(h), u_j - h)}{\|u_j - h\|} \to \infty \quad \text{as } j \to \infty.$$

Choose $H > \|F(h)\|$ and $R > \|h\|$ such that $(u - h, F(u) - F(h)) \geq H\|u - h\|$ if $\|u\| \geq R$, $u \in K$. For such u we have

$$\begin{aligned} &(u - h, F(u)) = (u - h, F(u) - F(h)) + (u - h, F(h)) \\ &\geq H\|u - h\| - \|F(h)\|\|u - h\| \geq (H - \|F(h)\|)(\|u\| - \|h\|) > 0 \end{aligned} \tag{74}$$

Now consider $K_R = K \cap B_R$, where $B_R = \{x \in X : \|x\| \leq R\}$ is the closed ball. By the proposition above, there exists $u_R \in K_R$ such that $(v - u_R, F(u_R)) \geq 0$ for all $v \in K_R$, so in particular $(u_R - h, F(u_R)) \leq 0$. Using equation (74) we conclude that $\|u_R\| < R$. We claim that this u_R is the one wanted in the proposition.

Indeed, fix $y \in K$ and for $\varepsilon > 0$ let $w_\varepsilon := u_R + \varepsilon(y - u_R)$. As $\|u\| < R$, we have that $w_\varepsilon \in K_R$ if ε is small enough. For such an ε we have

$$0 \leq (w_\varepsilon - u_R, F(u_R)) = (\varepsilon(y - u_R), F(u_R)) = \varepsilon(y - u_R, F(u_R))$$

and this gives $(y - u_R, F(u_R)) \geq 0$ for any $y \in K$ as we wanted. □

Recall that we note $L^p(\Omega, \mu)$ for the measurable functions in Ω respect to a general measure μ, and $L^p(\Omega)$ when μ is the lebuesgue measure which we will note $|\cdot|$.

Lemma 28. (Egorov) Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $\{f_i\} \subset L^p(\Omega, \mu)$ such that $f_i(x) \to f(x)$ for a.e. $x \in \Omega$. Then for every $\varepsilon > 0$ there exists $B_\varepsilon \subset \Omega$ such that $\mu(B_\varepsilon) < \varepsilon$ and $f_i \to f$ uniformly in $\Omega \backslash B_\varepsilon$.

Proof. Let n, m, k denote natural numbers. Define the sets

$$E_{n,k} := \bigcup_{m \geq n} \{x \in \Omega : |f_m(x) - f(x)| \geq \frac{1}{k}\}.$$

Fix k, and note that we have $E_{n+1,k} \subset E_{n,k}$. If $x \in \Omega$ is such that $f_i(x) \to f(x)$ then there exists $n(k,x)$ such that $x \notin E_{n,k}$. So in the set $A_k := \cap_n E_{n,k}$ there are no points x such that $f_i(x) \to f(x)$, and we conclude that $\mu(A_k) = 0$. Now, as $\mu(\Omega) < \infty$, we have that

$$0 = \mu(A_k) = \mu(\bigcap_n E_{n,k}) = \lim_n \mu(E_{n,k})$$

so for all k there exists n_k large enough such that $\mu(E_{n_k,k}) < \frac{\varepsilon}{2^k}$. Define $B := \cup_k E_{n_k,k}$, and clearly $\mu(B) \leq \varepsilon$. Now note that given k, we take $m \geq n_k$, and then for all $x \in \Omega \backslash B$, as $x \notin E_{n_k,k}$, it follows that $|f_m(x) - f(x)| < \frac{1}{k}$ if $m \geq n_k$, so taking the supreme on $x \in \Omega \backslash B$ we conclude that $\|f_m - f\|_{\infty,\Omega\backslash B} \leq \frac{1}{k}$ if $m \geq n_k$, and this proves the lemma. □

Lemma 29. Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $\{f_i\} \subset L^p(\Omega)$ such that
(1) $f_i(x) \to f(x)$ for a.e. $x \in \Omega$
(2) For some constant M we have $\|f_i\|_{L^p(\Omega)} \leq M$ for all i. Then $f_i \to f$ weakly in $L^p(\Omega)$.

Proof. Set $p' := \frac{p}{p-1}$ and let $g \in L^{p'}$. As the measure given by

$$A \to \mu_g(A) := \int_A |g(x)|^{p'} dx$$

for $A \in \mathcal{L} :=$ { lebesgue-measurable sets }, is subordinated to the Lebesgue measure, it follows that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|A| \leq \delta$ implies $\mu_g(A) \leq \varepsilon^{p'}$. Obviously we can assume $\delta \leq \varepsilon$. Fix such a δ. By the above lemma there exists B_δ with $|B_\delta| \leq \delta$ and such that $f_i \to f$ uniformly in $\Omega \backslash B_\delta$. Then we have

$$\begin{aligned} &|\int_\Omega (f - f_i) g dx| \leq \int_\Omega |(f - f_i)g| dx = \int_{\Omega\backslash B_\delta} |(f - f_i)g| dx + \int_{B_\delta} |(f - f_i)g| dx \\ &\leq (\int_{\Omega\backslash B_\delta} |f - f_i|^p dx)^{\frac{1}{p}} (\int_{\Omega\backslash B_\delta} |g|^{p'} dx)^{\frac{1}{p'}} + (\int_{B_\delta} |f - f_i|^p dx)^{\frac{1}{p}} (\int_{B_\delta} |g|^{p'} dx)^{\frac{1}{p'}} \\ &\leq |\Omega|^{\frac{1}{p}} \|f - f_i\|_{\infty,\Omega\backslash B_\delta} \|g\|_{L^{p'}(\Omega)} + (M + \|f\|_{L^p(\Omega)})\varepsilon < 2(M + \|f\|_{L^p(\Omega)})\varepsilon \end{aligned}$$

if we take $i \geq i_\varepsilon$ large enough. This proves the lemma. □

Note that the above result is valid as well for functions in $L^p(\Omega; \mathbb{R}^n)$, since we can work with each of the components and apply the above lemma to conclude the same thing.

Proposition 18. Let Ω be a bounded open subset of $\mathbb{R}^n$, and let $1 < p < \infty$. Consider a function $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that for some $\alpha, \beta > 0$ the following conditions are satisfied:
(1) The functions $x \to A(x, \xi)$ are measurable for all $\xi \in \mathbb{R}^n$
(2) The functions $\xi \to A(x, \xi)$ are continuous for a.e. $x \in \mathbb{R}^n$
(3) $A(x, \xi) \cdot \xi \geq \alpha |\xi|^p$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$
(4) $|A(x, \xi)| \leq \beta |\xi|^{p-1}$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$
(5) $(A(x, \xi) - A(x, \zeta))(\xi - \zeta) > 0$ for a.e. $x \in \Omega$ and for all $\xi \neq \zeta \in \mathbb{R}^n$.

Then, given $f \in W^{1,p}(\Omega)$ there exists $u \in W^{1,p}(\Omega)$ a weak solution of $div A(x, \nabla u(x)) = 0$ in Ω with $u - f \in W_0^{1,p}(\Omega)$, and u satisfies the estimate

$$\|u\|_{W^{1,p}(\Omega)} \leq C \|f\|_{W^{1,p}(\Omega)} \tag{75}$$

where C only depends on α, β, p and Ω.

Proof. First let us see the existence of u. Let $X := L^p(\Omega;\mathbb{R}^n)$, which is a reflexive Banach space. Set $Q_f := \{v \in W^{1,p} : v-f \in W_0^{1,p}\}$ and $K := \{\nabla v : v \in Q_f\} \subset X$. Then we claim that K is a convex and closed subespace of X. Indeed, given $\nabla v_1, \nabla v_2 \in K$ then $tv_1+(1-t)v_2-f = t(v_1-f)+(1-t)(v_2-f) \in W_0^{1,p}$, so $t\nabla v_1 + (1-t)\nabla v_2 \in K$ and K is convex.

Let us see that K is closed. Let $\nabla v_i \to G \in X$ with $\nabla v_i \in K$. As $v_i - f \in W_0^{1,p}(\Omega)$, by Poincare's inequality

$$\int_\Omega |v_i - f|^p dx \leq C \int_\Omega |\nabla v_i - \nabla f|^p dx \leq C(\|G\|_X^p + \|f\|^p_{W^{1,p}(\Omega)})$$

for some constant C depending on p, Ω. We conclude that $\{v_i\} \subset W^{1,p}$ is bounded, so by Banach-Alaoglu theorem there exists some $v \in W^{1,p}(\Omega)$ such that (taking a subsequence we call the same way) $v_i \to v$ weakly in $W^{1,p}(\Omega)$. In particular, $v_i \to v$ weakly in $L^p(\Omega)$ and $\nabla v_i \to \nabla v$ weakly in X. But $\nabla v_i \to G$ in X, so by uniqueness of the weak limit $G = \nabla v$.

Now we must see that $v - f \in W_0^{1,p}(\Omega)$. First note that $v_i - f \to v - f$ weakly in $W^{1,p}(\Omega)$. As the trace operator is bounded, it follows that $Tr(v_i - f) \to Tr(v - f)$ weakly in $L^p(\partial\Omega)$. Now, since $Tr(v_i - f) = 0 \in L^p(\partial\Omega)$ for all i we conclude that $Tr(v - f) = 0$, so $v \in Q_f$, and thus $G \in K$, so K is closed in X.

We define now $F : K \to X' : v \mapsto F(v)$ such that if $u \in X$

$$(u, F(v)) := \int_\Omega A(x, v(x)) \cdot u(x)dx$$

We want to apply proposition 17. Set $p' := \frac{p}{p-1}$. First note that $F(v) \in X' = L^{p'}(\Omega;\mathbb{R}^n)$ since we have

$$\begin{aligned}
&(u, F(v)) := \int_\Omega A(x, v(x)) \cdot u(x)dx \leq \int_\Omega |A(x, v(x))||u(x)|dx \\
&\leq \beta \int_\Omega |v(x)|^{p-1}|u(x)|dx \\
&\leq \beta(\int_\Omega |v(x)|^{(p-1)\frac{p}{p-1}} dx)^{\frac{p-1}{p}} (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}} = \|v\|_X^{p-1}\|u\|_X
\end{aligned}$$

where we have used the assumption (4) and Holder's inequality.

We shall see that F is monotone, coercive and weakly continuous. To see that F is monotone, note that

$$(u - v, F(u) - F(v)) = \int_\Omega (A(x, u(x)) - A(x, v(x))) \cdot (u(x) - v(x))dx > 0$$

if $u \neq v$ by assumption (5). Now, to see that F is coercive, fix $h \in K$, and we have

$$\begin{aligned}
&(u - h, F(u) - F(h)) = \int_\Omega (A(x, u(x)) - A(x, h(x))) \cdot (u(x) - h(x))dx \\
&= \int_\Omega (A(x, u(x)) \cdot u(x) + \int_\Omega A(x, h(x)) \cdot h(x) - \int_\Omega A(x, u(x)) \cdot h(x) - \int_\Omega A(x, h(x))) \cdot u(x) \\
&\geq \alpha(\int_\Omega |u(x)|^p dx + \int_\Omega |h(x)|^p dx) - \beta(\int_\Omega |u(x)|^{p-1}|h(x)|dx + \int_\Omega |h(x)|^{p-1}|u(x)|)dx \\
&\geq \alpha(\|u\|_X^p + \|h\|_X^p) - \beta(\|u\|_X^{p-1}\|h\|_X + \|h\|_X^{p-1}\|u\|_X)
\end{aligned}$$

so, using that for $\|u\|_X \geq max\{\|h\|_X, 1\}$ we have $\|u - h\| \leq \|u\|_X + \|h\|_X \leq 2\|u\|$ and the well known fact $a^p + b^p \approx (a + b)^p$, we have that for $\|u\|$ large

$$\frac{(u - h, F(u) - F(h))}{\|u - h\|} \geq C\{\alpha(\|u\|_X^{p-1} + \|h\|_X^{p-1}) - \beta(\|u\|_X^{p-2}\|h\|_X + \|h\|_X^{p-1})\} \to \infty$$

if $\|u\|_X \to \infty$. This shows that F is coercive.

Finally, to see that F is weakly continuous, let $\{u_i\} \subset K$ be a sequence such that $u_i \to u$ in X for some $u \in K$. We extract a subsequence u_{i_j} such that $u_{i_j}(x) \to u(x)$ a.e $x \in \Omega$, and by continuity of $A(x, \cdot)$, i.e. assumption (2), it follows that $A(x, u_{i_j}(x)) \to A(x, u(x))$ a.e. $x \in \Omega$. Also note that

$$\int_\Omega |A(x, u_{i_j}(x))|^{p'} dx \leq \beta \int_\Omega |u_{i_j}(x)|^p \leq C$$

for some constant independent of j. By lemma 29 we conclude that $A(\cdot, u_{i_j}(\cdot)) \to A(\cdot, u(\cdot))$ weakly in $L^{p'}(\Omega; \mathbb{R}^n)$. But we must see that the whole sequence $A(\cdot, u_i(\cdot))$ converges to $A(\cdot, u(\cdot))$ weakly in $L^{p'}(\Omega; \mathbb{R}^n)$.

To see this, suppose that $A(\cdot, u_i(\cdot))$ does not converge to $A(\cdot, u(\cdot))$ weakly. This means by definition that there exists $v \in L^p(\Omega; \mathbb{R}^n)$ such that the sequence of real numbers $\int_\Omega (A(x, u_i(x)) - A(x, u(x))) \cdot v dx$ does not tend to 0, and therefore there exists a subsequence i_j such that for all j

$$|\int_\Omega (A(x, u_{i_j}(x)) - A(x, u(x))) \cdot v dx| > \varepsilon.$$

But by the same argument given before, a subsequence $u_{i_{j_k}}$ of u_{i_j} must converge weakly to $A(\cdot, u(\cdot))$, and this is impossible. This proves that F is weakly continuous.

Now we can use proposition 17 and conclude that there exists $\nabla u \in K$ such that for every $\nabla v \in K$ we have

$$(\nabla v - \nabla u, F(u)) = \int_\Omega A(x, \nabla u(x))(\nabla v(x) - \nabla u(x)) dx \geq 0.$$

Now let $\phi \in W_0^{1,p}(\Omega)$. It is clear that $u + \phi - f$ and $u - \phi - f \in W_0^{1,p}$, so $u + \phi$ and $u - \phi \in Q_f$. Then, if $\epsilon \in \{0, 1\}$ we have

$$(\nabla(u + \epsilon\phi) - \nabla u, F(u)) = \epsilon \int_\Omega A(x, \nabla u(x)) \nabla \phi(x) dx \geq 0$$

and then u is a weak solution of $div(A(x, \nabla u(x))) = 0$.

Let us see now estimate (75). First, as $u - f \in W_0^{1,p}(\Omega)$ is a test function, we see that

$$\int_\Omega A(x, \nabla u(x)) \nabla u(x) dx = \int_\Omega A(x, \nabla u(x)) \nabla f(x) dx.$$

Now we compute

$$\begin{aligned}
&\alpha \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^n)} = \alpha \int_\Omega \|\nabla u\|^p dx \leq \int_\Omega A(x, \nabla u(x)) \cdot \nabla u(x) dx \\
&= \int_\Omega A(x, \nabla u(x)) \cdot \nabla f(x) dx \leq \beta \int_\Omega |\nabla u(x)|^{p-1} |f| dx \leq \beta \|\nabla u\|^{p-1}_{L^p(\Omega;\mathbb{R}^n)} \|\nabla f\|_{L^p(\Omega;\mathbb{R}^n)}
\end{aligned}$$

so it follows that

$$\|\nabla u\|_{L^p(\Omega;\mathbb{R}^n)} \leq \frac{\beta}{\alpha} \|\nabla f\|_{L^p(\Omega;\mathbb{R}^n)}.$$

On the other hand we have

$$\begin{aligned}
&\|u\|_{L^p(\Omega} - \|f\|_{L^p(\Omega} \leq |\|u\|_{L^p(\Omega} - \|f\|_{L^p(\Omega}| \leq \|u - f\|_{L^p(\Omega} \\
&\leq C(p, \Omega) \|\nabla u - \nabla f\|_{L^p(\Omega)} \leq C(p, \Omega)(\frac{\beta}{\alpha} + 1) \|\nabla f\|_{L^p(\Omega;\mathbb{R}^n)}
\end{aligned}$$

where we used Poincare's inequality. These two last inequalities combined give estimate 75, so the theorem is proved. □

In the next lemma and the remark below we show that the assumptions imposed on A in 18 are not restrictive, and the function A in which we are interested of course satisfies the mentioned assumptions.

Lemma 30. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $1 < p < \infty$. Consider a continuous function $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ that is C^1 in $\Omega \times \mathbb{R}^n \backslash \{0\}$ and assume that there exists $\delta > 0$ such that for every $x \in \Omega$, $t \geq 0$ and $\xi, \zeta \in \mathbb{R}^n$ we have

(1) $A(x, t\xi) = t^{p-1} A(x, \xi)$

(2) $(A(x, \xi) - A(x, \zeta)) \cdot (\xi - \zeta) \geq \delta(|\xi| + |\zeta|)^{p-2} |\xi - \zeta|^2$.

Then A satisfies the following: there exists constants $\beta > 0$ and $C_\delta > 0$ such that for every $x \in \Omega$, $\xi \in \mathbb{R}^n \backslash \{0\}, h \in \mathbb{R}^n$, $j = 1 \ldots, n$, we have

(a) $A(x, \xi) \cdot \xi \geq \delta |\xi|^p$. Actually condition (1) is not necessary for (a).

(b) $|A(x, \xi)| + |\partial_{x_j} A^k(x, \xi)| + |\xi| |\partial_{\xi_j} A(x, \xi)| \leq \beta |\xi|^{p-1}$. Condition (2) is not needed for (b).

(c) $\sum_{k,j} \partial_{\xi_j} A^k(x, \xi) h_j h_k \geq C_{\delta,p} |\xi|^{p-2} |h|^2$. Condition (1) is not necessary for (c).

Proof. The first estimate (a) follows by putting $\zeta = 0$ in the assumption (2), taking into account that by assumption (1) $A(x, 0) = 0$.

Let us see (b). Assume $\xi \neq 0$. By (2), $A(x, \xi) = |\xi|^{p-1} A(x, \xi |\xi|^{-1})$. From this (b) is a computation. First

$$A(x, \xi) \leq (\sup_{|\xi'|=1} |A(x, \xi')|) |\xi|^{p-1} = \beta_1 |\xi|^{p-1}.$$

Also we have

$$\partial_{x_j} A(x, \xi) = |\xi|^{p-1} \partial_{x_j} A(x, \xi |\xi|^{-1}) \leq (\sup_{|\xi'|=1} |\partial_{x_j} A(x, \xi')|) |\xi|^{p-1} = \beta_2 |\xi|^{p-1}$$

Finally, by the chain rule we have

$$\begin{aligned}
&\partial_{\xi_j} A(x, \xi) = \frac{1}{2}(p-1)(|\xi|^2)^{\frac{p-3}{2}} 2\xi_j A(x, |\xi|^{-1}\xi) \\
&+|\xi|^{p-1} (\partial_{\xi_j} A(x, |\xi|^{-1}\xi)) \cdot [-\frac{1}{2}(|\xi|^2)^{\frac{-3}{2}} 2\xi_j \xi + |\xi|^{-1} e_j] \\
&\leq (p-1)\beta_1 |\xi|^{p-2} + \beta_2 |\xi|^{p-2} + \beta_2 |\xi|^{p-2} \leq \beta |\xi|^{p-2}
\end{aligned}$$

for $\beta = max\{(p-1)\beta_1, \beta_2\}$. This gives (b).

To see (c), note that for each $x \in \Omega$, $\xi \neq 0$, as $A(x, \cdot)$ is differentiable, then by definition

$$A(x, \xi + h) - A(x, \xi) - D_\xi A(x, \xi)(h) = o(|h|)$$

as $|h| \to 0$. From this it follows that, if $|h| \to 0$, using (2) we have

$$\begin{aligned}
&\delta(|\xi + h| + |\xi|)^{p-2} |h|^2 \leq (A(x, \xi + h) - A(x, \xi)) \cdot h \\
&= h \cdot D_\xi A(x, \xi)(h) + o(|h|^2)
\end{aligned} \tag{76}$$

Besides, if $|h| \to 0$ then $(|\xi + h| + |\xi|)^{p-2} \to 2^{p-2} |\xi|^{p-2}$, so if $|h|$ is small enough we have

$$\frac{1}{2} 2^{p-2} \delta |\xi|^{p-2} |h|^2 \leq \delta(|\xi + h| + |\xi|)^{p-2} |h|^2$$

and inserting this in (76) we conclude that if $|h| \to 0$ then

$$2^{p-3} \delta |\xi|^{p-2} |h|^2 \leq h \cdot D_\xi A(x, \xi)(h) + o(|h|^2).$$

Neglecting the term $o(|h|^2)$ we see that for $0 < |h| < \varepsilon(\xi) := \varepsilon$ small enough

$$C_{\delta,p}|\xi|^{p-2}|h|^2 \leq h \cdot D_\xi A(x,\xi)(h) \tag{77}$$

for some constant $C_{\delta,p}$ independent of ξ and h.

Now, if $h \in \mathbb{R}^n \backslash \{0\}$ is arbitrary, we put $h = (\varepsilon|h|^{-1}h)|h|\varepsilon^{-1} = h_\varepsilon|h|\varepsilon^{-1}$ with $|h_\varepsilon| < \varepsilon(\xi)$ so by (77) we have

$$C_{\delta,p}|\xi|^{p-2}|h_\varepsilon|^2 \leq h_\varepsilon \cdot D_\xi A(x,\xi)(h_\varepsilon) \tag{78}$$

and finally if we multiply (78) by $(|h|\varepsilon^{-1})^2$ we obtain

$$C_{\delta,p}|\xi|^{p-2}|h|^2 \leq h \cdot D_\xi A(x,\xi)(h) = \sum_{k,j} \partial_{\xi_j} A^k(x,\xi)h_j h_k$$

and this yields (c). □

Remark 38. Note that for solving the p-Laplace equation on a riemannian manifold (M, g) we are interested in the function $A(x, q)$ given by (79), i.e, with components

$$A^j(x,q) := |g|^{\frac{1}{2}}(x)g^{ij}(x)(g^{ab}(x)q_a q_b)^{\frac{p-2}{2}}q_i = |g|^{\frac{1}{2}}(x)q^j|q|_g^{p-2} \tag{79}$$

defined for x in a coordinate patch $\Omega \subset \mathbb{R}^n$, $q \in \mathbb{R}^n$. Let us see that this $A(x, q)$ satisfies the hipothesis of lemma 30 if the metric g is C^1.

First note that it is clear that $A \in C^1(\Omega \times \mathbb{R}^n \backslash \{0\})$, and $A(x, tq) = t^{p-1}A(x, q)$ for $q \neq 0$ and $t > 0$. Besides

$$A(x,q) = |g|^{\frac{1}{2}}(x)q^j|q|_g^{p-2} \leq |g|^{\frac{1}{2}}(x)|q|_g^{p-1} \to 0$$

if $q \to 0$ so A is continuous in $\Omega \times \mathbb{R}^n$. Let us see condition (2). We can suppose that A is defined in the closure of $\Omega \subset \mathbb{R}^n$ an open bounded set. Therefore the metric satisfies $C_1 \leq |g| \leq C_2$, for some constants C_1 and C_2 so it is comparable to a constant, and this implies that the norm induced by g is uniformly comparable to the usual norm in $\mathbb{R}^n$.

So we can suppose directly we are in $\mathbb{R}^n$ with its flat metric, and $A(x, q) = A(q) = q|q|^{p-2}$, which is (up to a constant) the gradient of the function $f(q) = |q|^p$, so $A(q) = \nabla f(q)$. It is well known that for $p > 1$ the function f is uniformly, strictly convex and C^2 away from zero, so $Hess(f)$ is a positive definite matrix, and for every non-zero $\xi, \zeta \in \mathbb{R}^n$ we have

$$f(\xi) - f(\zeta) - (\nabla f(\zeta), \xi - \zeta) \geq Hess(f)(\xi)(\xi - \zeta, \xi - \zeta)$$
$$f(\zeta) - f(\xi) - (\nabla f(\xi), \zeta - \xi) \geq Hess(f)(\zeta)(\zeta - \xi, \zeta - \xi)$$

if we add both expressions, and taking into account that $hess(f)$ is positive definite, we obtain

$$(\nabla f(\xi) - \nabla f(\zeta), \xi - \zeta) \geq Hess(f)(\xi)(\xi - \zeta, \xi - \zeta) + Hess(f)(\zeta)(\xi - \zeta, \xi - \zeta)$$
$$\approx (|Hess(f)(\xi)| + |Hess(f)(\zeta)|)|\xi - \zeta|^2 \approx (|\xi|^{p-2} + |\zeta|^{p-2})|\xi - \zeta|^2 \geq C(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2$$

since $|Hess(f)(\xi)|$ involves two derivatives of f so that it behaves as $|x|^{p-2}$ (trivial in one dimension, and requires an easy computation in general dimension). We have also used the fact that $a^{p-2}+b^{p-2} \geq C(a + b)^{p-2}$ for some universal constant C.

6.2. Regularity of Weak Solutions

The next step to prove the existence of n-harmonic coordinates is to prove that the weak solution found in 18 is regular. This kind of results are always hard and technical, so we will need various previous results.

Proposition 19. (Chain rule for weak derivatives) Let $G : \mathbb{R}^m \to \mathbb{R}$ a continuous function such that $G \in C^1(\mathbb{R}^m)$. Assume ∇G is bounded in $\mathbb{R}^m$. Note $G(s_1, \dots, s_m)$ for G. Let $u = (u_1, \dots, u_m)$ be m functions in $W^{1,p}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ an open bounded set. Then the function $K = G \circ u : \Omega \to \mathbb{R}$ is in $W^{1,p}(\Omega)$ and besides

$$\frac{\partial K}{\partial x_i}(x) = \sum_{l=1}^{m} \frac{\partial G}{\partial s_l}(u(x)) \frac{\partial u_l}{\partial x_i}(x) \tag{80}$$

as distributions in Ω.

Proof. First we shall see that $K(x) \in L^p(\Omega)$. Let M be such that $|\nabla G(x)| \leq M$ for all x. Using Holder's inequality and the fundamental theorem of calculus, we have

$$\begin{aligned} |G(s) - G(s')| &= |\int_0^1 \partial_t[G(s' + t(s - s')]dt| \\ &\leq \int_0^1 |\nabla G(s' + t(s - s')]||s - s'|dt \leq M|s - s'|. \end{aligned} \tag{81}$$

Therefore, if we fix s', then $|G(s)| \leq |G(s')| + M|s - s'|$, so

$$|G(u(x))|^p \leq C(|G(s')|^p + |u(x)|^p + |s'|^p). \tag{82}$$

As Ω is bounded, the constants functions are in $L^1(\Omega)$, and we conclude from (82) that $K \in L^p(\Omega)$. For the functions on the righ hand side of (80) we have

$$\frac{\partial G}{\partial s_l}(u(x)) \frac{\partial u_k}{\partial x_i}(x) \leq M \frac{\partial u_k}{\partial x_i}(x)$$

so all of them are in $L^p(\Omega)$. It remains to see that (80) holds. For this, fix $\phi \in C_c^\infty(\Omega)$ and pick some open set U compactly contained in Ω such that $supp(\phi) \subset U$. Pick a sequence $\varphi^j = (\varphi_1^j, \dots, \varphi_m^j) \in C^\infty(\Omega; \mathbb{R}^m)$ such that $\varphi_l^j \to u_l$ in $W^{1,p}(U)$, and such that $\varphi_l^j(x) \to u_l(x)$, $\frac{\partial \varphi_l^j}{\partial x_i}(x) \to \frac{\partial u_l}{\partial x_i}(x)$ for a.e. $x \in U$. Set $K^j := F \circ \varphi^j \in C^1(\Omega)$. By the usual chain rule we have

$$\frac{\partial K^j}{\partial x_i}(x) = \sum_{l=1}^{m} \frac{\partial G}{\partial s_l}(\varphi^j(x)) \frac{\partial \varphi_l^j}{\partial x_i}(x)$$

and integrating by parts

$$\int_U \frac{\partial \phi}{\partial x_i}(x) K^j(x) dx = -\sum_{l=1}^{m} \int_U \phi(x) \frac{\partial G}{\partial s_l}(\varphi^j(x)) \frac{\partial \varphi_l^j}{\partial x_i}(x) dx \tag{83}$$

We shall see that (83) follows for K passing to the limit. First, by (81) we have $|K^j(x) - K(x)|^p \leq C|\varphi^j(x) - u(x)|^p$ so $K^j \to K$ in $L^p(\Omega)$ as $j\infty$. Then

$$\int_U \frac{\partial \phi}{\partial x_i}(x)(K^j(x) - K(x))dx \leq \|\frac{\partial \phi}{\partial x_i}\|_{L^{p'}(\Omega)} \|K^j - K\|_{L^p(\Omega)} \to 0$$

as $j \to \infty$, so the left hand side of (83) converges as we want. For the right hand side we note that

$$\begin{aligned}
&\int_U \phi(x)\frac{\partial G}{\partial s_l}(\varphi^j(x))\frac{\partial \varphi_l^j}{\partial x_i}(x)dx \\
&= \int_U \phi(x)\frac{\partial G}{\partial s_l}(\varphi^j(x))\frac{\partial u_l}{\partial x_i}(x)dx + \int_U \phi(x)\frac{\partial G}{\partial s_l}(\varphi^j(x))(\frac{\partial \varphi_l^j}{\partial x_i}(x) - \frac{\partial u_l}{\partial x_i}(x))dx \\
&\to \int_U \phi(x)\frac{\partial G}{\partial s_l}(u(x))\frac{\partial u_l}{\partial x_i}(x)dx + 0 \qquad \text{as } m \to \infty
\end{aligned}$$

For taking the limit we have used that U is bounded, so $L^p(U) \subset L^1(U)$, and by the boundness of ∇G we can use dominated convergence. This proves the proposition $\square$

Proposition 20. Let $1 < p < \infty$ an $\Omega \subset \mathbb{R}^n$ a bounded open set. Let $u \in W^{1,p}(\Omega)$. Then we have

(1) $|u| \in W^{1,p}(\Omega)$ and

$$|u|_{x_i} = \chi_{\{u \neq 0\}} sign(u) u_{x_i}$$

(2) u^+ and $u^- \in W^{1,p}(\Omega)$. Moreover,

$$(u^+)_{x_i} = \chi_{\{u>0\}} u_{x_i} \qquad \text{and} \qquad (u^-)_{x_i} = \chi_{\{u<0\}} u_{x_i}.$$

Proof. Let us see (1). Let $G_\varepsilon : \mathbb{R} \to \mathbb{R}$ given by $G_\varepsilon(t) = (t^2+\varepsilon^2)^{\frac{1}{2}} - \varepsilon$. Then $|G_\varepsilon(t)| \leq |t|$ and $G_\varepsilon \in C^1$ and

$$G_\varepsilon'(t) = \frac{t}{(t^2+\varepsilon^2)^{\frac{1}{2}}} \leq 1.$$

Let $v_\varepsilon = G_\varepsilon \circ u$. Note that $|v_\varepsilon| \leq |u|$, so $v_\varepsilon(x) \to |u(x)|$ as $\varepsilon \to 0$ for a.e. $x \in \Omega$ dominately. By proposition 19 we know that $v_\varepsilon \in W^{1,p}(\Omega)$ and

$$\frac{\partial v_\varepsilon}{\partial x_i} = G_\varepsilon'(u)u_{x_i} = \frac{u u_{x_i}}{(u^2+\varepsilon^2)^{\frac{1}{2}}}$$

so that $|\frac{\partial v_\varepsilon}{\partial x_i}| \leq u_{x_i}$ and then

$$\frac{\partial v_\varepsilon}{\partial x_i}(x) \to \chi_{\{u\neq 0\}} sign(u) u_{x_i}(x) \qquad \text{as } \varepsilon \to 0$$

a.e. $x \in \Omega$ dominately. By proposition 19, for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega v_\varepsilon \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi \frac{\partial v_\varepsilon}{\partial x_i} dx. \tag{84}$$

Now we apply the dominated convergence theorem to both sides of (84) to get

$$\int_\Omega |u| \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi \chi_{\{u\neq 0\}} sign(u) u_{x_i}(x) dx. \tag{85}$$

and this gives (1).

Now let us see (2) for $u^+(x)$. As $u^+(x) = \frac{1}{2}(|u(x)| + u(x))$, by (1) we have that $u^+(x) \in W^{1,p}(\Omega)$. In addition, $u^+_{x_i} = \frac{1}{2}(|u|_{x_i} + u_{x_i}) = \frac{1}{2}(\chi_{\{u\neq 0\}} sign(u) u_{x_i} + u_{x_i})$. Besides, note that $u^+ = |u^+|$ and then, as $\{u^+ \neq 0\} = \{u > 0\}$, we have by (1) that

$$u^+_{x_i} = |u^+|_{x_i} = \chi_{\{u>0\}} u^+_{x_i} = \chi_{\{u>0\}} \frac{1}{2}(\chi_{\{u\neq 0\}} sign(u) u_{x_i} + u_{x_i}) = \chi_{\{u>0\}} u_{x_i}. \tag{86}$$

For $u^-(x)$ is analogous. Indeed $u^-(x) = \frac{1}{2}(|u(x)| - u(x))$, so $u^-(x) \in W^{1,p}(\Omega)$, and $u^-_{x_i} = \frac{1}{2}(|u|_{x_i} - u_{x_i}) = \frac{1}{2}(\chi_{\{u\neq 0\}} sign(u) u_{x_i} - u_{x_i})$. As $u^- = |u^-|$ and $\{u^- \neq 0\} = \{u < 0\}$, then by (1) we have

$$u^-_{x_i} = |u^-|_{x_i} = \chi_{\{u<0\}} u^-_{x_i} = \chi_{\{u<0\}} \frac{1}{2}(\chi_{\{u\neq 0\}} sign(u) u_{x_i} - u_{x_i}) = -\chi_{\{u<0\}} u_{x_i}. \tag{87}$$

so (2) is proved. □

Corollary 12. Let $1 < p < \infty$ an $\Omega \subset \mathbb{R}^n$ a bounded open set. Fix $\alpha \in \mathbb{R}$. If $u \in W^{1,p}(\Omega)$ then $\nabla u(x) = 0$ for a.e. $x \in \{u = \alpha\}$.

Proof. We can suppose $\alpha = 0$ by considering $u - \alpha \in W^{1,p}(\Omega)$. With this assumption, as $u = u^+ - u^-$, then $\nabla(u) = \nabla(u^+) - \nabla(u^-)$. By the proposition above, both $\nabla(u^+)$ and $\nabla(u^-)$ are zero in the set $\{u = 0\}$, so also $\nabla(u)(x) = 0$ for a.e. $x \in \{u = 0\}$. □

Proposition 19 can be generalized as follows when dealing with functions G of just one real variable.

Proposition 21. Let $1 < p < \infty$, and let $G : \mathbb{R} \to \mathbb{R}$ be a continuous and piecewise C^1 function with corners in a finite number of points $t_1, \dots, t_n \in \mathbb{R}$. Suppose G' is bounded in $\mathbb{R}\backslash\{t_1, \dots, t_l\}$. Let $\Omega \subset \mathbb{R}^n$ be an open set, and $u \in W^{1,p}(\Omega)$. Put

$$A := \{u = t_1\} \cap \cdots \cap \{u = t_l\}$$

Then $G \circ u \in W^{1,p}(\Omega)$ and moreover

$$(G \circ u)_{x_i}(x) = G'(u(x)) u_{x_i}(x) \chi_{\Omega \backslash A}(x)$$

Proof. First observe that G is lipschitz because G' is bounded in $\mathbb{R}\backslash\{t_1, \dots, t_n\}$, say $|G'| \leq M$. Also, as G is piecewise C^1, it is inmediate that the distributional derivative of G exists and it is equal to G'. We conclude that $G \in W^{1,p}_{loc}(\mathbb{R})$. Now we take a family of smooth aproximations with compact support of the dirac delta $\{\rho_\varepsilon\}$ and denote $G_\varepsilon := G \star \rho_\varepsilon$. As $G' \in L^p_{loc}(\mathbb{R})$, we have that

$$(G_\varepsilon)'(t) = (G' \star \rho_\varepsilon)(t) = \int_{\mathbb{R}^n} G'(t-s)\rho_\varepsilon(s) ds. \tag{88}$$

This shows that $|(G_\varepsilon)'| \leq M$. On the other hand, standar aproximations theorems give that $G_\varepsilon \to G$ in $W^{1,p}_{loc}(\mathbb{R})$ and uniformly on compact sets as $\varepsilon \to 0$, and also $(G_\varepsilon)'(t) \to G'(t)$ as $\varepsilon \to 0$ pointwise for $t \in \mathbb{R}\backslash\{t_1, \dots, t_l\}$. This is a well known result, which proof can be found for example in [6], Section 5.3, Theorem 1, and Appendix C, Theorem 7.

Now let $v_\varepsilon := G_\varepsilon \circ u$ and then by 19 that

$$\frac{\partial v_\varepsilon}{\partial x_i}(x) = G'_\varepsilon(u(x)) u_{x_i}(x). \tag{89}$$

In particular note that by corollary 12 we have that $\frac{\partial v_\varepsilon}{\partial x_i}(x) = 0$ for a.e. $x \in A$. Then we have that

$$\frac{\partial v_\varepsilon}{\partial x_i}(x) \to G'(u(x)) u_{x_i}(x) \chi_{\Omega \backslash A}(x)$$

for a.e. $x \in \Omega$ dominately. Also, by (88), G_ε are Lipschitz with the same constant, and then $v_\varepsilon(x) \leq M|u(x)| + C$ for some constant C. As Ω is bounded, we conclude also that

$$v_\varepsilon(x) \to (G \circ u)(x)$$

for a.e. $x \in \Omega$ dominately.

Once this is seen we apply proposition 19 and we have that, given $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} v_\varepsilon = -\int_\Omega \varphi \frac{\partial v_\varepsilon}{\partial x_i} \tag{90}$$

and applying the dominated convergence theorem to both sides of (90) we conclude

$$\int_\Omega \varphi_{x_i}(G \circ u) = -\int_\Omega \varphi (G' \circ u) u_{x_i} \chi_{\Omega \setminus A}. \tag{91}$$

Finally, note that obviously the function $(G' \circ u) u_{x_i} \chi_{\Omega \setminus A} \in L^p(\Omega)$ since G' is bounded. This proves the proposition. $\square$

Let us fix some notation. We note $B_r := B(0,r)$ the open ball in Euclidean space. Also, if $1 \le q \le \infty$, we note $\|u\|_{q,r} := \|u\|_{L^q(B_r)}$. The following Theorem is a particular case of a result proved in [14]. Since the assumptions here are stronger and more comfortable, the proof becomes easier.

Theorem 15. Let $1 < p \le n$, and let $\Omega \subset \mathbb{R}^n$ be an open set and let $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ a measurable function such that for some $\beta, \delta > 0$ we have

1) $|A(x,\xi)| \le \beta |\xi|^{p-1}$

2) $A(x,\xi) \cdot \xi \ge \delta |\xi|^p$.

Let $u \in W^{1,p}(\Omega)$ be a weak solution of $div(A(x, \nabla u(x))) = 0$ in Ω. Then, if B_{2r} is compactly contained in Ω, we have the estimate

$$\|u\|_{\infty,r} \le C \|u\|_{p,2r} \tag{92}$$

for some constant depending on β, δ, p and r but not on u.

Proof. By definition of weak solution, for every test function $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega A(x, \nabla u(x)) \nabla \varphi(x) dx = 0. \tag{93}$$

The idea to prove that $u \in L^\infty(B_r)$ is to pick a suitable test function φ in (93) in such a way that we obtain a bound of the type $\|u\|_{q,B_{\alpha r}} \le C \|u\|_{p,B_{2r}}$ for some $q > p$ and $1 \le \alpha \le 2$. Then we can iterate this argument and get $q = \infty$.

Let $q, l \in \mathbb{R}$ such that $q, l \ge 1$. Define the function $F : \mathbb{R}_+ \to \mathbb{R}$ depending on p, q and given by

$$F_{l,q}(t) = F(t) = \begin{cases} t^q & si \quad 0 \le t \le l \\ \\ q l^{q-1} t - (q-1) l^q & si \quad\quad t \ge l \end{cases} \tag{94}$$

Note that $F(t)$ is linear in $t \ge l$. Besides we have

$$F'_{l,q}(t) = F'(t) = \begin{cases} q t^{q-1} & si \quad 0 \le t \le l \\ \\ q l^{q-1} & si \quad\quad t \ge l \end{cases} \tag{95}$$

so $F \in C^1(\mathbb{R}_+)$ and F' is bounded. Besides F' is piecewise C^1 with possible corners in $t = l$ and $t = 0$, and F'' is bounded wherever it exits since it is zero for $t \ge l$. Now let $G(t) = G_{l,q}(t) = sign(t) F(|t|) F'(|t|)^{p-1}$, which is given by

$$G(t) = \begin{cases} sign(t) |t|^{q+(q-1)(p-1)} q^{p-1} & si \quad 0 \le |t| \le l \\ \\ sign(t) q^{p-1} l^{(p-1)(q-1)} [q l^{q-1} |t| - (q-1) l^q] & si \quad\quad t \ge l \end{cases} \tag{96}$$

Note that $G:\mathbb{R}\to\mathbb{R}$ is continuous and piecwise C^1, with corners in $t=+l, t=-l$, and

$$G'(t)=\begin{cases}[q+(q-1)(p-1)]|t|^{(q-1)p}q^{p-1} & si \quad 0\le|t|\le l\\ q^{p-1}l^{p(q-1)+1} & si \qquad t\ge l\end{cases} \tag{97}$$

Therefore $G':\mathbb{R}\to\mathbb{R}$ is bounded where it exists. Then we use proposition 21 to conclude $G(u)\in W^{1,p}(\Omega)$. Also, from the definition $G(t)=G_{l,q}(t)=sign(t)F(|t|)F'(|t|)^{p-1}$, we compute

$$\begin{aligned}G'(t)&=sign(t)F'(|t|)sign(t)F'(|t|)^{p-1}+sign(t)F(|t|)(p-1)F'(|t|)^{p-2}F''(|t|)sign(t)\\&=F'(|t|)^p+(p-1)F(|t|)F'(|t|)^{p-2}F''(|t|)\end{aligned} \tag{98}$$

Since $F(|t|),F'(|t|)$, and $F''(|t|)$ are all ≥ 0, we see from this that $G'(t)\ge F'(|t|)^p$.

Now we take as a test function $\varphi(x):=\eta(x)^pG(u(x))$ for $\eta\in C_c^\infty(U)$ a cut off non-negative function supported on certain open set U compactly contained in Ω. We shall determine η and U later. Note that $\varphi\in W_0^{1,p}(U)$ since it has compact support contained in U. Besides we have

$$\nabla\varphi(x)=p\eta(x)^{p-1}\nabla\eta(x)G(u(x))+\eta(x)^pG'(u(x))\nabla u(x)$$

Now we compute that

$$\begin{aligned}&A(x,\nabla u(x))\cdot\nabla\varphi(x)=A(x,\nabla u(x))\cdot p\eta(x)^{p-1}\nabla\eta(x)\\&+A(x,\nabla u(x))\cdot\eta(x)^pG'(u(x))\nabla u(x)\\&\ge -p\eta(x)^{p-1}|\nabla\eta(x)||G(u(x))|\beta|\nabla u(x)|^{p-1}+\delta\eta(x)^pF'(|u(x)|)^p|\nabla u(x)|^p\\&=\delta|\eta(x)F'(|u(x)|)\nabla u(x)|^p-p\eta(x)^{p-1}|\nabla\eta(x)||F(|u(x)|)||F'(|u(x)|)|^{p-1}\beta|\nabla u(x)|^{p-1}\\&=\delta|\eta(x)F'(|u(x)|)\nabla u(x)|^p-p\beta|F(|u(x)|)\nabla\eta(x)||\eta(x)F'(|u(x)|)\nabla u(x)|^{p-1}\end{aligned}$$

If we let $v:=F(|u|)$, then $\nabla v=sign(u)F'(|u|)\nabla u$, so if we integrate over U, the expression above becomes

$$0\ge\delta\int_U|\eta(x)\nabla v(x)|^pdx-p\beta\int_U|v(x)\nabla\eta(x)||\eta(x)\nabla v(x)|^{p-1}dx \tag{99}$$

Now, from (99) we deduce

$$\begin{aligned}\|\eta\nabla v\|^p_{L^p(U}&=\int_U|\eta(x)\nabla v(x)|^pdx\le\frac{p\beta}{\delta}\int_U|v(x)\nabla\eta(x)||\eta(x)\nabla v(x)|^{p-1}dx\\&\le\frac{p\beta}{\delta}(\int_U|v(x)\nabla\eta(x)|^pdx)^{\frac{1}{p}}(\int_U|\eta(x)\nabla v(x)|^pdx)^{\frac{p-1}{p}}\\&=C_{p,\beta,\delta}\|v\nabla\eta\|_{L^p(U)}\|\eta\nabla v\|^{p-1}_{L^p(U)}\end{aligned} \tag{100}$$

and this yields

$$\|\eta\nabla v\|_{L^p(U)}\le C\|v\nabla\eta\|_{L^p(U)} \tag{101}$$

Now we distinguish cases.

Case1: Suppose $p<n$. Then by Poincare's inequality, setting $p^*:=\frac{np}{n-p}$ we have

$$\|\eta v\|_{L^{p^*}(U)}\le C\|\nabla(\eta v)\|_{L^p(U)}\le C(\|v\nabla\eta\|_{L^p(U)}+\|\eta\nabla v\|_{L^p(U)})\le C\|v\nabla\eta\|_{L^p(U)} \tag{102}$$

for some constant $C=C(p,\beta,\delta,U)$ that depens on U, where we used (101) in the last inequality. Note that C is not necessarily equal in each stage. Note also that this C, although depends on U (and later we will use this estimate taking different $U's$) is valid for every compactly supported function of U, so if we make our estimates for functions supported in a fixed U big enough (as we will), C can be taken to be the same.

Now take $h' < h \leq 2r$ and take η such that $\eta = 1$ in $B_{h'}$ and $\eta = 0$ in $\mathbb{R}^n \backslash B_h$. Furthermore, we can take η to be almost linear on $B_h \backslash B_{h'}$ in such a way that $\|\nabla\eta\|_{L^\infty(\mathbb{R}^n)} \leq 2|h-h'|^{-1}$. Inserting this in (101) and (102) we obtain the main estimates

$$\|\nabla v\|_{p,h'} = \|\eta\nabla v\|_{p,h'} \leq \|\eta\nabla v\|_{p,h} \leq C\|v\nabla\eta\|_{p,h} \leq C|h-h'|^{-1}\|v\|_{p,h} \tag{103}$$

$$\|v\|_{p^*,h'} = \|\eta v\|_{p^*,h'} \leq \|\eta v\|_{p^*,h} \leq C\|v\nabla\eta\|_{p,h} \leq C|h-h'|^{-1}\|v\|_{p,h}. \tag{104}$$

At this point we remind that $v = F(|u|) = F_{l,q}(|u|)$. Now recall that if $l_1 \leq l_2$ then $F_{l_1,q}(|u|) \leq F_{l_2,q}(|u|)$. This can be easily seen by noting that $F_{l,q}(t)$ is obtained by gluing at the point (l, l^q) the parabola t^q and its tangent line. As $q \geq 1$, we know that t^q is a convex function, so the tangent line stays below the graphic, and this gives $F_{l_1,q}(t) \leq F_{l_2,q}(t)$.

Besides, $v_{l,q}(x) = F_{l,q}(|u(x)|) \to |u(x)|^q$ for a.e. $x \in \Omega$ as $l \to \infty$. By the monotone convergence Theorem we can take limits as $l \to \infty$ in (104) to get $\||u|^q\|_{p^*,h'} \leq C|h-h'|^{-1}\||u|^q\|_{p,h}$, and this is the same to say that $\|u\|^q_{qp^*,h'} \leq C|h-h'|^{-1}\|u\|^q_{qp,h}$, which is equivalent to

$$\|u\|_{qp^*,h'} \leq (C|h-h'|^{-1})^{\frac{1}{q}}\|u\|_{qp,h} \tag{105}$$

We shall make some manipulations on notation in this inequality. Set $m := pq$, $\chi := \frac{p^*}{p} = \frac{n}{n-p} > 1$ and 105 reads

$$\|u\|_{\chi m,h'} \leq (C|h-h'|^{-1})^{\frac{p}{m}}\|u\|_{m,h}. \tag{106}$$

Recall now that in (106) the number $q \geq 1$ is a parameter, so we can put now $q_\nu = \chi^\nu$, $m = p\chi^\nu$, $h = h_\nu := r(1+2^{-\nu})$, $h' = h_{\nu+1} = r(1+2^{-\nu-1})$ for ν a natural number. Note that $|h_\nu - h_{\nu+1}| = r2^{-\nu-1}$. Inserting this in (106) we have

$$\begin{aligned}
\|u\|_{p\chi^{\nu+1},r} &\leq \|u\|_{p\chi^{\nu+1},h_{\nu+1}} \leq (C|h_\nu - h_{\nu+1}|^{-1})^{\chi^{-\nu}}\|u\|_{p\chi^\nu,h_\nu} \\
&= (Cr^{-1}2^{\nu+1})^{\chi^{-\nu}}\|u\|_{p\chi^\nu,h_\nu} = C^{\chi^{-\nu}}r^{-\chi^{-\nu}}2^{(\nu+1)\chi^{-\nu}}\|u\|_{p\chi^\nu,h_\nu} \\
&\leq \cdots \leq C^{\sum_{i=0}^\nu \chi^{-i}}r^{-\sum_{i=0}^\nu \chi^{-i}}2^{\sum_{i=0}^\nu (i+1)\chi^{-i}}\|u\|_{p,2r}
\end{aligned} \tag{107}$$

where we have iterated the inequality ν times.

Finally we note that, since $\chi > 1$, we have $\sum_{i=0}^\infty \chi^{-i} < \infty$ and $\sum_{i=0}^\infty (i+1)\chi^{-i} < \infty$ by the $i^{th} - square$ criterion for series. Then , as $\chi^{\nu+1} \to \infty$ as $\nu \to \infty$, letting $\nu \to \infty$ in (107) we conclude $\|u\|_{\infty,r} \leq C\|u\|_{p,2r}$ for some C depending on r, p, β, δ. Recall that C depended on U but we took U such that B_{2r} is compacty contained in U, so the dependence on U becomes dependence on r. This proves the theorem if $p < n$.

Case 2: Suppose $p = n$. Then we have, by (101) we have

$$\|\eta\nabla v\|_{L^n(U)} \leq C\|v\nabla\eta\|_{L^n(U)} \tag{108}$$

Now let $\alpha = (1-\varepsilon)n$ for $\varepsilon > 0$ small, and then $\alpha^* = \frac{n\alpha}{n-\alpha} = n\frac{1-\varepsilon}{\varepsilon}$. Recall that for U bounded we have that if $f \in L^n(U)$ then

$$\|f\|_{L^\alpha(U)} \leq |U|^{\frac{n-\alpha}{n\alpha}}\|f\|_{L^n(U)} \tag{109}$$

Now, by poincare's inequality, (109) and (108), we have

$$\begin{aligned}
\|\eta v\|_{L^{\alpha^*}(U)} &\leq C\|\nabla(\eta v)\|_{L^\alpha(\Omega)} \leq C(\|\eta\nabla v\|_{L^\alpha(U)} + \|v\nabla\eta\|_{L^\alpha(U)}) \\
&\leq C(\|\eta\nabla v\|_{L^n(U)} + \|v\nabla\eta\|_{L^n(U)}) \leq C\|v\nabla\eta\|_{L^n(U)}
\end{aligned} \tag{110}$$

where C depends on $\varepsilon, \beta, \delta, U, n$. We will work in some fixed U and for fixed ε, so this does not matter. Let $0 < h' < h < 2r$, and as done before, take η to be a nice cut off function such that $\eta = 1$ in $B_{h'}$, $\eta = 0$ in $\mathbb{R}^n \backslash B_h$ and $\|\nabla \eta\|_{L^\infty(\mathbb{R}^n)} \leq 2|h-h'|^{-1}$. Set $\chi := \frac{1-\varepsilon}{\varepsilon} > 1$ if ε is small, and insert this into (110) to obtain

$$\|v\|_{n\chi,h'} \leq \|\eta v\|_{n\chi,h} \leq C|h-h'|^{-1}\|v\|_{n,h}. \tag{111}$$

As before, $v = F_{l,q}(|u|)$, and letting $l \to \infty$ in (111), we get $\|u\|^q_{nq\chi,h'} \leq C|h-h'|^{-1}\|u\|^q_{nq,h}$, equivalent to

$$\|u\|_{nq\chi,h'} \leq (C|h-h'|^{-1})^{q^{-1}}\|u\|_{nq,h}. \tag{112}$$

Put $m = nq$ and this reads

$$\|u\|_{m\chi,h'} \leq (C|h-h'|^{-1})^{\frac{n}{m}}\|u\|_{m,h}. \tag{113}$$

this is the same estimate that we obtained in case one, see (106). The rest of the proof is exactly the same technical trick. Take $q = \chi^\nu$, so $m = \chi^\nu n$. Take $h_\nu = r(1+2^{-\nu})$, $h' = h_{\nu+1} = r(1+2^{-\nu-1})$. Insert this in (113), with ν a natural number, to get

$$\|u\|_{n\chi^{\nu+1},h_{\nu+1}} \leq (C|h_\nu - h_{\nu+1}|^{-1})^{\chi^{-\nu}}\|u\|_{n\chi^\nu,h_\nu} \tag{114}$$

iterating this process an letting $\nu \to \infty$ as done in (107), we conclude $\|u\|_{\infty,r} \leq C\|u\|_{n,2r}$ and the theorem is proved. □

Now we state, without proof, a regularity result that we will need. The proof can be found in [11]. The hipothesis in [11] seem to be different than the hipothesis we have put here, but in [11] it is also proved that our assumptions are stronger and implies the hipothesis of [11].

Theorem 16. Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 < p < \infty$, and $A : \Omega \times \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n$, be a C^1 function noted as $A(x,\xi)$. Assume that there exists numbers $\delta, \beta > 0$ such that we have the following:

(1) For each $x \in \Omega$, $\xi, h \in \mathbb{R}^n$ we have the degenerate ellipticity type inequality

$$\sum_{k,j=1}^{n} \partial_{\xi_j} A^k(x,\xi) h_j h_k \geq \delta|\xi|^{p-2}|h|^2 \tag{115}$$

(2) For each $x \in \Omega$, $\xi \in \mathbb{R}^n$, $j,k \in \{1,\dots,n\}$, we have $|\partial_{\xi_j} A^k(x,\xi)| \leq \beta|\xi|^{p-2}$

(3) For each $x \in \Omega$, $\xi \in \mathbb{R}^n$, $j,k \in \{1,\dots,n\}$, we have $|\partial_{x_j} A^k(x,\xi)| \leq \beta|\xi|^{p-1}$.

Then, if $u \in W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ is a weak solution of $div(A(x,\nabla u(x))) = 0$ in Ω, we have that u is locally $C^{1+\alpha}$ in Ω.

By this we mean that for each open set U compactly contained in Ω there exist constants C, α, depending on U, β, δ, n, p and u, such that $\|u\|_{C^{1,\alpha}(U)} \leq C$.

Now we give a lemma which allows to gain one order of weak differentiation under certain reasonable conditions.

Proposition 22. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $1 < p < \infty$. Consider a continuous function $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ that is C^1 in $\Omega \times \mathbb{R}^n \backslash \{0\}$ and assume that there exists $\delta > 0$ such that for every $x \in \Omega$, $t \geq 0$ and $\xi, \zeta \in \mathbb{R}^n$ we have

(1) $|A(x,\xi)| + |\partial_{x_j} A^k(x,\xi)| + |\xi||\partial_{\xi_j} A(x,\xi)| \leq \beta|\xi|^{p-1}$

(2) $(A(x,\xi) - A(x,\zeta)) \cdot (\xi - \zeta) \geq \delta(|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2$.

Let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of $div(A(x,\nabla u(x))) = 0$ such that $\nabla u(x) \neq 0$ for a.e. $x \in \Omega$. Then $u \in W^{2,2}_{loc}(\Omega)$.

Proof. First we remind that by 30 we also have

$$[D_\xi A(x,\xi)h]\cdot h = \sum_{k,j} \partial_{\xi_j} A^k(x,\xi) h_j h_k \geq C_{\delta,p}|\xi|^{p-2}|h|^2 \tag{116}$$

Then we can use theorem 15 to get that $u \in L^\infty_{loc}(\Omega)$, so we can use 16 to obtain that actually $u \in C^{1+\alpha}_{loc}(\Omega)$ for some $0<\alpha<1$. Then, ∇u is continuous after redefined in a null set.

For any function f we note $\Delta_h f(x) := f(x+h) - f(x)$. A basic and well known result tell us that $f \in W^{1,p}_{loc}(\Omega)$ if and only if $f \in L^p_{loc}(\Omega)$ and for every open set U compactly contained in Ω there exists a constant C independent of h (depending of U) such that $\|\nabla_h f\|_{L^p(U)} \leq C|h|$ if h is small enough. The proof of this is easy and can be found in [6], Section 5.8, Theorem 3.

Then, to prove the proposition we need to show that

$$\|\Delta_h \nabla u\|_{L^2(U)} \leq C|h| \tag{117}$$

for any U compactly supported in Ω and h small. The constant C can depend on the function u and the domain U, but not on h.

First note that if $\varphi \in W^{1,p}(\Omega)$ has compact support then we have

$$\int_\Omega A(x+h, \nabla u(x+h))\nabla\varphi(x)dx = \int_\Omega A(x,\nabla u(x))\nabla\varphi(x-h)dx = 0$$

since $supp(\varphi(\cdot - h) = supp(\varphi) + h \subset \Omega$ for h small enough. Then we have, for h small

$$\int_\Omega [A(x+h,\nabla u(x+h)) - A(x,\nabla u(x))]\nabla\varphi(x)dx = 0. \tag{118}$$

We shall take a suitable test function in this expression to see that (117) is true. Take $U \subset\subset V \subset\subset \Omega$, (by $\subset\subset$ we mean being compactly contained), and $\eta \in C^\infty_c(V)$ a cutoff function such that $\eta = 1$ in U, $0 \leq \eta \leq 1$. Take as test function $\varphi := \eta^2 \Delta_h u$. We shall insert φ into (118) and derive some estimates. To do this we shall manipulate (118). First note that

$$\begin{gathered} A(x+h,\nabla u(x+h)) - A(x,\nabla u(x)) = I_1(x,h) + I_2(x,h) \\ I_1(x,h) := A(x,\nabla u(x+h)) - A(x,\nabla u(x)) \\ I_2(x,h) := A(x+h,\nabla u(x+h)) - A(x,\nabla u(x+h)) \end{gathered} \tag{119}$$

For the first term we have, denoting $\Delta^t_h \nabla u(x) := t\nabla u(x+h) + (1-t)\nabla u(x)$, that

$$\begin{aligned} I^k_1(x,h) &= A^k(x,\nabla u(x+h)) - A^k(x,\nabla u(x)) \\ &= \textstyle\int_0^1 \partial_t[A^k(x, t\nabla u(x+h) + (1-t)\nabla u(x))]dt \\ &= \int_0^1 \nabla_\xi A^k(x,\Delta^t_h \nabla u(x))\Delta_h \nabla u(x)dt \\ &= \textstyle\sum_{j=1}^n \int_0^1 \partial_{\xi_j}[A^k(x,\Delta^t_h\nabla u(x))]\Delta_h \partial_j u(x)dt \end{aligned} \tag{120}$$

Denote $D_\xi A(x,\xi) := (\partial_{\xi_j} A^k(x,\xi))^n_{j,k=1}$. With $\varphi := \eta^2\Delta_h u$ as before, we have $\nabla\varphi = \eta^2\nabla\Delta_h u + 2\eta\nabla\eta\Delta_h u$. Note that clearly $\nabla\Delta_h u = \Delta_h \nabla u$. Motivated by (120) we compute now some terms that will appear in (118) later

$$\begin{aligned} &[D_\xi A(x,\Delta^t_h\nabla u(x))\Delta_h\nabla u(x)]\cdot\nabla\varphi(x) \\ &= \eta^2(x)[D_\xi A(x,\Delta^t_h\nabla u(x))\Delta_h\nabla u(x)]\cdot\Delta_h\nabla u(x) \\ &+2\eta(x)\Delta_h u[D_\xi A(x,\Delta^t_h\nabla u(x))\Delta_h\nabla u(x)]\cdot\nabla\eta(x) = S_1(t,h,x) + S_2(x) \end{aligned} \tag{121}$$

We estimate $S_1(t,h,x)$ using the ellipticity condition in (116) as follows

$$\begin{aligned} S_1(t,h,x) &= \eta^2(x)[D_\xi A(x,\Delta_h^t\nabla u(x))\Delta_h\nabla u(x)]\cdot\Delta_h\nabla u(x) \\ &\geq\ C\eta^2|\Delta_h^t\nabla u(x)|^{p-2}|\Delta_h\nabla u(x)|^2 \end{aligned} \tag{122}$$

To estimate $S_2(t,h,x)$ we use the Cauchy inequality with ε: if a,b are positive numbers then $2ab\leq a^2+b^2$, so given $\varepsilon>0$

$$ab=\sqrt{\varepsilon}a\frac{b}{\sqrt{\varepsilon}}\leq\frac{1}{2}[\varepsilon a^2+\frac{b^2}{\varepsilon}].$$

Then we have

$$\begin{aligned} &|S_2(t,h,x)| = 2\eta(x)|\Delta_h u(x)[D_\xi A(x,\Delta_h^t\nabla u(x))\Delta_h\nabla u(x)]\cdot\nabla\eta(x)| \\ &\leq 2|D_\xi A(x,\Delta_h^t\nabla u(x))|_{\mathcal{M}_{n\times n}}|\eta\Delta_h\nabla u(x)||\Delta_h u(x)\nabla\eta(x)| \end{aligned} \tag{123}$$

We claim now that there exists a compact set B of $\mathbb{R}^n$ such that $0\notin B$ and for some $\delta_0>0$ small we have

$$\{\Delta_h^t\nabla u(x): x\in\overline{V}, t\in[0,1], |h|\leq\delta_0\}\subset B.$$

The existence of B is justified as follows. To achieve that $0\notin B$ we need the fact that ∇u never vanishes in Ω, so $|\nabla u|\geq\gamma$ for some $\gamma>0$. Then by uniform continuity of ∇u in $V+B_{\delta_0}$, we can get that

$$max\{|\nabla u(x+h)-\nabla u(x)|: x\in V\}\leq\frac{1}{2}\gamma$$

for every h small enough. This inmediatly implies that $\Delta_h^t\nabla u(x)=t\nabla u(x+h)+(1-t)\nabla u(x)\neq 0$ for $x\in V$. The fact that B can be taken compact is just continuity of $\Delta_h^t\nabla u(x)$ respect x,h and t. Therefore we have

$$\inf\{|\Delta_h^t\nabla u(x)|: x\in V, t\in[0,1], |h|\leq\delta_0\}\geq C$$

for some constant C. If we look at (122), this yields that

$$S_1(t,h,x)\geq C_1\eta^2|\Delta\nabla u(x)|^2 \tag{124}$$

for $x\in V$, with C_1 indepent of $t\in[0,1], h\leq\delta_0$.

On the other hand, if we look at (123), the existence of B and the continuity of $D_\xi A$ in $\Omega\times\mathbb{R}^n\backslash\{0\}$ implies

$$S_2(t,h,x)\leq C_2(\varepsilon|\eta(x)\Delta_h\nabla u(x)|^2+\varepsilon^{-1}|\Delta_h u(x)\nabla\eta(x)|^2) \tag{125}$$

for $x\in V$, with $C_2=\|D_\xi A\|_{L^\infty(V\times B)}$ independent $t\in[0,1], h\leq\delta_0$.

Now, from (120), we can compute

$$\begin{aligned} &\int_\Omega I_1(x)\varphi(x)=\int_V\int_0^1\sum_{k,j=1}^n\partial_{\xi_j}A^k(x,\Delta_h^t\nabla u(x))\Delta_h\nabla_j u(x)\partial_k\varphi(x)dx \\ &=\int_V\int_0^1[D_\xi A(x,\Delta_h^t\nabla u(x))\Delta_h\nabla u(x)]\cdot\varphi(x)dtdx=\int_V\int_0^1 S_1(t,h,x)+S_2(t,h,x)dtdx \\ &\geq\int_V\int_0^1 C_1\eta^2|\Delta_h\nabla u(x)|^2dtdx-\int_V\int_0^1 C_2(\varepsilon|\eta(x)\Delta_h\nabla u(x)|^2+\varepsilon^{-1}|\Delta_h u(x)\nabla\eta(x)|^2)dtdx \\ &\geq C_1\|\eta\Delta_h\nabla u\|^2_{L^2(V)}-C_2\varepsilon\|\eta\Delta_h\nabla u\|^2_{L^2(V)}-C_2\varepsilon^{-1}\|\eta\Delta_h u\nabla\eta\|^2_{L^2(V)} \end{aligned} \tag{126}$$

It remains to estimate the second term I_2. Denote $D_xA(x,\xi) = (\partial_{x_j} A^k(x,\xi))_{j,k=1}^n$. Using (116) we obtain

$$\begin{aligned} &|I_2(x,h)\cdot\nabla\varphi(x)| \le |A(x+h,\nabla u(x+h)) - A(x,\nabla u(x+h))||\nabla\varphi(x)| \\ &\le [\text{ mean value theorem }] \le |h|\sup\{|D_xA(y,\nabla u(x+h))| : y\in[x,x+h]\}|\nabla\varphi(x)| \\ &\le |h||\nabla u(x+h)|^{p-1}|\nabla\varphi(x)| \le C|h||\nabla\varphi(x)| \end{aligned} \tag{127}$$

for every $x\in V$, $h\le\delta_0$, where the last inequality is because ∇u is continuous in $V+B_{\delta_0}$. Therefore, using that $0\le\eta\le1$, we have

$$\begin{aligned} &\int_\Omega I_2(x,h)\nabla\varphi(x,h)dx \ge -\int_V C|h|\eta|\Delta_h\nabla u|dx - \int_V 2C|h|\eta|\Delta_h u||\nabla\eta|dx \\ &\ge -C_3|h|\|\eta\Delta_h\nabla u\|_{L^2(V)} - C_4|h|\|\Delta_h u\|_{L^2(V)} \end{aligned} \tag{128}$$

the last inequality because in bounded domains $\|\cdot\|_{L^1}\le C\|\cdot\|_{L^2}$.

Finally we sum up (126) and (128) to obtain

$$\begin{aligned} 0 &= \int_V (I_1(x,h)+I_2(x,h))\nabla\varphi(x,h) \\ &\ge C_1\|\eta\Delta_h\nabla u\|^2_{L^2(V)} - C_2\varepsilon\|\eta\Delta_h\nabla u\|^2_{L^2(V)} - C_2\varepsilon^{-1}\|\eta\Delta_h u\nabla\eta\|^2_{L^2(V)} \\ &-C_3|h|\|\eta\Delta_h\nabla u\|_{L^2(V)} - C_4|h|\|\Delta_h u\|_{L^2(V)} \\ &\ge (C_1-\varepsilon C_2-|h|C_3)\|\eta\Delta_h\nabla u\|^2_{L^2(V)} - C_5\varepsilon^{-1}\|\Delta_h u\|^2_{L^2(V)} - C_4|h|\|\Delta_h u\|_{L^2(V)} \end{aligned} \tag{129}$$

where we have used that $C_2\varepsilon^{-1}\|\eta\Delta_h u\nabla\eta\|^2_{L^2(V)} \le C_5\varepsilon^{-1}\|\Delta_h u\|^2_{L^2(V)}$, because the constants are allowed to depend on η.

Now we choose ε and $|h|$ small so that $C_1-\varepsilon C_2-|h|C_3 = C_6 > 0$. We fix ε from now on, so it is eaten by the constants. Also note that since $u\in C^{1+\alpha}_{loc}(\Omega)$, then also $u\in W^{1,2}_{loc}(\Omega)$, and then we have that $\|\Delta_h u\|^2_{L^2(V)} \le C_7|h|$. Inserting this into (129) we get

$$0 \ge C_6\|\eta\Delta_h\nabla u\|^2_{L^2(V)} - C_8h^2 - C_4h^2$$

and finally this gives $\|\Delta_h\nabla u\|^2_{L^2(U)} \le \|\eta\Delta_h\nabla u\|^2_{L^2(V)} \le C_9|h|^2$, which proves the proposition. □

In the following proposition we could assume for simplicity that $\Omega = B_1$. For our purposes it is enough to consider B_1 since the results we are seeking are local and up to diffeomorphism. However we still work in a general Ω, since the arguments barely change. The only new requeriment we ask is that $\partial\Omega$ is regular in order to apply Sobolev embedding theorems. We remind that, by definition, $\partial\Omega\in C^k$ if it is locally the graph of a C^k function.

Proposition 23. Let $\Omega\subset\mathbb{R}^n$ an open bounded set with $\partial\Omega\in C^1$, and let $1<p<\infty$. Consider a continuous function $A:\Omega\times\mathbb{R}^n\to\mathbb{R}^n$ that is C^1 in $\Omega\times\mathbb{R}^n\backslash\{0\}$ and assume that there exists $\delta,\beta>0$ such that for every $x\in\Omega$, $t\ge0$ and $\xi,\zeta\in\mathbb{R}^n$ we have

(1) $|A(x,\xi)| + |\partial_{x_j}A^k(x,\xi)| + |\xi||\partial_{\xi_j}A(x,\xi)| \le \beta|\xi|^{p-1}$

(2) $(A(x,\xi)-A(x,\zeta))\cdot(\xi-\zeta) \ge \delta(|\xi|+|\zeta|)^{p-2}|\xi-\zeta|^2$.

Let $u\in W^{1,p}(\Omega)$ be a weak solution of $div(A(x,\nabla u(x))=0$ in Ω. Then for every open set $U\subset\subset\Omega$ there are constants $C>0$ and $\alpha\in(0,1)$, depending on n,p,δ,β,U and $\|u\|_{W^{1,p}(\Omega)}$ such that $u\in C^{1,\alpha}(U)$ and $\|u\|_{C^{1,\alpha}(U)}\le C$.

Suppose adittionaly that $A\in C^r(\Omega\times\mathbb{R}^n\backslash\{0\})$ for some $r>1$. Set $G=\{x\in\Omega : \nabla u(x)\ne0\}$. Then $u\in C^{r+1}_*(V)$ in any open set $V\subset\subset G$.

Proof. First of all we claim that $u \in L^{\infty}_{loc}(\Omega)$. We distinguish cases.

Case (a). If $p > n$, by the Sobolev embedding theorem we have

$$\|u\|_{L^\infty(\Omega)} \leq \|u\|_{C^{1-n/p}(\Omega)} \leq C_{n,p}\|u\|_{W^{1,p}(\Omega)}.$$

Case (b). If $p \leq n$ we use Theorem 15 to conclude that for every ball $B_{2r} \subset\subset \Omega$ we have that

$$\|u\|_{L^\infty(B_r)} \leq C_{n,p,\delta,\beta,r}\|u\|_{L^p(B_{2r})} \tag{130}$$

Let $\hat{U}$ be an open set such that $\hat{U} \subset\subset \Omega$. Take $r = \frac{1}{3}dist(\hat{U}, \partial\Omega)$. Then $\hat{U}$ can be covered by a finite number of balls of radious r, i.e, $\hat{U} \subset B_r(y_1) \cup \cdots \cup B_r(y_l)$, with $y_1, \ldots y_l \in \hat{U}$. Besides, by election of r, $B_{2r}(y_j) \subset \Omega$, so using (130) we see that $\|u\|_{L^\infty(B_r(y_j))} \leq C_{n,p,\delta,\beta,r}\|u\|_{L^p(\Omega)}$, and taking the maximum over $j = 1, \ldots, l$ we conclude that

$$\|u\|_{L^\infty(\hat{U})} \leq C_{n,p,\delta,\beta,r}\|u\|_{L^p(\Omega)} \tag{131}$$

so case (b) is over. Note that the constant depends on $\hat{U}$ via r, so it only depens on $dist(\hat{U}, \partial\Omega)$.

Now, if $U \subset\subset \Omega$, let $\hat{U}$ be an open set such that $U \subset\subset \hat{U} \subset\subset \Omega$ and such that $dist(\hat{U}, \partial\Omega) \approx \frac{1}{2}dist(U, \partial\Omega)$. In any of cases (a) and (b) we have concluded that

$$\|u\|_{L^\infty(\hat{U})} \leq C_{n,p,\delta,\beta,U}\|u\|_{W^{1,p}(\Omega)} \tag{132}$$

Note that the constant C depends on U via $dist(U, \partial\Omega)$.

Therefore we can use Theorem 16 to conclude that there exists constants C' and α depending on $n, p, \delta, \beta, dist(U, \partial\Omega)$ and $\|u\|_{W^{1,p}(\Omega)}$ such that $u \in C^{1+\alpha}(U)$ and $\|u\|_{C^{1+\alpha}(U)} \leq C'$. Note that we can use Theorem 16 because, by Proposition 30, we have

$$\sum_{k,j} \partial_{\xi_j} A^k(x,\xi) h_j h_k \geq C_{\delta,p}|\xi|^{p-2}|h|^2 \tag{133}$$

and therefore all the hipothesis are satisfied. This proves the first part.

Now suppose that $A \in C^r(\Omega \times \mathbb{R}^n \backslash \{0\})$ for some $r > 1$, and let $V \subset\subset \Omega$ be an open set such that $\nabla u(x) \neq 0$ for every $x \in \overline{V}$. Let us see that $u \in C^{r+1}_*(V)$. As ∇u is continuous, the set $G := \{\nabla u \neq 0\}$ is open and contains the compact $\overline{V}$. By Proposition 22 we know that $u \in W^{2,2}_{loc}(G)$, and therefore $u \in W^{2,2}(V)$.

Besides, as $div(A(x, \nabla u(x))) = 0$ in Ω, we have that $\partial_j[A^j(x, \nabla u(x))] = 0$ in the weak sense. Now, as $u \in W^{2,2}(V)$, we can apply the chain rule proved in Proposition 19, and we conclude that, for $x \in V$,

$$\sum_j \partial_{x_j} A^j(x, \nabla u(x)) + \sum_{k,j} \partial_{\xi_k} A^j(x, \nabla u(x)) \partial_j \partial_k u = 0. \tag{134}$$

As ∇u never vanishes in the compact $\overline{V}$ we have that for any $x \in V$, $|\nabla u(x)| > \varepsilon > 0$ for some positive number ε. Inserting this into (133) we get that

$$\sum_{k,j} \partial_{\xi_k} A^j(x, \nabla u(x)) h_j h_k \geq C_{\delta,p}\varepsilon^{p-2}|h|^2. \tag{135}$$

This shows that (134) can be interpreted as a linear elliptic equation that u satisfies in V. For clarity we write

$$a^{jk}(x) := \partial_{\xi_k} A^j(x, \nabla u(x))$$
$$f(x) := -\textstyle\sum_j \partial_{x_j} A^j(x, \nabla u(x))$$

and then (134) becomes, for $x \in V$,

$$\sum_{j,k=1}^{n} a^{jk}(x)\partial_j\partial_k u(x) = f(x) \tag{136}$$

This is a differential operator defined in V, whose symbol $p(x,\xi) = \sum_{k,j=1}^{n} a^{jk}\xi_j\xi_k$ satisfies $p(x,\xi) \geq C|\xi|^2$ for $x \in V$.

Now we have two cases.

Case (a). Suppose $1 < r < 2$. Then $0 < r-1 < 1, 0 < \alpha < 1$. Since $u \in C^{1,\alpha}(V)$ and $A \in C^r(\Omega \times \mathbb{R}^n \backslash\{0\})$, we see that the function $x \to (x, \nabla u(x))$ is $C^\alpha(V)$ and the partial derivatives of A are $C^{r-1}(\Omega \times \mathbb{R}^n\backslash\{0\})$. Then, by lemma 18, both a^{jk} and $f \in C^\sigma(V)$, with $\sigma := (r-1)\alpha$. Also $u \in W^{2,2}(V)$, so by elliptic regularity , Theorem 7, we conclude that $u \in C^{2+\sigma}(V)$.

Therefore ∇u is $C^{1,\sigma}(V)$, in particular Lipschitz, so by lemma 18 again, we see that both a^{jk} and $f \in C^{r-1}(V)$. Applying again elliptic regularity we see that $u \in C^{r+1}(V)$, as claimed. This finishes the case $1 < r < 2$.

Case (b). Suppose $r \geq 2$ and $r \notin \mathbb{N}$. From now on all function spaces C_* are considered in V, unless explicit mention. Write $r = k + \delta$ for some $k \in \mathbb{N}$, $k \geq 2$, and some $0 < \delta < 1$. We are going to see first that

$$u \in C^{k+1+\delta'} \text{ for some } 0 < \delta' \leq \delta \tag{137}$$

As the derivatives of A are $C^{r-1}(\Omega \times \mathbb{R}^n\backslash\{0\})$, they are Lipschitz. On the other hand $\nabla u \in C^\alpha(V)$. By the lemma again we see that both a^{jk} and $f \in C^\alpha(V)$. By elliptic regularity, $u \in C^{2+\alpha}(V)$, so $\nabla u \in C^{1,\alpha}(V)$. Therefore, as a^{jk}, f are compositions of C^1 functions, they are C^1. Let b be any of a^{jk}, f. We see that $b = g \circ h$ with $g \in C^{k-1+\delta}$ is some first derivative of $A(x,\xi)$, and $h \in C^{1+\alpha}$ is $x \to (x, \nabla u(x))$. Therefore the first partial derivatives of b are sums of functions of a very especific form, i.e,

$$\partial^1 b = (\partial^1 g \circ h)\partial^1 h^j \tag{138}$$

being h^j a $C^{1,\alpha}$ function (a component of h). Then, as $k \geq 2$, $\partial^1 g \circ h$ is at least C^δ and $\partial^1 h$ is C^α. Therefore by lemma 14 we conclude that $\partial^1 b = (\partial^1 g \circ h)\partial^1 h^j$ is $C^{\delta'}$, for $\delta' = min\{\delta, \alpha\} > 0$. We see that b is $C^{1,\delta'}$. Therefore $u \in C^{3+\delta'}$. If $k = 2$, this proves the claim (137).

Let $k \geq 3$. We more or less repit the argument above. First, with b any of a^{jk}, f as before, now we have $b = g \circ h$ with $g \in C^{k-1+\delta}$, $h \in C^{2+\delta'}$, (note that h has one more derivative now) so b is C^2. If we differentiate in (138) we obtain

$$\partial^2 b = (\partial^2 g \circ h)\partial^1 h^i \partial^1 h^j + (\partial^1 g \circ h)\partial^2 h^j = C + D \tag{139}$$

being as before h^i, h^j components of h. Now, $\partial^2 g \circ h$ is at least C^δ since $k \geq 3$, and $\partial^1 h^i \partial^1 h^j$ is $C^{1+\delta'}$, so C is at least C^δ. Besides, $\partial^1 g \circ h$ is C^2 and $\partial^2 h^j$ is $C^{\delta'}$, so D is at least $C^{\delta'}$. We conclude that $b \in C^{2,\delta'}$, so $u \in C^{4+\delta'}$. If $k = 3$ the claim is proved.

If $k \geq 4$, we repeat this procedure with the new information that $h \in C^{3+\delta'}$, so differentiating in (139) we see that $\partial^3 b \in C^{\delta'}$, so $b \in C^{3+\delta'}$ and $u \in C^{5+\delta'}$, which proves the claim in case $k = 4$. We can repeat this process until we have $u \in C^{k+1+\delta'}$, which gives (137).

We are almost done. We have seen that $u \in C^{r+1-\lambda}(V)$ with $\lambda := \delta - \delta'$, so $0 \leq \lambda < 1$. Therefore $\nabla u \in C^{r-\lambda} \subset C^{r-1}$, so both $a^{jk}, f \in C^{r-1}(V)$. Now we are going to make use of proposition 10. Note that the linear elliptic equation (136) can be interpreted as a non-smooth pseudodifferential operator $p(x,D)$ acting on u, i.e,

$$p(x,D)u = \sum_{k,j} a^{jk}(x)\partial_j\partial_k u(x) = f(x)$$

and the symbol $p(x,\xi)$ of $p(x,D)$ is $p(x,\xi) = \sum a^{kj}(x)\xi_k\xi_j$. Note that we have seen before that $p(x,D)$ is uniformly elliptic of order 2 in V, being V and arbitrary open set compactly contained in $G = \{\nabla u \neq 0\} \subset \Omega$.

Now take B any ball compactly contained in G, and take an open set V such that $B \subset\subset V \subset\subset G$. We shall verify the hipothesis of proposition 10. We have $a^{kj} \in C^{r-1}(V)$, $u \in C^{r+1-\lambda}(V) = C^{2-(r-1)+\epsilon}(V)$, being $\epsilon = 2r-2-\lambda > 0$ since $r \geq 2$ and $0 \leq \lambda < 1$. Also, $f \in C^{r-1}(V)$ and then by 10 we conclude that $u \in C^{r+1}_*(B)$.

Now, every $V \subset\subset G$ can be covered with a finite numbers of balls B compactly contained in G, and u is C^{r+1}_* in each ball, so we see that $u \in C^{r+1}_*(V)$.

Case (c). Suppose $r \geq 2$, $r \in \mathbb{N}$. Then $A(x,\xi) \in C^{r-\eta}$ for all $0 < \eta < \frac{1}{2}$. Take any ball $B \subset\subset G$ and let V such that $B \subset\subset V \subset\subset G$. As $r - \eta \notin \mathbb{N}$, from the previous cases we get $u \in C^{r+1-\eta}_*(V) = C^{2-(r-1)+\epsilon}_*(V)$, being $\epsilon = 2r-2-\eta > 0$ if η is small, since $r \geq 2$. As $a^{kj}, f \in C^{r-1}(V)$, we apply 10 and conclude that $u \in C^{r+1}_*(B)$. As before, as every $V \subset\subset G$ can be covered with a finite numbers of balls B compactly contained in G, we see that $u \in C^{r+1}_*(V)$. This proves the proposition. □

Now we prove an interpolation result that we will need later.

Lemma 31. (Interpolation) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $f \in C^\alpha(\overline{\Omega})$ for some $0 < \alpha \leq 1$. Let also $K \subset \Omega$ be a compact set, and put $\delta_0 := dist(K, \mathbb{R}^n \backslash \Omega)$. Let $1 \leq p < \infty$ and suppose that for some $M > 0$ we have

$$[f]_{C^\alpha(\overline{\Omega})} \leq M$$

$$\|f\|_{L^p(\Omega)} \leq \delta_0^{\frac{n+\alpha p}{p}} M$$

Under this hipothesis we have that

$$\|f\|_{L^\infty(K)} \leq C_{n,p,\alpha} M^{\frac{n}{n+\alpha p}} \|f\|_{L^p(\Omega)}^{\frac{\alpha p}{n+\alpha p}} \tag{140}$$

Proof. Let $x_0 \in K$ and $B_\delta(x_0)$ for some $0 < \delta < \delta_0$ that we will choose later. The triangle inequality yields

$$\|f(x_0)\|_{L^p(B_\delta(x_0))} \leq \|f(x_0) - f\|_{L^p(B_\delta(x_0))} + \|f\|_{L^p(B_\delta(x_0))}$$

so we have

$$\begin{aligned}
&\|f\|_{L^p(B_\delta(x_0))} \geq |f(x_0)||B_\delta(x_0)| - (\int_{B_\delta(x_0)} |f(x) - f(x_0)|^p dx)^{\frac{1}{p}} \\
&\geq C_{n,p}\delta^{\frac{n}{p}}|f(x_0)| - [f]_{C^\alpha(\Omega)}(\int_{B_\delta}(x_0)|x - x_0|^{\alpha p} dx)^{\frac{1}{p}} \\
&\geq C_{n,p}\delta^{\frac{n}{p}}|f(x_0)| - M(\int_{\rho=0}^{\delta} \rho^{\alpha p+n-1} d\rho)^{\frac{1}{p}} = C_{n,p}\delta^{\frac{n}{p}}|f(x_0)| - MC_{n,p,\alpha}\delta^{\frac{\alpha p+n}{p}}
\end{aligned}$$

Sinse $0 < \delta < \delta_0 = dist(K, \mathbb{R}^n \backslash \Omega)$, this δ does not depend on $x_0 \in K$. Therefore we can take the supremum on $x_0 \in K$ to see that

$$\|f\|_{L^p(\Omega)} \geq C_{n,p}\delta^{\frac{n}{p}}\|f\|_{L^\infty(K)} - MC_{n,p,\alpha}\delta^{\alpha+\frac{n}{p}}$$

and thus we conclude that

$$\|f\|_{L^\infty(K)} \leq C_{n,p}\delta^{-\frac{n}{p}}\|f\|_{L^p(\Omega)} + MC_{n,p,\alpha}\delta^\alpha \tag{141}$$

by hipothesis we can take

$$\delta = \left[\frac{\|f\|_{L^p(\Omega)}}{M}\right]^{\frac{p}{n+\alpha p}} \leq \delta_0$$

inserting this δ into (141) we have

$$\|f\|_{L^\infty(K)} \leq C_{n,p}[\|f\|_{L^p(\Omega)}]^{\frac{\alpha p}{n+\alpha p}} M^{\frac{n}{n+\alpha p}} + C_{n,p,\alpha} M^{\frac{n}{n+\alpha p}} [\|f\|_{L^p(\Omega)}]^{\frac{\alpha p}{n+\alpha p}}$$

and this gives (140) and proves the lemma. □

Now we are ready to prove the existence of A-harmonic coordinates.

Theorem 17. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $1 < p < \infty$. Consider a continuous function $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ that is C^1 in $\Omega \times \mathbb{R}^n \backslash \{0\}$ and assume that there exists $\delta > 0$ such that for every $x \in \Omega$, $t \geq 0$ and $\xi, \zeta \in \mathbb{R}^n$ we have

(1) $A(x, t\xi) = t^{p-1} A(x, \xi)$

(2) $(A(x, \xi) - A(x, \zeta)) \cdot (\xi - \zeta) \geq \delta(|\xi| + |\zeta|)^{p-2} |\xi - \zeta|^2$.

Under these hipothesis, given any point $x_0 \in \Omega$ there is a nieghborhood $U := U^{x_0}$ of x_0 and there is a C^1 (in fact $C^{1,\alpha}$ for some $0 < \alpha < 1$) diffeomorphism

$$\Phi = (\phi^1, \ldots, \phi^n) : U \to V := \Phi(U) \subset \mathbb{R}^n$$

such that all its components $\phi^j : U \to \mathbb{R}$ are A-harmonic in U (in the weak sense), and such that, given any invertible matrix S, the differential $D\Phi$ of Φ satisfies $\|D\Phi(x_0) - S\| < \eta$ for any $\eta > 0$ previously fixed.

Besides, any C^1 diffeomorphism defined near x_0 whose components are A-harmonic is in fact a $C^{1,\alpha}$ diffeomorphism near x_0 for some $0 < \alpha < 1$.

Moreover, if A is C^r for some $r > 1$ then we can achieve that Φ is a C^{r+1}_* diffeomorphism (and therefore its components are strongly A-harmonic). In this case, any C^1 diffeomorphism defined near x_0 whose components are A-harmonic (in the weak sense) is in fact a C^{r+1}_* diffeomorphism near x_0, so its components are strongly A-harmonic.

Proof. Take first S as the identity matrix. We can assume that $x_0 = 0$ via a translation, and for simplicity we shall suppose that $B_1 \subset \Omega$. If we did not suppose that $B_1 \subset \Omega$, the proof would go the same way changing B_1 by B_r for some fixed $r > 0$ small enough so that $B_r \subset \Omega$.

The coordinates ϕ^j will be obtained by solving, for ε small, the Dirichlet problem

$$\begin{aligned} &div(A(x, \nabla\phi^j(x))) = 0 \qquad \text{in } B_\varepsilon \\ &\phi^j|_{\partial B_\varepsilon} = x^j \end{aligned} \tag{142}$$

Recall that by 30, the hipothesis of propositions 18 and 23 are satisfied. Thus, for $0 < \varepsilon \leq 1$, the problem (142) has a weak solution $\phi^j \in W^{1,p}(B_\varepsilon)$ such that $\phi^j - x^j \in W^{1,p}_0(B_\varepsilon)$. Moreover we know that in any subdomain of B_ε, ϕ^j is $C^{1,\alpha}$ for $0 < \alpha < 1$ depending on the subdomain. We will show that if ε is small enough, the Jacobian matrix $D\Phi$ is invertible at the origin. Then we will use the inverse function Theorem.

We write $u := \phi^j$ below, for some fixed j, being ϕ^j the mentioned solution of (142). Define the dilated coordinates $\hat{x} := x/\varepsilon$ and let

$$\hat{u}(\hat{x}) := \varepsilon^{-1} u(\varepsilon\hat{x})$$

Note that $\hat{u}$ depends on ε and is defined in B_1. Similarly, given $\varphi \in C_c^\infty(B_\varepsilon)$, let $\hat{\varphi}(\hat{x}) := \varepsilon^{-1}\varphi(\varepsilon\hat{x})$, and $\hat{\varphi}(\hat{x}) \in C_c^\infty(B_\varepsilon)$. By the chain rule we have that

$$\nabla\hat{u}(\hat{x}) = \nabla u(\varepsilon\hat{x})$$

so $\nabla\hat{u}(x/\varepsilon) = \nabla u(x)$. Then it follows that

$$\int_{B_\varepsilon} A(x, \nabla u(x)) \cdot \nabla\varphi(x)dx = \int_{B_\varepsilon} A(x, \nabla\hat{u}(x/\varepsilon)) \cdot \nabla\varphi(x/\varepsilon)dx$$
$$= \varepsilon^n \int_{B_1} A(\varepsilon\hat{x}, \nabla\hat{u}(\hat{x})) \cdot \nabla\varphi(\hat{x})d\hat{x} = \varepsilon^n \int_{B_1} A_\varepsilon(\hat{x}, \nabla\hat{u}(\hat{x})) \cdot \nabla\varphi(\hat{x})d\hat{x}$$

being $A_\varepsilon(\hat{x}, \hat{q}) := A(\varepsilon\hat{x}, \hat{q})$. Therefore the function u solves (142) if and only if $\hat{u}$ solves

$$\begin{aligned} &div(A_\varepsilon(\hat{x}, \nabla\hat{u}(\hat{x})) = 0 \qquad \text{in } B_1 \\ &\hat{u}|_{\partial B_1} = \hat{x}^j \end{aligned} \tag{143}$$

We have, thus, translated the original Dirichlet problem in B_ε to another in B_1, but the function A has been changed for A_ε. Note that A_ε is defined for x in some neighborhhod of B_1 and satisfies the assumptions on this Theorem with the same constant δ uniformly in $0 < \varepsilon \leq 1$. Therefore, by proposition (30), we see that A_ε also satisfies the hipothesis of propositions 18 and 23 for the same constants β, δ, etc, uniformly in ε. We conclude that

$$\begin{aligned} &\|\hat{u}\|_{W^{1,p}(B_1)} \leq C_1\|\hat{x}^j\|_{W^{1,p}(B_1)} \leq C \\ &\|\hat{u}\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \leq C' \end{aligned} \tag{144}$$

where C depends on δ, p, B_1 and C' depends on $\delta, p, B_1, \|\hat{u}\|_{W^{1,p}(B_1)}$. In any case, we can change in the above estimate C' for some constant C independent of ε. Now write $\hat{u} = \hat{u}_0 + \hat{u}_1$, where $\hat{u}_0(\hat{x}) := \hat{x}^j$ is the j-th coordinate. Note that

$$\hat{u}_1 = \hat{u} - \hat{u}_0 \in W_0^{1,p}(B_1).$$

We want to see that $\nabla\hat{u}$ is near $\nabla\hat{u}_0$ if ε is small. More precisely we caim that

$$\|\nabla\hat{u}_1\|^p_{L^p(B_1)} = \int_{B_1} |\nabla\hat{u}_1|^p d\hat{x} \leq C\varepsilon^{min\{p,p'\}} \tag{145}$$

with p' such that $\frac{1}{p} + \frac{1}{p'} = 1$, i.e, $p' = \frac{p}{p-1}$. To see (145) define

$$I := \int_{B_1} (|\nabla\hat{u}| + |\nabla\hat{u}_0|)^{p-2}|\nabla\hat{u}_1|^2 d\hat{x}.$$

Estimate (145) will follow if we see first that

$$I \leq C\varepsilon\|\nabla\hat{u}_1\|_{L^p(B_1)} \tag{146}$$

Indeed, suppose we know (146). Then, if $p \geq 2$ we have

$$\int_{B_1} |\nabla\hat{u}_1|^p d\hat{x} = \int_{B_1} |\nabla\hat{u}_1|^{p-2}|\nabla\hat{u}_1|^2 d\hat{x}$$
$$\leq \int_{B_1} (|\nabla\hat{u}| + |\nabla\hat{u}_0|)^{p-2}|\nabla\hat{u}_1|^2 d\hat{x} = I \leq C\varepsilon(\int_{B_1} |\nabla\hat{u}_1|^p d\hat{x})^{1/p}$$

so $\|\nabla\hat{u}_1\|^p_{L^p(B_1)} \leq \varepsilon^{p'}$. On the other hand, if $1 < p < 2$, take $a = \frac{2}{p} > 1$ and $a' = \frac{2}{2-p}$ so that $\frac{1}{a} + \frac{1}{a'} = 1$. The Holder inequality with a and a' implies

$$\begin{aligned}
&\int_{B_1} |\nabla\hat{u}_1|^p d\hat{x} = \int_{B_1} (|\nabla\hat{u}| + |\nabla\hat{u}_0|)^{-\frac{p(p-2)}{2}} (|\nabla\hat{u}| + |\nabla\hat{u}_0|)^{\frac{p(p-2)}{2}} |\nabla\hat{u}_1|^p d\hat{x} \\
&= [\int_{B_1} \{(|\nabla\hat{u}| + |\nabla\hat{u}_0|)^{-\frac{p(p-2)}{2}}\}^{\frac{2}{2-p}} d\hat{x}]^{\frac{2-p}{2}} [\int_{B_1} \{(|\nabla\hat{u}| + |\nabla\hat{u}_0|)^{\frac{p(p-2)}{2}} |\nabla\hat{u}_1|^p\}^{\frac{2}{p}} d\hat{x}]^{\frac{p}{2}} \\
&= [\int_{B_1} \{(|\nabla\hat{u}| + |\nabla\hat{u}_0|)^p d\hat{x}]^{\frac{2-p}{2}} I^{\frac{p}{2}} \leq C I^{\frac{p}{2}}
\end{aligned}$$

in the last step we used that $u_0 = \hat{x}^j \in W^{1,p}(B_1)$ is a fixed function, and $\|\hat{u}\|_{W^{1,p}(B_1)} \leq C$ by (144). Therefore

$$\int_{B_1} |\nabla\hat{u}_1|^p d\hat{x} \leq C I^{\frac{p}{2}} \leq \varepsilon^{\frac{p}{2}} [\int_{B_1} |\nabla\hat{u}_1|^p d\hat{x}]^{\frac{1}{2}}$$

and therefore $\|\nabla\hat{u}_1\|^p_{L^p(B_1)} \leq \varepsilon^p$.

Now it remains to see (146). We compute:

$$\begin{aligned}
I &= \int_{B_1} (|\nabla\hat{u}| + |\nabla\hat{u}_0|)^{p-2} |\nabla\hat{u}_1|^2 d\hat{x} \\
&\leq \frac{1}{\delta} \int_{B_1} [A_\varepsilon(\hat{x}, \nabla\hat{u}) - A_\varepsilon(\hat{x}, \nabla\hat{u}_0)] \cdot [\nabla\hat{u} - \nabla\hat{u}_0] d\hat{x} \\
&= -\frac{1}{\delta} \int_{B_1} A_\varepsilon(\hat{x}, \nabla\hat{u}_0) \cdot [\nabla\hat{u} - \nabla\hat{u}_0] d\hat{x} \\
&= -\frac{1}{\delta} \int_{B_1} [A_\varepsilon(\hat{x}, \nabla\hat{u}_0) - A_0(\hat{x}, \nabla\hat{u}_0)] \cdot [\nabla\hat{u} - \nabla\hat{u}_0] d\hat{x}
\end{aligned} \tag{147}$$

In the first line of (147) we used that A_ε satisfies the hipothesis of the theorem. In the second line we used that $\hat{u}$ satisfies $div(A_\varepsilon(\hat{x}, \nabla\hat{u}(\hat{x}))) = 0$ in B_1 in the weak sense, and that $\hat{u}_1 = \hat{u} - \hat{u_0} \in W^{1,p}_0(B_1)$ is a test function, so we have

$$\int_{B_1} A_\varepsilon(\hat{x}, \nabla\hat{u}) \cdot [\nabla\hat{u} - \nabla\hat{u}_0] d\hat{x} = 0$$

In the third line we have used that $\nabla\hat{u}_0 = e_j$ is constant so $A_0(\hat{x}, \nabla\hat{u}_0) = A(0, \nabla\hat{u}_0) := v \in \mathbb{R}^n$ is also a constant, and then we have

$$\int_{B_1} A_0(\hat{x}, \nabla\hat{u}_0) \cdot [\nabla\hat{u} - \nabla\hat{u}_0] d\hat{x} = \int_{B_1} v \cdot \nabla\hat{u}_1 d\hat{x} = \sum_j \int v_j \partial^j [\hat{u}_1] = \sum_j \int \hat{u}_1 \partial^j [v_j] = 0$$

since, as $\hat{u}_1 \in W^{1,p}_0(B_1)$, the boundary terms are zero when we integrate by parts.

Now we remind that by proposition (30), for all $l = 1, \ldots, n$ we have $|\partial_{x_l} A_\varepsilon(\hat{x}, \xi)| \leq \beta |\xi|^{p-1}$. By the mean value Theorem it follows that if we denote $e_j = \nabla\hat{u}_0$ then

$$|A_\varepsilon(\hat{x}, e_j) - A_0(\hat{x}, e_j)| = |A(\varepsilon\hat{x}, e_j) - A(0, e_j)| \leq |e_j|^{p-1} |\varepsilon\hat{x}| \leq \varepsilon$$

for every $\hat{x}$ in B_1. Therefore, coming back to (147), we see that

$$I \leq \frac{1}{\delta} \int_{B_1} |A_\varepsilon(\hat{x}, \nabla\hat{u}_0) - A_0(\hat{x}, \nabla\hat{u}_0)| \cdot |\nabla\hat{u}_1| d\hat{x} \leq C\varepsilon \|\nabla\hat{u}_1\|_{L^p(B_1)}$$

and this proves (146). Collecting the results so far we have, by (145) and (144) that

$$\begin{aligned} &\|\nabla\hat{u} - \nabla\hat{u}_0\|_{L^p(B_1)} \le C\varepsilon^{min\{1,\frac{1}{p-1}\}} \\ &\|\nabla\hat{u} - \nabla\hat{u}_0\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \le C \end{aligned} \tag{148}$$

Now we can interpolate these two inequalities using Lemma 141. We use the notations of that Lemma. Let $f = \nabla\hat{u} - \nabla\hat{u}_0$. We know that $\|f\|_{L^p(B_{\frac{1}{2}})} \le C\varepsilon^{min\{1,\frac{1}{p-1}\}}$, $\|f\|_{C^{1,\alpha}(B_{\frac{1}{2}})} \le M$ for some constant M. Take $K = \overline{B_{\frac{1}{8}}}$, so $\delta_0 = \frac{3}{8}$. If we take ε small enough we can achieve that $\|f\|_{L^p(B_{\frac{1}{2}})} \le [\delta_0]^{\frac{n+\alpha p}{p}} M$, so we conclude that, for some exponent $\gamma > 0$, we have

$$\|\nabla\hat{u} - \nabla\hat{u}_0\|_{L^\infty(B_{\frac{1}{8}})} \le C\varepsilon^\gamma = o(1) \qquad \text{as } \varepsilon \to 0 \tag{149}$$

Note that this process was made for $\hat{u}(\hat{x}) = \varepsilon^{-1}u(\varepsilon\hat{x}) = \varepsilon^{-1}\phi^j(\varepsilon\hat{x})$, so $\phi^j(x) = \varepsilon\hat{u}(\varepsilon^{-1}x)$, and $\nabla\phi^j(x) = \nabla\hat{u}(\varepsilon^{-1}x)$, so from (149), and taking into account that $\nabla\hat{u}_0 = e_j$, we see that

$$\|\nabla\phi^j - e_j\|_{L^\infty(B_{\frac{\varepsilon}{8}})} \le C\varepsilon^\gamma = o(1) \qquad \text{as } \varepsilon \to 0 \ .$$

Repeating this argument with each component and denoting $\Phi = (\phi_1, \ldots, \phi_n) : B_\varepsilon \to \mathbb{R}^n$, and Id for the identity $n \times n$ matrix, then in particular

$$|D\Phi(0) - Id| \le \varepsilon^\gamma = o(1) \qquad \text{as } \varepsilon \to 0$$

So $D\Phi(0)$ is an invertible matrix (and arbitrarily close to Id) for ε small enough.

Now recall that $A(x,\xi)$ is $C^1(\Omega \times \mathbb{R}^n\backslash\{0\})$ and the diffeomorphism Φ is $C^{1,\alpha}$ for some $0 < \alpha < 1$. By the inverse function Theorem we get that there exists an open neighborhood U of $x_0 = 0$ such that $\Phi : U \to V = \Phi(U)$ is a C^1 diffeomorphism. Let us see that actually it is a $C^{1,\alpha}$ diffeomorphism. The inverse function theorem also tell us that

$$D(\Phi^{-1}) = [D\Phi \circ \Phi^{-1}]^{-1} : V \to Gl(n) \tag{150}$$

Note that, maybe in a smaller U, we can achieve that Φ^{-1} is Lipschitz, and that the determinant Det of $D\Phi \circ \Phi^{-1}$ is $\ge c > 0$ for some c. Note $D\Phi \circ \Phi^{-1}$ is a composition of a Lipschitz and a C^α function, and Det is a sum of products of C^α functions, so both of them are C^α. We see then that $\frac{1}{Det}$ is also C^α by Lemma 17. We conclude by the formula of inverting matrixes that $D(\Phi^{-1})$ is C^α.

Now let F be any other C^1 diffeomorphism defined near x_0 whose components f^j are A-harmonic. By Proposition 23, we know that the f^j are in fact $C^{1,\alpha}$ for some $0 < \alpha < 1$, and by the formula for the differential of the inverse given in (150) we see, with the same arguments as before, that F is a $C^{1,\alpha}$ diffeomorphism near x_0.

Finally, assume that $A(x,\xi)$ is C^r with $r > 1$. We know then that Φ is C^{r+1}_* by 23. Put $r = k + \delta$ for $0 \le \delta < 1$ and note that if $k = 1$ then $\delta > 0$, and if $\delta = 0$ then $k \ge 2$. If $\delta > 0$, by the IFT we know that Φ^{-1} is $C^{k+1} \subset C^r$. If $\delta = 0$, then $\Phi \in C^{k+1}_* \subset C^k = C^r$ so $\Phi^{-1} \in C^r$ also by the IFT. We see that in any case $\Phi^{-1} \in C^r$.

As $D\Phi$ is C^r_*, the composition $D\Phi\circ\Phi^{-1}$ is C^r_* by Lemma 19. As C^r_* is an algebra with the pointwise multiplication by Proposition 16, we see that Det is also C^r_*. Now, by Lemma 17, we see that $\frac{1}{Det}$ is also C^r_*. We conclude from the formula (150) that $D(\Phi^{-1})$ is C^r_*, and so Φ^{-1} is $C^{r+1}_*(V)$, and this yields that Φ is a C^{r+1}_* diffeomorphism as we wanted.

Let F be any other C^1 diffeomorphism near x_0 whose components f^j are A-harmonic. By Proposition 23, we know that the f^j are in fact C^{r+1}_* near x_0, and using the formula of the differential of the inverse as above, we see that F is a C^{r+1}_* diffeomorphism near x_0.

The proof of the Theorem in case $S = I$ is done. If S is any invertible matrix, we change the initial data of (142) and choose ϕ^j to be the solutions of

$$\begin{aligned} &div(A(x, \nabla\phi^j(x))) = 0 \qquad \text{in } B_\varepsilon \\ &\phi^j|_{\partial B_\varepsilon} = (Sx)^j \end{aligned} \tag{151}$$

Repeating the argument we see (with the same notations as above) that the only change is that now $\hat{u_0}(\hat{x}) = (S\hat{x})^j$ and thus $\nabla\hat{u_0}$ is the j-th column of the matrix S. This gives that $D\Phi(0)$ is arbitrarily close to S as $\varepsilon \to 0$. $\square$

From this Theorem, the existence of p-harmonic coordinates on a Riemannian manifold is an inmediate consequence.

Theorem 18. Let $r > 1$, and let (M, g) be a Riemannian manifold whose metric is of class C^r in some C^{r+1}_* local coordinate chart φ near a point $x_0 \in M$. Let also $1 < p < \infty$. Then, there exists a local coordinate chart F defined in some open neigborhood U of x_0 such that

(1) $F : U \to F(U)$ is a C^{r+1}_* diffeomorphism onto an open set of $\mathbb{R}^n$.

(2) The coordinate functions of F are p-harmonic.

(3) The metric expressed in these p-harmonic coordinates satisfies $|g(x_0) - Id| < \varepsilon$ for any given $\varepsilon > 0$ previously fixed.

Moreover, all C^1 p-harmonic coordinates (in the weak sense) near x_0 are C^{r+1}_* diffeomorphisms, and every C^1 p-harmonic function defined near x_0 such that $du(x_0) = \nabla u(x_0) \neq 0$ have C^{r+1}_* regularity near x_0.

Proof. Take a C^{r+1}_* coordinate system $\varphi = (x_1, \dots, x_n)$ in an open neighborhood U of x_0, so that the metric is C^r in the φ coordinates. We shall work in these coordinates $\varphi : U \to \Omega$, whose range is a certain open set $\Omega \subset \mathbb{R}^n$. Take any function u defined near x_0 and express u in these coordinates putting $u_\varphi = u \circ \varphi^{-1}$, so u_φ is defined in Ω.

As discussed at the begining of this section, $u_\varphi := u \circ \varphi^{-1}$ is p-harmonic near $\varphi(x_0)$ if and only if $div(A(x, \nabla u_\varphi(x))) = 0$ where $A(x, q)$ is given in 38, so is C^r regular in $\Omega \times \mathbb{R}^n \backslash \{0\}$ and satisfies the hipothesis of Theorem 17. Therefore, if u is a C^1 p-harmonic function defined near x_0 such that $\nabla u(x_0) \neq 0$, then in the φ coordinates u_φ is a C^1 A-harmonic function near $\varphi(x_0)$ such that $\nabla u(\varphi(x_0)) \neq 0$, so ∇u does not vanish near x_0 by continuity.

Then, by Theorem 17, we see that u_φ is C^{r+1}_* near $\varphi(x_0)$. As φ is C^{r+1}_*, we see that $u = u \circ \varphi^{-1} \circ \varphi$ is C^{r+1}_* near x_0 as a function defined on an open set of *the manifold* M . Note that u_φ is C^{r+1}_* near $\varphi(x_0)$ no matter of regular is the coordinate system φ, but if we want u to have that regularity as a function defined on M, we have to demand the coordinates φ to be C^{r+1}_*. This proves the last asertion of the Theorem.

To see the others assertions, we apply Theorem 17 and find a C^{r+1}_* diffeomorphism $\Phi : G \to \Phi(G)$ from some open neighborhood $G \subset \Omega$ of $\varphi(x_0)$ onto $\Phi(G) \subset \mathbb{R}^n$ whose components are A-harmonic. Then $F := \Phi \circ \varphi : \varphi^{-1}(G) \to \Phi(G)$ is a C^{r+1}_* diffeomorphism from some open neighborhood of x_0 onto an open set of $\mathbb{R}^n$ whose coordinate functions are p-harmonic near x_0 by construction (since they are p-harmonic in some coordinate system and the p-harmonic equation does not depend on coordinates). If $G_0 := (g_{ij}(\varphi(x_0)))_{i,j}$ is the metric at x_0 in the φ coordinates, the metric at x_0 expressed in the F coordinates is given by

$$G_1 = [D(\Phi^{-1})(F(x_0))]^t G_0 [D(\Phi^{-1})(F(x_0))] = \left([D\Phi(\varphi(x_0))]^{-1}\right)^t G_0 [D\Phi(\varphi(x_0))]^{-1}.$$

In Theorem 17 we saw that $D\Phi((\varphi(x_0))$ can be chosen arbitrarily close to any matrix. Let A be an invertible matrix such that $A^t G_0 A = Id$. Then, if we take $D\Phi((\varphi(x_0))$ close enough to A^{-1}, by continuity of the operations to compute the inverse of a matrix, we will have that G_1 is arbitrarily close to Id.

The next assertions of the Theorem have already been proved in Theorem 17, and this concludes the proof. $\square$

Remark 39. Suppose that we weaken the hipothesis of the Theorem above by only asking that $\varphi : U^{x_0} \to \Omega \subset \mathbb{R}^n$ is a C^1 coordinate system defined near some point $x_0 \in M$, such that the pull-back $g_\varphi = (\varphi^{-1})^* g$ is C^r for $r > 1$. Then we have a C^{r+1}_* diffeomorphism $F : \Omega' \to V \subset \mathbb{R}^n$, being $\Omega' \subset \Omega$ a smaller neighborhood of $\varphi(x_0)$, such that the coordinates of F are p-harmonic functions for the Riemannian manifold (Ω', g_φ). But note that the lift $\hat{F} := F \circ \varphi : \varphi^{-1}(\Omega') \to V$ of F to a function on M defined near x_0 is *not* C^{r+1}_* regular, but only C^1 regular as the coordinates φ.

This remark will be more important to us than the Theorem above, since we are going to give results about specific and fixed localitations (Ω, g_φ) of (M, g) via local coordinates $\varphi : U^{x_0} \to \Omega$ defined locally on a manifold M. This coordinates will not necessarily be smooth as functions defined on M. This way, we can suppose that $(M, g) = (\Omega, g_\varphi)$ is an open set of $\mathbb{R}^n$ equipped with a possibly nonregular metric g_φ, and forget about the differentiable structure of M.

6.3. Some properties of p-harmonic coordinates.

One property of p-harmonic coordinates is that when we express the metric in these coordinates, we can be sure that no other coordinates will make the metric more regular, as we will see.

Proposition 24. Let (M, g) be a Riemannian manifold. Suppose that the metric g is C^r for some $r > 1$ in some C^1 local coordinates φ near a point $x_0 \in M$. If a tensor field T is C^s for some $s \geq r$ expressed in the φ coordinates then it is of class at least C^r_* in any p-harmonic coordinates near x_0.

Proof. Suppose $\varphi : U \to \Omega$, with U a neighborhood of x_0 and $\Omega \subset \mathbb{R}^n$. By Theorem 18 we know that there exists a system of p-harmonic coordinates $F : V \to G \subset \mathbb{R}^n$ defined in some open neighborhood $V \subset \Omega$ of $\varphi(x_0)$. By that Theorem we also know that F is a C^{r+1}_* diffeomorphism. Let $\phi := F \circ \varphi : \varphi^{-1}(V) \to G$. Denote the coordinate representations of T in the φ and ϕ coordinates by $T_\varphi = (\varphi^{-1})^* T$ and $T_\phi = (\phi^{-1})^* T$. They are related by the pullback, i.e,

$$T_\phi = (F^{-1})^* T_\varphi.$$

This pull back only involves first derivatives of F^{-1}, which is C^{r+1}_* regular. Therefore $(F^{-1})^* T_\varphi$ is of class $C^r_*(G)$, since by hipothesis T_φ is $C^s(V)$ with $s \geq r$. This proves that T_ϕ is $C^r_*(G)$.

Moreover, we know that any p-harmonic coordinate system near x_0 is a C^{r+1}_* diffeomorphism, and the same argument gives that T is C^r_* expressed in any p-harmonic coordinate system. $\square$

Remark 40. In the Proposition above we did not require any regularity condition on the system of coordinates φ since the claim was for the expression in coordinates of the tensor T, and not for the tensor T itself defined in the manifold M.

In the following Corollary we state a remarkable feature of p-harmonic coordinates. It boils down to say that among all the possible expressions in coordinates of a metric g near a point $x_0 \in M$, the expression of g in any p-harmonic coordinate system has the maximal regularity possible.

Corollary 13. Let (M, g) be a Riemannian manifold, $r > 1$ and fix $x_0 \in M$. For any C^1 coordinate system $\alpha : U^{x_0} \to V \subset \mathbb{R}^n$, defined in some neighborhood U^{x_0} of x_0 (which may depend on α), denote

$g_\alpha := (\alpha^{-1})^* g$ the expression of g in the α coordinates. Suppose that the metric g is of class C^r when expressed in some C^1 coordinate system φ near a point x_0. Fix any p-harmonic coordinate system ϕ near x_0. Denote

$$max(g) := \sup\{t > 1 : \text{ exists a } C^1 \text{ coordinate system } \alpha \text{ such that } g_\alpha \in C^t \text{ near } x_0\} \leq \infty.$$

We have:

(1) If $max(g) \notin \mathbb{N}$ and there exists a C^1 coordinate chart α^* such that $g_{\alpha^*} \in C^{\max(g)}$ near x_0 then also $g_\phi \in C^{\max(g)}$ near x_0. Therefore g has maximal regularity expressed in any p-harmonic coordinates (we mean by this that g can not be made any smoother by changing coordinates).

(2) If $max(g) \in \mathbb{N}$ and there exists a C^1 coordinate chart α^* such that $g_{\alpha^*} \in C^{\max(g)}$ near x_0, then $g_\phi \in C_*^{\max(g)}$ near x_0. In this case g may have not maximal regularity in p-harmonic coordinates, but almost.

(3) If for every C^1 coordinate chart α we have that $g_\alpha \notin C^{\max(g)}$, then $g_\phi \in C^{\max(g)-\varepsilon}$ near x_0 for every $\varepsilon > 0$. Therefore g has maximal regularity expressed in any p-harmonic coordinates.

Proof. We know by Proposition 24 that if for some coordinate chart α we have $g_\alpha \in C^t$ for some $t > 1$, then $g_\phi \in C_*^t$. This proves cases (1) and (2). To see (3), take a sequence of coordinate charts α_n and numbers $t_n > 1$ such that $g_{\alpha_n} \in C^{t_n}$, $t_n \to max(g)$ as $n \to \infty$, and $t_n \notin \mathbb{N}$. Then $g_\phi \in C^{t_n}$ for each n, so necessarily $g_\phi \in C^{max(g)-\varepsilon}$. This proves (3) and we are done. □

Note that in Proposition 24 above and the Corollary 13 below we can substitute p-harmonic coordinates by A-harmonic coordinates with $A(x, \xi)$ satisfaying the hipothesis of the Theorem 17. The proof is totally analogous, but we are only interested in p-harmonic coordinates here.

Now we see an interesting property of n-harmonic coordinates, related to conformal geometry. It says that whenever a metric is conformally flat, the coordinate chart that makes the metric a multiple of the identity has to be n-harmonic necessarily.

Proposition 25. Let (M, g) be a Riemannian manifold, g not necessarily regular. Suppose that in a C^1 coordinate chart $x = (x_1, \dots, x_n)$ the metric g has the expression $g_{jk} = c\delta_{jk}$ for some bounded function. Then the chart x is n-harmonic. This does not follow for any other value of p.

Proof. We work in the coordinate system x. We must see that $\delta(|dx^l|^{p-2}dx^l) = 0$ for any l if and only if $p = n$. We have, by (68) that

$$\begin{aligned}\delta(|dx^l|^{p-2}dx^l) &= -|g(x)|^{-\frac{1}{2}}\partial_j[|g(x)|^{\frac{1}{2}}g^{jk}(x)(g^{ab}(x)\delta_{al}\delta_{bl})^{\frac{p-2}{2}}\delta_{kl}] \\ &= -c^{-\frac{n}{2}}\partial_j[c^{\frac{n}{2}}g^{jl}(g^{ll})^{\frac{p-2}{2}}] = -c^{-\frac{n}{2}}\partial_l[c^{\frac{n-2}{2}-\frac{p-2}{2}}]\end{aligned}$$

And the last term is zero no matter who is c if and only if $p = n$. This proves the proposition. Note that, although the n-harmonic equation seems to require $g \in C^1$ to make sense, in this case it is trivially satisfied no matter how regular is the metric. Nevertheless, if one want to avoid the miracle, we always have a well defined weak formulation for the p-laplace equation which does not require regularity on the metric, given in (79). The computation above shows that $A(x, \nabla x^l) = e_l$, so $div(A(x, \nabla x^l) = 0$ in the weak sense.

Note that without C^r, $r > 1$, regularity of the metric we do not know whether a weak solution u of $div(A(x, \nabla u) = 0$ is regular or not. However, in this case x^l is obviously C^∞ in the x-coordinates (it is only C^1 expressed in other coordinates, but that is another history). □

Example 5. Consider the metrics in $\mathbb{R}^2$ given in standard coordinates of $\mathbb{R}^2$ by

$$\begin{aligned}g^1_{Id}(x, y) &= (1 + 3x|x|)^2 dx^2 + [1 + (x + |x|^3)^2]dy^2 \\ g^2_{Id}(x, y) &= (1 + 3x|x|)^2(dx^2 + dy^2)\end{aligned}$$

The function $x|x|$ is $C^{1,1}$, but it is not C^2 near the point $x=0$. So $f := (1+3x|x|)^2 = 1+6x|x|+9x^4$, as a sum of two smooth functions and a not C^2 function, cannot be C^2. We conclude that both g^1 and g^2 are $C^{1,1}$ and not C^2 in standard coordinates, and his first derivatives have a corner in the line $\{x=0\} \subset \mathbb{R}^2$. One might wonder is there is a C^1 diffeomorphism $\eta : B_\varepsilon \to V \subset \mathbb{R}^2$ near the point $x_0 := (0,0)$ such that the pull-back $g^j_\eta = (\eta^{-1})^* g^j$ is more regular than g^j_{Id}, $j=1,2$.

Consider the $C^{2,1}$ (and not C^3) diffeomorphism given by

$$\eta(x,y) = (x+|x|^3, y) := (t,y)$$

A trivial computation shows that $g^1_\eta(t,y) = dt^2 + (1+t^2)dy^2$, so g_1 is smooth expressed in the η coordinates. So there exists a coordinate chart η such that g^1_η is smooth. This implies that in any p-harmonic coordinates ϕ the pull-back g^1_ϕ is also smooth. However, an easy computation shows that the η coordinates are not p-harmonic for any value of $p>1$. Indeed, putting $x^1 := t$, x^1 is not p-harmonic, since, after a little computation in the (t,y) coordinates (recall that $(g^1_\eta)^{11} = 1$), we see that

$$\Gamma^1 = \frac{t}{1+t^2} \neq \frac{p-2}{2} g^{1i} \partial_i [log((g^1_\eta)^{11})] = 0 \quad \text{for any } p.$$

This shows that, though expressing the metric in p-harmonic coordinates gives maximal regularity, the converse is not true, i.e, a metric can be smooth in coordinates which are not p-harmonic for any p.

On the other hand, there is no C^1 diffeomorphism η such that g^2_η has more than C^2 regularity. Indeed, if there were such an η, then g^2_η would be of class at least $C^{2,\alpha}$ for some $0<\alpha<1$, and then we know by Proposition 24 that in any p-harmonic coordinate system ϕ the pullback g_ϕ would be $C^{2,\alpha}_* = C^{2,\alpha}$. But, since the metric g^2 is a multiple of the identity in standard coordinates, we know by Proposition 25 that the standard coordinates are 2-harmonic (i.e, harmonic). This implies that g^2_{Id} should be $C^{2,\alpha}$, which is not true.

The next proposition shows that n-harmonic coordinates are invariant under conformal change of the metric. This is also true for any n-harmonic function, but we are mainly interested in this specific case, for which the proof is easier.

Proposition 26. Let (M,g) be Riemannian manifold, let $r>1$, and let $x_0 \in M$. Let $\phi = (x^1,\dots,x^n) : U^{x_0} \to V \subset \mathbb{R}^n$ be a C^1 n-harmonic coordinate system near x_0, and suppose that the pull-back g_ϕ is C^1. Let c be a positive function defined near x_0 such that the pull-back c_ϕ is C^1, and define a conformal metric near x_0 by $\hat{g} := cg$.

Then, the coordinate system ϕ keeps being n-harmonic for the metric $\hat{g}$. Moreover, no other value of p has this property.

Proof. We shall work in the ϕ coordinates. First remind from Proposition 15 that a coordinate function x^k is p-harmonic if and only if

$$A(g) := g^{ij}\frac{1}{2}g^{lk}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) = \frac{1}{2}(p-2)g^{ki}\frac{\partial_i g^{kk}}{g^{kk}} := B(g)$$

therefore this is true for the metric g with $p=n$. We have to see that it is true also for the metric $\hat{g}$.

We compute

$$\begin{aligned} A(\hat g) &= \frac{1}{2c^2} g^{ij} g^{lk} (g_{jl}\partial_i c + c\partial_i g_{jl} + g_{il}\partial_j c + c\partial_j g_{il} - g_{ij}\partial_l c - c\partial_l g_{ij}) \\ &= \frac{1}{c} A(g) + \frac{1}{2c^2} g^{ij} g^{lk} (g_{jl}\partial_i c + g_{il}\partial_j c - g_{ij}\partial_l c) \\ B(\hat g) &= \frac{p-2}{2c} g^{ki} \frac{\partial_i [c^{-1} g^{kk}]}{c^{-1} g^{kk}} = \frac{n-2}{2} \frac{g^{ki}}{g^{kk}} (-g^{kk} \frac{1}{c^2} \partial_i c + \frac{1}{c} \partial_i g^{kk}) \\ &= \frac{1}{c} B(g) + \frac{1}{2c^2} (2-p) g^{ki} \partial_i c \end{aligned}$$

Therefore, $A(\hat g) = B(\hat g)$ if and only if $C = D$, being

$$\begin{aligned} C &:= g^{ij} g^{lk} (g_{jl}\partial_i c + g_{il}\partial_j c - g_{ij}\partial_l c) = g^{ik}\partial_i c + g^{jk}\partial_j c - n g^{lk}\partial_l c = (2-n) g^{ik}\partial_i c \\ D &:= (2-p) g^{ki}\partial_i c \end{aligned}$$

which is true if and only if $p = n$. □

§7. Gauge Conditions for Ellipticity

In a riemannian manifold (M, g), consider a tensor T which expression in coordinates depends on the metric g. To fix ideas, suppose T is a $(2, 0)$ tensor. Fix a coordinate system x, and consider the equation $T_{ab} = \hat T_{ab}(g)$ in the coordinate system x, where the left hand side is a fixed function (the components of T in the x coordinates), and the right hand side is a diffrential operator acting on the fixed metric g, which depends on how the components g_{ij} combine in the x coordinates to give the component T_{ab} of the tensor T.

This equation is an identity, i.e. valid for every g, if we do not imposse any condition on the coordinate system x. However, if we imposse, for example, that the system x is p-harmonic for the metric g, then the differential operator $\hat T(g)$ changes in the x coordinates due to possible cancelations, and the equation $T_{ab} = \hat T_{ab}(g)$ in the x coordinates is true for the metric g, but is not true for a metric $\hat g$ in which the x coordinates are not p-harmonic.

Imagine that, choosing an appropiate system of coordinates x, we can achieve that the operator $\hat T_{ab}(g)$ becomes an elliptic operator. Then, by elliptic regularity results, we would be able to deduce regularity for g in terms of regularity of the tensor T.

Note that we are not saying that the tensor T regarded as an operator acting on metrics g is elliptic, because in that case the identity $\hat T_{ab}(g) = T_{ab}$ would be elliptic in any coordinates x. We are saying that if we fix a metric g we can choose some appropiate coordinates x for that metric g in order to make the equation $\hat T_{ab}(g) = T_{ab}$ elliptic in that coordinates.

An specific choice of coordinates in the equation $T_{ab} = \hat T_{ab}(g)$ is called a local Gauge Condition for that equation. In this section we shall see how, for certain tensors T, p-harmonic coordinates can be used as local Gauge conditions that results in an elliptic equation.

7.1. Ricci Tensor and Harmonic Coordinates

In this subsection we see a simple example of how the techniques mentioned above work for the Ricci tensor.

Lemma 32. Let (M, g) be a riemannian manifold such that g is C^2 expressed in some coordinates. Denote $\Gamma^l := \Gamma^l_{ab} g^{ab}$. Then

$$Ric_{ij} = -\frac{1}{2} g^{kl}\partial_{kl} g_{ij} + \frac{1}{2}[g_{li}\partial_j \Gamma^l + g_{lj}\partial_i \Gamma^l] + T_1(g) \tag{152}$$

where for $k \in \mathbb{N}$ we define $T_k(g)$ for expressions that depend smoothly only on g_{ab} and derivatives up to order k of g_{ab}.

Proof. First note that by formula 8, we have that

$$\begin{aligned} R_{iklj} &= R_{ikl}{}^a g_{ja} = \{\partial_i \Gamma^a_{kl} - \partial_k \Gamma^a_{il}\} g_{ja} + T_1(g) \\ &= \{\partial_i [\frac{1}{2} g^{ab}(\partial_k g_{lb} + \partial_l g_{kb} - \partial_b g_{kl})] - \partial_k [\frac{1}{2} g^{ab}(\partial_i g_{lb} + \partial_l g_{ib} - \partial_b g_{il})]\} g_{ja} + T_1(g) \\ &= \frac{1}{2}[\underline{\partial_{ik} g_{lj}} + \partial_{il} g_{kj} - \partial_{ij} g_{kl} - \underline{\partial_{ik} g_{lj}} - \partial_{kl} g_{ij} + \partial_{kj} g_{il}] + T_1(g) \end{aligned}$$

Therefore it follows that

$$Ric_{ij} = -\frac{1}{2} g^{kl} \partial_{kl} g_{ij} + \frac{1}{2} g^{kl}[\partial_{kj} g_{il} + \partial_{il} g_{kj} - \partial_{ij} g_{kl}] + T_1(g) \tag{153}$$

On the other hand

$$\partial_j \Gamma^l = \partial_j [\frac{1}{2} g^{ab} g^{lc} [\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab}] = \frac{1}{2} g^{ab} g^{lc} [\partial_{aj} g_{bc} + \partial_{bj} g_{ac} - \partial_{cj} g_{ab}] + T_1(g)$$

This implies that

$$\begin{aligned} & g_{li} \partial_j \Gamma^l + \tfrac{1}{2} g^{lk} \partial_{ij} g_{lk} = \tfrac{1}{2} g_{li} g^{ab} g^{lc} [\partial_{aj} g_{bc} + \partial_{bj} g_{ac} - \partial_{cj} g_{ab}] + \tfrac{1}{2} g^{lk} \partial_{ij} g_{lk} + T_1(g) \\ & = \tfrac{1}{2} g^{ab} [\partial_{aj} g_{bi} + \partial_{bj} g_{ai} - \partial_{ij} g_{ab}] + \tfrac{1}{2} g^{lk} \partial_{ij} g_{lk} + T_1(g) = g^{ab} \partial_{aj} g_{bi} + T_1(g) \end{aligned}$$

The last equality because the two first sums are the same by simmetry, and the last two sums cancel out. We conclude that

$$g_{li} \partial_j \Gamma^l + g_{lj} \partial_i \Gamma^l = g^{ab} \partial_{aj} g_{bi} + g^{ab} \partial_{ai} g_{bj} - g^{lk} \partial_{ij} g_{lk} + T_1(g)$$

Inserting the last equation in 153 we obtain (152) and the lemma is proved. □

Taking into account equation (70) we have the following.

Corollary 14. In harmonic coordinates the expression for the Ricci tensor becomes

$$Ric_{ij} = -\frac{1}{2} g^{kl} \partial_{kl} g_{ij} + T_1(g) \tag{154}$$

From this and the classic elliptic regularity results it follows readily the following Theorem. We require the metric to be C^2 in order to define the Ricci tensor in a classical way, though this hipothesis can be weakened as we shall see.

Theorem 19. Let (M, g) a Riemannian manifold, and let $r > 1$. Suppose that the metric g is C^r expressed in some system of coordinates φ defined near x_0. Let also $k \in \mathbb{N}$ and $0 < \alpha \leq 1$. We have

(1) If in any system of harmonic coordinates ϕ near x_0 the expression of the metric g_ϕ is C^2 and the expression of the Ricci tensor Ric_ϕ is $C^{k,\alpha}$, then in fact g_ϕ is $C^{k+2,\alpha}$.

(2) If in some system of coordinates φ' the metric $g_{\varphi'}$ is $C^{k,\alpha}$ for some $k \geq 2$ and the Ricci tensor $Ric_{\varphi'}$ is $C^{l,\alpha}$ for $l \geq k$ in these coordinates, then g is $C^{k+2,\alpha}$ in harmonic coordinates.

Proof. First, as g_φ is C^r with $r > 1$ we know that harmonic coordinates exists, and the metric has maximal regularity in such coordinates.

The claim (1) follows by Corollary 14. Indeed, in these coordinates we have

$$Ric_{ij} - f = a^{lk} \partial_{lk} g_{ij}$$

with $a^{lk} := g^{lk} \in C^2$, and $f = T_1(g)$. Since $f = T_1(g)$ depends smoothly on the metric and its first derivatives, $f = T_1(g) \in C^1 \subset C^{1-\varepsilon}$, and $Ric_{ij} \in C^{k,\alpha}$ by hipothesis. This is a linear elliptic operator acting on g_{ij}. If $k+\alpha \leq 1-\varepsilon$, by elliptic regularity $g \in C^{k+2,\alpha}$ according to Theorem 7.

If $k+\alpha > 1-\varepsilon$ we see that $g_{ij} \in C^{3-\varepsilon}$ for all $\varepsilon > 0$. Repeating the argument, now $f = T_1(g) \in C^{2-\varepsilon}$ and $a^{lk} \in C^{3-\varepsilon}$. If $k+\alpha \leq 2-\varepsilon$, we are done.

If $k+\alpha > 2-\varepsilon$, we see $g_{ij} \in C^{4-\varepsilon}$, so $f = T_1(g) \in C^{3-\varepsilon}$. If $k+\alpha \leq 3-\varepsilon$ we are done, and otherwise we keep going.

Repeating the argument $k+1$ times we obtain that both $a^{kl}, T_1(g) \in C^{k+1-\varepsilon} \subset C^{k+\alpha}$ in ε is small enough, so $g_{ij} \in C^{k+2,\alpha}$ as desired.

Let us see (2). As both the g and Ric are $C^{k,\alpha}$ for $k \geq 2$ and $\alpha > 0$ in some coordinates, we know by Proposition 24 that both of them are $C^{k,\alpha}$ in harmonic coordinates. Now we use (1) to see that $g \in C^{k+2,\alpha}$. This yields (2) and proves the Theorem. □

From this Theorem we see that if in harmonic coordinates ϕ the Ricci tensor is smooth and the metric is $C^{2,\alpha}$, it follows that g is also smooth in the ϕ coordinates. In arbitrary coordinates the Theorem only says that we gain two derivatives on the metric when we express it in harmonic coordinates, but with respect to the regularity of the metric in the original coordinates, not to the regularity of the Ricci tensor. The following example ilustrates this.

Example 6.

(a) Consider $(\mathbb{R}^n, g_0)$ being $g_0 = Id$ the flat usual metric. Let φ be a diffeomorphism of class C^3, so that the pull-back metric $g_1 := (\varphi^{-1}) * g_0$ is C^2. As the Ricci tensor is invariant under isometries, $Ric(g_1) = 0$, but the metric g_1 is not smooth, not even C^3. This happens because the coordinates given by φ are not harmonic.

In particular, if $n = 2$ we see that $g_1(x,y) = D(\varphi^{-1})^t(x,y)D(\varphi^{-1})(x,y)$ can not be the identity matrix multiplied by a function unless $\varphi \in C^\infty$, by Proposition 25. This is a curious conclusion.

(b) Suppose that in harmonic coordinates the metric g is C^2, and the curvature tensor r is zero. Then in these coordinates $Ric = 0$, so g is smooth.

7.2. Definition of Tensors with Low Regular Metrics.

Now we adress the question of giving a similar result that Theorem 19 but for C^r metrics with $r > 1$. The strategy will be to define the Ricci tensor in the weak sense, and deduce elliptic regularity results as before. To give a characterization of conformal flatness for C^r metrics we will do more or less the same thing with the Weyl and Cotton tensors. We will define the Weyl and Cotton tensors in the weak sense, and then we will use elliptic regularity to prove that if such weak tensors vanish then the metric is regular. Finally, when we know that the metric is regular, conformal flatness will follow from the classic Weyl-Schouten Theorem.

From now on we will call $E := W^{1,2} \cap L^\infty$. We will see that the condition $g \in E$ in some coordinate patch is enough to define the Weyl, Riemann curvature and Ricci tensors in coordinates. Call also $E' := W^{2,2} \cap W^{1,\infty}$. We will see that $g \in E'$ in some coordinate patch is enough to define the Cotton tensor in coordinates. First we see that E is an algebra under pointwise multiplication.

Lemma 33. Let $\Omega \subset \mathbb{R}^n$. Let $u, v \in E_{loc}(\Omega)$. Then $uv \in E_{loc}(\Omega)$ and $\nabla(uv) = v\nabla u + u\nabla v$.

If we only suppose $u, v \in W^{1,2}_{loc}(\Omega)$ then we have $uv \in W^{1,1}_{loc}(\Omega)$ and $\nabla(uv) = v\nabla u + u\nabla v$.

If we additionally suppose that for some $c > 0$ we have $|v(x)| \geq c > 0$ a.e. $x \in \Omega$, then $v^{-1} \in E_{loc}(\Omega)$ and $\nabla(v^{-1}) = v^{-2}\nabla v$.

Remark 41. If we suppose that $u, v \in E$ then from this it follows that $uv \in E$, since the distributional derivative keeps being the same. The same applies for v^{-1} under the additional assumption of non-vanishing.

Proof. First suppose that $v \in C^1(\Omega)$. Let $\varphi \in C_0^1(\Omega)$ be a text function, and $K := Supp(\varphi)$. Note that $\varphi_{x_i} v \in C_0^1(\Omega)$ is also a test function, for which the usual product rule applies. Therefore

$$\int_\Omega uv\varphi_{x_i}dx = \int_\Omega u[(v\varphi)_{x_i} - v_{x_i}\varphi]dx = -\int_\Omega (u_{x_i}v + uv_{x_i})\varphi dx.$$

For general $v \in E(\Omega)$, take $v^n \in C^1(\Omega)$ such that $v^n \to v$ in $W^{1,2}(K)$. By the previous case and Holder's inequality we have

$$\begin{aligned}
&|\int_\Omega uv\varphi_{x_i}dx + \int_\Omega (u_{x_i}v + uv_{x_i})\varphi dx| \\
&= |\int_\Omega (v - v^n)u\varphi_{x_i}dx + \int_\Omega uv^n\varphi_{x_i}dx + \int_\Omega (u_{x_i}v + uv_{x_i})\varphi dx| \\
&= |\int_\Omega (v - v^n)u\varphi_{x_i}dx + \int_\Omega \{u_{x_i}(v - v^n) + u(v_{x_i} - v^n_{x_i})\}\varphi dx| \\
&\le \|v - v^n\|_{L^2(K)}\left[\|u\varphi_{x_i}\|_{L^2(\Omega)} + \|u_{x_i}\varphi\|_{L^2(\Omega)}\right] + \|v_{x_i} - v^n_{x_i}\|_{L^2(K)}\|\|u\varphi\|_{L^2(\Omega)} \to 0
\end{aligned}$$

as $n \to \infty$. This proves that the distributional derivative of uv is $\nabla(uv) := v\nabla u + u\nabla v$ (note that this is true only supposing that $u, v \in W^{1,2}_{loc}(\Omega)$). Now, as $u, v \in L^\infty_{loc}(\Omega)$, then $\nabla(uv) \in L^2_{loc}(\Omega)$, so $uv \in E(\Omega)$.

Now suppose that $v(x) \ge c > 0$ a.e. $x \in \Omega$, and let $v^n \in C^1(\Omega)$ such that $v^n \to v$ in $W^{1,2}(K)$ and $v^n(x) \to v(x)$ a.e. $x \in K$. Since almost everywhere subsets are dense, by continuity $|v^n(x)| \ge c$ and $|v^n(x)| \le \|v\|_{L^\infty(K)}$ for all $x \in K$, so the usual derivative of $(v^n)^{-1}$ is $-(v^n)^{-2}\nabla v^n$. Now we compute

$$\begin{aligned}
&\left|\int_\Omega [v^{-1}\varphi_{x_i} + v^{-2}v_{x_i}\varphi]dx\right| \\
&= \left|\int_\Omega [v^{-1} - (v^n)^{-1}]\varphi_{x_i}dx + \int_\Omega (v^n)^{-1}\varphi_{x_i}dx + \int_\Omega v^{-2}v_{x_i}\varphi dx\right| \\
&= \left|\int_\Omega [v^{-1} - (v^n)^{-1}]\varphi_{x_i}dx + \int_\Omega [v^{-2}v_{x_i} - (v^n)^{-2}v^n_{x_i}]\varphi dx\right| \\
&\le \left|\int_\Omega \frac{v^n - v}{vv^n}\varphi_{x_i}dx\right| + \left|\int_\Omega \frac{(v^n)^2 v_{x_i} - v^2 v^n_{x_i}}{v^2(v^n)^2}\varphi dx\right| \\
&\le \left\|\frac{v^n - v}{vv^n}\right\|_{L^2(K)}\|\varphi_{x_i}\|_{L^2(\Omega)} + \left\|\frac{(v^n)^2 v_{x_i} - v^2 v^n_{x_i}}{v^2(v^n)^2}\right\|_{L^2(K)}\|\varphi\|_{L^2(\Omega)} := A_n + B_n
\end{aligned}$$

Now note that both sequences $f_n := \|[v^n - v](vv^n)^{-1}\|$ and $g_n := \left\|[(v^n)^2 v_{x_i} - v^2 v^n_{x_i}](vv^n)^{-2}\right\|$ converge almost everywhere to 0 in K, and moreover f_n and g_n are uniformly bounded functions in K, so they are dominated by an $L^2(K)$ function. By the Dominated Convergence Theorem we see that both A_n and B_n tend to 0, as desired. This proves the Lemma. □

Definition 44. Given $\Omega \subset \mathbb{R}^n$ an open let and $1 \le p \le \infty$ and p' its conjugate. We denote $W^{-1,p'}(\Omega) := W^{1,p}(\Omega)'$ for the dual space of $W_0^{1,p}(\Omega)$.

Using the isometric embbeding $\iota : W_0^{1,p}(\Omega) \to F := L^p(\Omega)^{n+1} : u \to (u, \nabla u)$ we can have an idea of how acts an element T of $W^{-1,p'}(\Omega)$. Indeed, we see that $T : W_0^{1,p}(\Omega) \to \mathbb{R}$ induces a linear continuous map $\hat{T} := T \circ \iota^{-1}$ on its image $\iota(W_0^{1,p}(\Omega)) \subset F$. By the Hanh-Banach Theorem we can extend $\hat{T}$ to a continuous functional T^* on F, i.e, $T^* \in F' = L^{p'}(\Omega)^{n+1}$. Therefore, when we restrict T^* to $\iota(W_0^{1,p}(\Omega))$, we see that there exists functions $v_0, \ldots, v_n \in L^{p'}(\Omega)$ such that

$$T(u) = \int_\Omega u v_0 dx + \sum_{i=1}^n \int_\Omega u_{x_i} v_i dx$$

Note that the functions v_i are not unique, as neither is the extension T^* of $\hat{T}$.

Remark 42. Given $f \in L^{p'}(\Omega) = W^{0,p'}(\Omega)$, we can consider f as a distribution, i.e, $f \in (L^p(\Omega))' \subset \mathcal{D}'(\Omega)$. Then, as $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ it is obvious that its distributional derivatives $f_{x_i} \in D'(\Omega)$ can be uniquely extended to an element to $W^{-1,p'}$ and this element is of course the one that acts by $(f_{x_i}, u) := -(f, u_{x_i})$ fro any $u \in W_0^{1,p}(\Omega)$. So we see that derivatives of $L^{p'}$ functions fall in $W^{-1,p'}$. This should explain a little bit the notation.

We prove now a couple of technical propositions that will be useful later.

Proposition 27. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let $E = W^{1,2} \cap L^\infty(\Omega)$ as before. Given $f, g \in E$, we have

$$\begin{aligned} &\partial_l[fg] = f\partial_l g + g\partial_l f \in L^1 \\ &\partial_k[f\partial_l g] = \partial_k f \partial_l g + f\partial_{kl} g \qquad \text{as elements of } W^{-1,q'} \text{ for any } q > n. \end{aligned} \tag{155}$$

Proof. The first equality follows from Proposition 33. Let us see the second. First note that, as commented above, $\partial_k f \partial_l g + f \partial_{kl} g \in L^1(\Omega) + W^{-1,q'}(\Omega) \subset W^{-1,q'}(\Omega)$ for any $q > n$, , being q' the Holder conjugate of q. On the other hand, $f\partial_l g \in L^2$, so $\partial_k[f\partial_l g] \in W^{-1,2}(\Omega) \subset W^{-1,q'}(\Omega)$. We shall see that both are equal in $W^{-1,q'}(\Omega)$. Indeed, given $\varphi \in W_0^{1,q}(\Omega)$, on one hand we have

$$(\partial_k[f\partial_l g], \varphi) = -\int_\Omega f\partial_l g \partial_k \varphi dx$$

On the other hand, recall that as $q > n \geq 2$ we have $W_0^{1,q}(\Omega) \subset W_0^{1,2}(\Omega)$ so by Lemma 33 we see that the pointwise product $f\varphi \in E$ and $\nabla(f\varphi) = \varphi\nabla f + f\nabla\varphi \in L^2$, so we have

$$\begin{aligned} (\partial_k f \partial_l g + f\partial_{kl} g, \varphi) &= \int_\Omega \varphi \partial_k f \partial_l g dx - \int_\Omega \partial_l g \partial_k [f\varphi] dx \\ = \int_\Omega \varphi \partial_k f \partial_l g dx - \int_\Omega \varphi \partial_l g \partial_k f dx &- \int_\Omega f \partial_l g \partial_k \varphi dx = -\int_\Omega f\partial_l g \partial_k \varphi dx \end{aligned}$$

and this proves the Proposition. □

Proposition 28. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be open and bounded. Given $f \in E(\Omega)$ and $g = g_1 + g_2 \in W^{-1,2}(\Omega) + L^1(\Omega)$ we can define the product fg in such a way that $fg \in W^{-1,q'}(\Omega) + L^1(\Omega)$ and besides fg is the pointwise product of functions when g is a $L^p(\Omega)$ function for some $p \geq 2$.

Remark 43. If $h \in W^{-1,q'}(\Omega)$ can be expressed as $h = eh'$ for some $e \in E(\Omega)$ and $h' \in W^{-1,2}(\Omega) + L^1(\Omega)$, then for $f \in E$ we can still define $fh := (fe)h'$.

Proof. Let $f \in E(\Omega), g = g_1 + g_2 \in W^{-1,2}(\Omega) + L^1(\Omega)$. It is clear that we can define the pointwise product $fg_2 \in L^1(\Omega)$. It remains to see how we define the product fg_1. This will be similar to what it is done for distributions. Let $q > n \geq 2$. Note that, by the Sobolev Embeding Theorem and the definition of norms, we know that $W_0^{1,q} \subset C^{0,\delta} \subset L^\infty$ for some $0 < \delta < 1$, with continuous inclusions. Taking duals we have $L^p(\Omega) \subset L^1(\Omega) \subset W^{-1,q'}(\Omega)$ for $p \geq 1$. Also note that $W^{1,q}(\Omega) \subset E(\Omega)$.

That said, we claim now that we have a well defined map

$$\cdot f : W_0^{1,q}(\Omega) \to W_0^{1,2}(\Omega) : g \mapsto fg.$$

First note that, as g is continuous and has zero trace, then g is zero restricted to the boundary of the coordinate patch Ω.

Let us see that the map above is well defined. As $g \in W_0^{1,q}(\Omega) \subset E(\Omega)$, by Lemma 33 we see that $\nabla(fg) = f\nabla g + g\nabla f \in L^2(\Omega)$, since $f, g \in L^\infty(\Omega)$. From this we see that $fg \in W^{1,2}(\Omega)$. Note that this fails for $q < n$ as we are not sure that $g\nabla f \in L^2(\Omega)$.

Finally let us see that $Tr(fg) = 0$. Let $\phi_n \to f$ in $W^{1,2}(\Omega)$, $\phi_n \in C(\Omega)$. We can suppose $\phi_n \in L^\infty(\Omega)$ since we can change ϕ_n by

$$\phi'_n = \phi_n \chi_{\{|\phi_n| \le \|f\|_{L^\infty(\Omega)}\}} + sign(\phi_n)\|f\|_{L^\infty(\Omega)}\chi_{\{|\phi_n| > \|f\|_{L^\infty(\Omega)}\}}$$

which satisfies $\phi'_n \in C(\Omega)$, $\|\phi'_n\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)}$, and $|\phi'_n(x) - f(x)| \le |\phi_n(x) - f(x)|$ so ϕ'_n are a better approximation of f. So we directly suppose $\phi_n \in E(\Omega)$. Recall that this implies $\phi_n g \in W^{1,2}(\Omega)$.

We claim in addition that $\phi_n g \to fg$ in $W^{1,2}(\Omega)$. To see this first note that $\|\phi_n g - fg\|_{L^2(\Omega)} \le \|\phi_n - f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. Also we know that the product rule applies so

$$\|\nabla(\phi_n g) - \nabla(fg)\|_{L^2(\Omega)} \le \|g\|_{L^2(\Omega)}\|\nabla\phi_n - \nabla f\|_{L^2(\Omega)} + \|\nabla g\|_{L^2(\Omega)}\|\phi_n - f\|_{L^2(\Omega)}.$$

So, as $Tr : W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is continuous and $Tr(\phi_n g) = (\phi_n g)|_{\partial\Omega} = 0$, we see that $Tr(fg) = 0$, so $fg \in W_0^{1,2}(\Omega)$ as we wanted. This proves that $\cdot f$ is well defined as claimed. Therefore we have the transpose map $(\cdot f)' : W^{-1,2}(\Omega) \to W^{-1,q'}(\Omega)$.

Suppose now that for some function g we have that $I_g \in W^{-1,2}(\Omega)$, (where I_g acts on $W_0^{1,2}(\Omega)$ by integrating against g), and suppose we have also that $I_{fg} \in W^{-1,q'}(\Omega)$. For example this is true if $g \in L^p(\Omega)$ for some $p \ge 2$. We might wonder whether $(\cdot f)' I_g = I_{fg}$, and we claim this is true. Indeed, given $h \in W_0^{1,q}(\Omega)$ we have

$$((\cdot f)' I_g, h) = (I_g, fh) = \int_\Omega gfh dx = (I_{fg}, h)$$

So, as $(\cdot f)'$ coincides with $\cdot f$ when that makes sense, we denote simply $(\cdot f)' = \cdot f$, and the proposition is proved. $\square$

Now we shall see how to define tensors that involve two derivatives of the metric if we only suppose $g \in E$, and tensors that involve three derivatives supposing $g \in E'$.

First we write explicitly the expressions of the $(3,1)$ curvature tensor R, the Ricci tensor Ric, the Schouten tensor s, the Weyl tensor w and the $(3,1)$-Weyl tensor W. In Einstein notation the tensors obtained by lowering and raising indexes are denoted with the same letter, so now we denote all tensors with capital letters. Moreover the tensors obtained taking traces are denoted also with the same letter. In particular note that the Ricci tensor is denoted R_{ij} since it is obtained by taking the trace from the curvature, $R_{ij} = R_{aij}{}^a = R_{aijl}g^{al}$. In analogy, the scalar curvature is obtained as the trace of the Ricci tensor, so it is denoted by $R = R^a_a = R_{ab}g^{ab}$. That said, according to Definition 10, we have

$$\begin{aligned} R_{abc}{}^d &= \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^m_{bc}\Gamma^d_{am} - \Gamma^m_{ac}\Gamma^d_{bm} \\ S_{ab} &= \frac{1}{2-n}\left[R_{ab} + \frac{1}{2(1-n)} g^{rs} R_{rs} g_{ab}\right] \\ W_{abcd} &= R_{abcd} - g_{ac}S_{bd} + g_{ad}S_{bc} - S_{ac}g_{bd} + S_{ad}g_{bc} \end{aligned} \tag{156}$$

$$
\begin{aligned}
W_{abc}{}^{l} &= W_{abcd}g^{ld} = R_{abc}{}^{l} - g_{ac}S_{bd}g^{ld} + \delta^{l}_{a}S_{bc} - S_{ac}\delta^{l}_{b} + S_{ad}g^{ld}g_{bc} \\
&= R_{abc}{}^{l} + \frac{1}{2-n}\left[-g_{ac}R_{bd}g^{ld} + \delta^{l}_{a}R_{bc} - \delta^{l}_{b}R_{ac} + R_{ad}g^{ld}g_{bc}\right] \\
&+\frac{1}{2(1-n)(2-n)}g^{rs}R_{rs}\left[-g_{ac}g_{bd}g^{ld} + \delta^{l}_{a}g_{bc} - \delta^{l}_{b}g_{ac} + g_{ad}g^{ld}g_{bc}\right] \\
&= R_{abc}{}^{l} + \frac{1}{2-n}\left[-g_{ac}R_{bd}g^{ld} + \delta^{l}_{a}R_{bc} - \delta^{l}_{b}R_{ac} + R_{ad}g^{ld}g_{bc}\right] \\
&+\frac{1}{(1-n)(2-n)}g^{rs}R_{rs}\left[\delta^{l}_{a}g_{bc} - \delta^{l}_{b}g_{ac}\right]
\end{aligned}
\tag{157}
$$

The next proposition shows how to define the Weyl tensor if the metric has just one derivative.

Proposition 29. Given a Riemmannian manifold (M, g), and a system of coordinates (φ, Ω), if the metric $g_\varphi \in E(\Omega) = W^{1,2}(\Omega) \cap L^\infty(\Omega)$, then we can make sense of the components of the Riemann curvature, Weyl, and Ricci tensors in these coordinates (as distributions), and moreover if $\lambda \in E(\Omega)$ is a positive function, and we define $g' = \lambda g \in E(\Omega)$, then the Weyl tensors w and w' of g and g' satisfy the same relation than in the case where g and g' are C^2, i.e, $w' = \lambda w$.

Proof. With the formulae above in mind, fix a coordinate patch $\Omega \subset \mathbb{R}^n$. In the following discussion all the spaces are considered in Ω unless explicit mention. Suppose that $g_{ij} \in E$. Then $|g|$ and $|g|^{-1} \in E$ by Lemma 33, so its inverse matrix $g^{ij} \in E$. Therefore we have

$$\Gamma^c_{ab} = \frac{1}{2}g^{cl}[\partial_a g_{bl} + \partial_b g_{al} - \partial_l g_{ab}] \in L^2$$

since $L^\infty L^p \subset L^p$ for all $p \geq 1$. Also, if we consider $\Gamma^c_{ab} \in \mathcal{D}'$, then looking at (156) we see that both $R_{abc}{}^{d}$ and $R_{bc} \in W^{-1,2} + L^1$.

In order to make sense of S_{ab} for $n \geq 3$, we would want to make sense of $g^{rs}R_{rs}$. Looking at (156), we have $g^{rs}g_{ab} \in E$ and $R_{rs} \in W^{-1,2} + L^1$ so by Proposition 28 we have $g^{rs}g_{ab}R_{rs} \in W^{-1,q'} + L^1$ and $R_{abcd} = R_{abc}{}^{l}g_{ld} \in W^{-1,q'} + L^1$. As $W^{-1,2} \subset W^{-1,q'}$, and we see that both $S_{ab}, W_{abc}{}^{d}$ can be regarded in $W^{-1,q'} + L^1$.

Also, we see that $W_{abcd} = W_{abc}{}^{l}g_{ld} \in W^{-1,q'} + L^1$, because $W_{abc}{}^{d} \in E(W^{-1,2} + L^1)$ so we can still multiply W^d_{abc} by functions on E (see Remark 43). It is important to note that with this definition the curvature R_{abcd} has all the symmetries of the Riemann curvature tensor. To see this, look at the expression of $R_{abc}{}^{d}$ in (156), and note that we can expand, multiply and differentiate the expression of the Christoffel symbols in terms of the metric as in the regular case, i.e,

$$
\begin{aligned}
&\Gamma^c_{ab}\Gamma^f_{de} = \tfrac{1}{4}g^{cl}g^{fl}\partial_a g_{bl}\partial_d g_{el} + \tfrac{1}{4}g^{cl}g^{fl}\partial_a g_{bl}\partial_e g_{dl} + \cdots \in L^1 \\
&\partial_k\Gamma^c_{ab} = \partial_k[\tfrac{1}{2}g^{cl}][\partial_a g_{bl} + \partial_b g_{al} - \partial_l g_{ab}] + \tfrac{1}{2}g^{cl}[\partial_{ka} g_{bl} + \partial_{kb} g_{al} - \partial_{kl} g_{ab}] \in L^1 + W^{-1,q'}
\end{aligned}
$$

Note that after expanding these expressions and inserting them in the formula for $R_{abc}{}^{d}$, we have lost a bit a regularity, since now $R_{abc}{}^{d} \in W^{-1,q'} + L^1$, but this is irrelevant. The simmetries of $R_{abc}{}^{d}$ derive from its formal expression in terms of the metric, so they are the same as in the smooth case. Note also that, regarding $R_{abc}{}^{d} \in W^{-1,q} + L^1$, we can still make sense of $R_{abcd} = g_{ld}R_{abc}{}^{l}$ because $R_{abc}{}^{l}$ is a sum of things that lie in $E(W^{-1,2} + L^1)$, see Remark 43.

Therefore R_{abcd} has the same expression in terms of g_{ij} that in the smooth case, so necessarily it has the same simmetries. Exactly the same argument applies to see that W_{abcd} has the same simmetries as in the smooth case, so in particular its Ricci contraction is zero, i.e, $Ric(W)_{bc} = W_{abc}{}^{a} = 0$.

Finally note that $L^1 \subset (L^\infty)' \subset W^{-1,q'}$, since $W^{1,q} \subset L^\infty$ with continuous inclusion by the Sobolev Embeding. We see that all the tensors in (156) and (157) can be regarded as elements of $W^{-1,q'} + L^1 \subset W^{-1,q'}$.

Not let $\lambda \in E$, and define the conformal metric $g' := \lambda g \in E$ being $\lambda \geq k > 0$ for some constant k. We claim that the conformal behaviour of the weekly defined Weyl tensor remains the same, i.e, $W'_{abcd} = \lambda W_{abcd}$. The proof of this boils down to check that all the computations we did in 3 and 1 to compare W'_{abcd} and W_{abcd} in the regular case (i.e, $\lambda, g \in C^2$) are valid also in this less regular setting.

Note that in the regular case the equality $W'_{abcd} = \lambda W_{abcd}$ holds, and it is derived from the fact that, as $g'_{ij} = \lambda g_{ij}$, we can apply Leibnitz's rule when taking derivatives up to order 2, and this gives us a relation between both metrics. We will use Proposition 27 to show that we can do the same thing for metrics in E.

In first place note that if $g \in E$, it is still true that the Riemann curvature tensor r admits a unique descomposition $r = w + s \owedge g$ being w a curvature tensor with Ricci contraction 0. This descomposition was proved in 1 using the simmetries of r and taking traces twice. We have seen that these simmetries are the same here and we can take traces, so nothing changes.

Given this, it suffices to see that ${R_{abc}}^d$ and ${R'_{abc}}^d$ are related in the same way than they are in the regular case, because in that case we are done by Proposition 3 and the Corollary 1 below, since the same arguments apply. To see that ${R_{abc}}^d$ and ${R'_{abc}}^d$ are related as if they were regular, it is enough to see that we can apply the Leibnitz rule when computing

$$ {R'_{abc}}^d = \partial_a \Gamma'^d_{bc} - \partial_b \Gamma'^d_{ac} + \Gamma'^m_{bc}\Gamma'^d_{am} - \Gamma'^m_{ac}\Gamma'^d_{bm}. $$

To see this note that

$$ \Gamma'^m_{bc} = \frac{1}{2}(\lambda)^{-1} g^{ml}\{\partial_b[\lambda g_{cl}] + \partial_c[\lambda g_{bl}] - \partial_l[\lambda g_{bc}]\} = \frac{1}{2}(\lambda)^{-1} g^{ml} g_{cl}\partial_b \lambda + \frac{1}{2}(\lambda)^{-1} g^{ml}\lambda \partial_b g_{cl} + \dots \tag{158} $$

since by Proposition 27 we can apply Leibnitz's rule to each of the sumands and also we can apply the distributive law. Note that each of the summands is in L^2. So the products $\Gamma'^m_{bc}\Gamma'^d_{am}$ can be manipulated as if the metric were smooth.

Also, by the algebra properties of E in Lemma 33 and Proposition 27, we can apply Leibnitz's rule to each of the summands of the expanded expression of Γ'^m_{bc} in (158) as if these summands were smooth so it follows that

$$ \partial_a \Gamma'^m_{bc} = \frac{1}{2}\partial_a[(\lambda)^{-1} g^{ml} g_{cl}\partial_b\lambda] + \dots = (\lambda)^{-1} g^{ml} g_{cl}\partial_{ab}\lambda - \lambda^{-2} g^{ml} g_{cl}\partial_a\lambda\partial_b\lambda + \dots \in W^{-1,q'} $$

Note that after expanding the expression of Γ'^m_{bc} we lose a bit of information because we know (by another argument) that $\partial_a \Gamma'^d_{bc}$ actually lies in $W^{-1,2}$.

We conclude from this that ${R_{abc}}^d$ and ${R'_{abc}}^d$ are related in the same way as in the smooth case, this time laying both in $W^{-1,q'}$ (not in $W^{-1,2} + L^1$ as before). From this we contract with the metric and we see that also $R_{abcd} = g_{ld}{R_{abc}}^l$ and $R'_{abcd} = \lambda g_{ld}{R'_{abc}}^l$ are related as if they were smooth. We must see that this contraction with the metric is valid. Although we a priori cannot make sense of $g_{ld}h$ for general $h \in W^{-1,q'}$, if h has the special form $h = eh'$ with $e \in E$ and $h' \in W^{-1,2}$, then we define $g_{ld}h := (g_{ld}e)h'$ and everything works fine, by Remark 43. Note that by the computations above, both ${R_{abc}}^d$ and ${R'_{abc}}^d$ are sums of functions $h_i = e_i h'_i$ with this special form, so we can contract with the metric without any problem.

Thus, as in the regular setting, $R'_{abcd} = \lambda[R_{abcd} + (b_u \owedge g)_{abcd}]$, being $u := \frac{1}{2}log(\lambda)$, and $b_u = Hess(u) - du \otimes du + \frac{1}{2}|grad(u)|^2_g g$. Note that $u \in E$ and $\partial_a u = \frac{1}{2}\lambda^{-1}\partial_a[\lambda]$ by the Chain rule for weak derivatives. Now we apply the argument given in Corollary 1 and we see that $W'_{abcd} = \lambda W_{abcd}$. This completes the proof of the proposition. □

Remark 44. One might wonder if it is really necessary to ask $g \in E$ to define the Weyl tensor in the weak sense. Of course the problem is not on defining the derivatives of g, since we can differentiate g as many times as we want regarding g as a distribution. The problem lies on defining the products

$\Gamma^c_{ab}\Gamma^k_{ij}$ which appear in $R_{abc}{}^d$, since we cannot multiply distributions in a satisfactory way (we do multiply distributions with functions, but not with proper distributions).

Note for example that the tensorial product of distributions $u_1, u_2 \in \mathcal{D}'(\Omega)$ given by $(u_1 \otimes u_2, \varphi) := (u_1, \varphi)(u_2, \varphi)$ for $\varphi \in \mathcal{D}(\Omega)$ does not work for two reasons. First it does not extend the product on functions, i.e, $(I_f \otimes I_g, \varphi) = \int f\varphi dx \int g\varphi dx \neq \int fg\varphi dx = (I_{fg}, \varphi)$. The second reason, is that $u_1 \otimes u_2$ is not linear, it is bilinear, so it does not belong to $\mathcal{D}'(\Omega)$.

In fact, it is known that we cannot define a product on distributions that extend the pointwise product of functions (see [18]). It is enough to note that $x^{-\frac{1}{2}}\chi_{(-1,1)}$ is a distribution in $\Omega = (-1, 1)$, but its square is $x^{-1}\chi_{(-1,1)}$ is not integrable near 0.

So we are not able to generalize the Riemann curvature tensor (and thus any of the above tensors) unless the Christoffel symbols are regarded as functions, and this demands on g to have at least one weak derivative.

Remark 45. Suppose now that the metric g is C^r_* for some $r > 1$ in some local coordinates. Then g^{jk} are C^r_* by the algebra properties of Zigmund spaces, and thus it is clear that Γ^c_{ab} are C^{r-1}_*. This implies that $R_{abc}{}^d$ and R_{ab} are C^{r-2}_*, since differentiation (in distributional sense) subtracts one exponent in the Zigmund spaces (see Remark after 9). Moreover, we can define the product $g^{jk}R_{ab} \in C^{r-2}_*$ by the Corollary after Lemma 15, since $r > 1$ implies $r > |r - 2|$. Therefore $P_{ab} \in C^{r-2}_*$ and for the same reason $W_{abc}{}^d$ and $W_{abcd} \in C^{r-2}_*$.

Now we see that if we demand one more derivative on the metric, the Cotton tensor can also be defined in the weak sense in a satisfactory manner. This time we need some properties of the space $E' = W^{2,2} \cap W^{1,\infty}$ in some open bounded set $\Omega \subset \mathbb{R}^n$, which the next propositions summarize.

Proposition 30. Let $\Omega \subset \mathbb{R}^n$. Let $f, g \in E'(\Omega) = W^{2,2}(\Omega) \cap W^{1,\infty}(\Omega)$. Then $fg \in E'(\Omega)$. Therefore $E'(\Omega)$ is an algebra.

Moreover, if we suppose that $f(x) \geq k > 0$ a.e. $x \in \Omega$ for some constant k then $f^{-1} \in E'(\Omega)$.

Proof. As $E'(\Omega) \subset E(\Omega) = W^{1,2}(\Omega) \cap L^\infty(\Omega)$, we use Proposition 33 to see that $(fg)_{x_i} = gf_{x_i} + f_{x_i}g$. Now we note that $f_{x_i}, g_{x_i} \in E(\Omega)$, so $(fg)_{x_i} \in E$, and this gives $fg \in E'(\Omega)$.

If now we suppose $f(x) \geq k > 0$, by Proposition 33 we know that $f^{-1} \in E$ and that $(f^{-1})_{x_i} = -f^{-2}f_{x_i}$. From here we see that $(f^{-1})_{x_i} \in E$, since both $f^{-2} = f^{-1}f^{-1}$ and f_{x_i} lie in E, which is an algebra. □

Proposition 31. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set, let $f \in E'(\Omega) = (W^{2,2} \cap W^{1,\infty})(\Omega)$ and $g \in L^2(\Omega)$. Then $fg \in L^2$ and moreover $(fg)_{x_i} = gf_{x_i} + fg_{x_i}$ as elements in $W^{-1,2}(\Omega)$ (in particular we are saying that $fg_{x_i} \in W^{-1,2}(\Omega)$).

Proof. First note that the multiplication by f satisfies

$$\cdot f : W^{1,2}_0(\Omega) \to W^{1,2}_0(\Omega) : \varphi \mapsto f\varphi \tag{159}$$

Recall Proposition 28, where we only supposed that $f \in E(\Omega) = (W^{1,2} \cap L^\infty)(\Omega)$ and then the multiplication by f was worse behaved since it mapped the smaller space $W^{1,q}_0(\Omega)$ (for $q > n$) into $W^{1,2}_0(\Omega)$. However, if $f \in E'(\Omega)$, $\varphi \in W^{1,2}_0(\Omega)$, then by Lemma 155 we know that the distributional derivative of $f\varphi$ is $\nabla(f\varphi) = \varphi\nabla f + f\nabla\varphi$. As $f \in L^\infty(\Omega), \nabla f \in L^\infty(\Omega; \mathbb{R}^n)$, we see that $\nabla(f\varphi) \in L^2(\Omega; \mathbb{R}^n)$, so $f\varphi \in W^{1,2}(\Omega)$.

Now let us see that $Tr(f\varphi) = 0 \in L^2(\partial\Omega)$. As $\varphi \in W^{1,2}_0(\Omega)$ we can find $\phi_n \in \mathcal{D}(\Omega)$ such that $\phi_n \to \varphi$ in $W^{1,2}(\Omega)$. By the Sobolev Embeding, as $f \in W^{1,\infty}(\Omega)$, f is continuous (in fact Lipschitz), so $f\phi_n \in C_c(\Omega)$ and therefore $Tr(f\phi_n) = 0$. If we see that $f\phi_n \to f\varphi$ in $W^{1,2}(\Omega)$, by the continuity of the trace we will conclude that $Tr(\varphi f) = 0$ as we want. To see this first note that $\|f\phi_n - f\varphi\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|\phi_n - \varphi\|_{L^2(\Omega)} \to 0$ as $n \to \infty$. Also, aplaying the product rule, we have

$$\|\nabla(f\phi_n) - \nabla(f\varphi)\|_{L^2(\Omega)} \leq \|\nabla f\|_{L^2(\Omega)}\|\phi_n - \varphi\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|\nabla\phi_n - \nabla\varphi\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty$$

So we have seen the mapping property (159). Now we can consider the transpose map $(\cdot f)'$ of $\cdot f$, which we shall denote also by $(\cdot f)' := \cdot f$, since it is an extension of the original $\cdot f$ regarding $W_0^{1,2}(\Omega) \subset W^{-1,2}(\Omega)$ as usual. Therefore we have a well defined map

$$\cdot f : W^{-1,2}(\Omega) \to W^{-1,2}(\Omega) : T \mapsto fT \tag{160}$$

This shows in particular that for $g \in L^2(\Omega)$ we can define fg_{x_i}. Now let us see $(fg)_{x_i} = gf_{x_i} + fg_{x_i}$ en $W^{-1,2}(\Omega)$. Take $\varphi \in W_0^{1,2}(\Omega)$, and note that the product rule applies to $f\varphi$. It follows that

$$\begin{aligned} ((fg)_{x_i}, \varphi) &= -(fg, \varphi_{x_i}) = -\int_\Omega fg\varphi_{x_i}dx \\ (f_{x_i}g + g_{x_i}f, \varphi) &= \int_\Omega f_{x_i}g\varphi dx - \int_\Omega g(f\varphi)_{x_i}dx = -\int_\Omega fg\varphi_{x_i}dx \end{aligned}$$

and this completes the Proposition. □

Now let us write the expression of the Cotton tensor C_{abc} in local coordinates.

$$\begin{aligned} &(\nabla_{\partial_a}S)(\partial_b, \partial_c) = \partial_a S_{bc} - S(\Gamma^l_{ab}\partial_l, \partial_c) - S(\partial_b, \Gamma^l_{ac}\partial_l) = \partial_a S_{bc} - \Gamma^l_{ab}S_{lc} - \Gamma^l_{ac}S_{bl} \\ &C_{abc} = (\nabla_{\partial_a}S)(\partial_b, \partial_c) - (\nabla_{\partial_b}S)(\partial_a, \partial_c) = \partial_a S_{bc} - \partial_b S_{ac} + \Gamma^l_{bc}S_{al} - \Gamma^l_{ac}S_{bl} \end{aligned} \tag{161}$$

Proposition 32. Given a Riemmannian manifold (M, g), and a system of coordinates (φ, Ω), if the metric $g_\varphi \in E'(\Omega) = (W^{2,2} \cap W^{1,\infty})(\Omega)$, then we can make sense of the components of the Cotton tensor in these coordinates (as distributions), and moreover if $\lambda \in E'(\Omega)$ is a positive function, and we define $g' = \lambda g \in E'(\Omega)$, then the Cotton tensors c and c' of g and g' satisfy the same relation than in the case where g and g' are C^3, i.e, $c' = c - w(\cdot,\cdot,\cdot, grad(u))$, with $u := \frac{1}{2}log(\lambda)$. In particular if $n = 3$ we have $c' = c$.

Proof. All spaces will be considered in Ω unless explicit mention. First note that by Proposition 30 the inverse $g^{ij} \in E'$ and therefore $\Gamma^c_{ab} = \frac{1}{2}g^{cl}[\partial_a g_{bl} + \partial_b g_{al} - \partial_l g_{ab}] \in E = W^{1,2} \cap L^\infty$. Also, from the formula of ${R_{abc}}^d$ given in (156) we see that ${R_{abc}}^d \in L^2 + E \subset L^2$, so also $R_{bc} \in L^2$, and $S_{ab} \in L^2$. From the formula 161 we readily see that $C_{abc} \in W^{-1,2} + L^2 \subset W^{-1,2}$.

It remains to see the transformation of the Cotton tensor under conformal change. Put $g' = \lambda g$ with $\lambda \in E'$. We shall see that we can apply the Leibnitz rule for the derivatives of the metric up to order 3. If we see this, then the components C'_{abc} and C_{abc} of the Cotton tensors associated to g' and g will be related by the same formal expression (in terms of the metric g and λ) than in the smooth case (this time making sense as distributions), and this will give the claim.

Recall that we already saw in Proposition 29 that we can make operations with the components of tensors that involve ≤ 2 derivatives of the metric as if the metric were smooth, and it remains to see that in this case we can do it with components of tensors that involve up to three derivatives of the metric.

More concisely, to analize how the Cotton tensor change we are interested in the components S'_{ab} of the Schouten tensor, which depend on the second derivatives of the metric via the Ricci tensor R'_{ab} and the scalar curvature $R'_{rs}{g'}^{rs}g'_{ab}$ (so via the Ricci tensor). The higher order terms (i.e, less regular, with more derivatives) of the Ricci tensor R'_{ab} are things of the type $\partial_a {\Gamma'}^d_{bc} \in L^2$. More concisely we have

$$\begin{aligned}
{\Gamma'}^c_{ab} &= \tfrac{1}{2}\lambda^{-1} g^{cl}\partial_a[\lambda g_{lb}] + \text{ similar things } = \tfrac{1}{2}\lambda^{-1}g^{cl}g_{lb}\partial_a\lambda + \tfrac{1}{2}g^{cl}\partial_a g_{lb} + \cdots \in E \\
\partial_d {\Gamma'}^c_{ab} &= \tfrac{1}{2}\lambda^{-1}g^{cl}g_{lb}\partial_{ad}\lambda + \tfrac{1}{2}g^{cl}\partial_{ad}g_{lb} + \text{ similar higher order terms } + \text{ things in } E
\end{aligned}$$

From this we see that, in order to apply the Leibnitz rule again when computing $\partial_{ed}{\Gamma'}^c_{ab}$, we just need to check that it is valid for things of the form fg with $f \in E'$ and $g \in L^2$. This follows from Proposition 31, where we saw that under these assumptions we have $\nabla(fg) = g\nabla f + f\nabla g$ in $W^{-1,2}$. Therefore we can apply the Leibnitz rule all the times we need in order to compute C'_{abc} in terms of C_{abc}, and thus they are related by the same formulas here that in the regular case. This proves the proposition. $\square$

Remark 46. Suppose now that the metric g is C^r_* for some $r > 2$ in some local coordinates. Then g^{jk} are C^r_*, so Γ^c_{ab} are C^{r-1}_* by the algebra properties of Zigmund spaces and therefore ${R_{abc}}^d$ and R_{ab} are C^{r-2}_*. Also, the product $g^{jk}R_{ab} \in C^{r-2}_*$, so $P_{ab} \in C^{r-2}_*$ and analogously ${W_{abc}}^d$ and $W_{abcd} \in C^{r-2}_*$. Looking at the components of C_{abc} we see that the Cotton tensor is in C^{r-3}_*, since differentiation substracts one exponent as mentioned before.

7.3. Ellipticity of the Ricci Tensor in Harmonic Coordinates.

Having defined tensors in the weak sense we can give now some regularity properties of the type: if the metric is C^r, for $r > 1$ and the tensor is more regular than expected (as a distribution), then the metric gain almost all the derivatives one can expect. Let us start with the Ricci tensor. This case will be the easiest.

Proposition 33. Let (M, g) be a Riemannian manifold and (y, Ω) a system of coordinates. Suppose that the metric $g \in C^r$ for some $r > 1$ in the y-coordinates near a point $x_0 \in M$. Then the expression of the Ricci tensor in any system of harmonic coordinates near x_0 can be regarded as an elliptic matrix coefficient linear differential operator acting on g.

Proof. Take $(\varphi)^{-1} = x$ any system of harmonic coordinates near x_0, and recall that $g \in C^r_*$ expressed in the φ coordinates. Note that as the x coordinates are harmonic, by Corollary 14 we know that in the x coordinates we have

$$R_{ij}(g) = a^{lk}(x)\partial_{lk}g_{ij} + A_1(x)D^1(g) + A_0(x)g$$

being $a^{lk} := g^{lk}$ and $A_i(x) = P_i(g, D^1(g))$ for $i = 1, 0$ are $1 \times n^2$ rows which act on the $n^2 \times 1$ column g. The entries of A_i are algebraic expression involving the derivatives of g up to order 1. It is clear that the entire operator $Ric = (R_{ij}(g))_{ij}$ has as principal symbol the $n^2 \times n^2$ matrix $P_2(x, \xi) = a^{lk}\xi_{lk}Id_{n^2\times n^2} = |\xi|^2 Id_{n^2\times n^2}$, so its principal symbol is an isomorphism of $\mathbb{R}^{n^2}$ for each $\xi \neq 0$. This shows that the Ricci operator is elliptic in harmonic coordinates.

Note that this is not really the differential operator associated to the Ricci tensor acting on g, since we are regarding as coefficcients some expresions that depend on g. This formal issue can be fixed by considering the linearization of the Ricci tensor at g. Later we shall make some comments about this. $\square$

The ellipticity of the ricci tensor in harmonic coordinates results in the corresponding elliptic regularity result.

Theorem 20. Let (M, g) be a Riemannian manifold. Suppose that in some system of coordinates ϕ the expression of the metric g_ϕ is C^r for some $r > 1$. Suppose furthermore that the expression R_{ab} of the Ricci tensor in some system of harmonic coordinates φ satisfies $R_{ab} \in C^s_*$ for some $s > r - 2$. Then the expression g_{ij} of the metric in the φ coordinates actually satisfies $g_{ij} \in C^{s+2}_*$.

Proof. We know by Proposition 24 that $g_{ij} \in C^r_*$ expressed in the φ coordinates. Recall that this automatically implies that $R_{ab} \in C^{r-2}_*$ in harmonic coordinates by Remark 45, which is why are supposing that $R_{ab} \in C^s_*$ for some $s > r - 2$ in the hipothesis of this Theorem.

That said, as the φ coordinates are harmonic, by Corollary 14 we know that in the φ coordinates we have

$$\tilde{f} := R_{ij} - f = a^{lk}\partial_{lk} g_{ij}$$

with $a^{lk} := g^{lk} \in C^r_*$ by the algebra properties of Zigmund spaces, and $f = T_1(g) \in C^{r-1}_*$, since $f = T_1(g)$ depends algebraically (and therefore smoothly) on the metric and its first derivatives. This is a linear elliptic operator of order 2 acting on g_{ij}. Besides, $\tilde{f} = R_{ij} - f \in C^\sigma_*$ where $\sigma = \min\{s, r-1\}$. We have $-r < r - 2 < \sigma < r$ and $g_{ij} \in C^{2-r+\varepsilon}_*$ where $\varepsilon = 2r - 2 > 0$. Therefore Theorem 12 applies. If $\sigma = s$, we get $g_{ij} \in C^{s+2}_*$ and we are done.

If $\sigma = r - 1$ we get $g_{ij} \in C^{r+1}_*$, so $a^{lk} = g^{lk} \in C^{r+1}_*$ and $\tilde{f} \in C^{\sigma_1}_*$, where $\sigma_1 = \min\{s, r\}$. We have $-r - 1 < \sigma_1 < r + 1$ and $g_{ij} \in C^{2-(r+1)+\varepsilon}_*$ where $\varepsilon = 2r > 0$ so again by Theorem 12 we get $g_{ij} \in C^{\sigma_1+2}_*$. If $\sigma_1 = s$ we are done.

If $\sigma_1 = r$ we get $g_{ij} \in C^{r+2}_*$, so $a^{lk} \in C^{r+2}_*$ and $\tilde{f} \in C^{\sigma_2}_*$, where $\sigma_2 = \min\{s, r + 1\}$, so we can apply again Theorem 12 (note that at each stage the hipothesis are satisfied more comfortably) and get $g_{ij} \in C^{\sigma_2+2}_*$.

Repeating this process m times for m large enough we get $\sigma_j = \min\{s, r + m - 1\} = s$ and $g_{ij} \in C^{s+2}_*$. This proves the Theorem. $\square$

Remark 47. Note that this Theorem weakens the hipothesis of Theorem 19 in the sense that it works for C^r metrics for $r > 1$ while in Theorem 19 we had to suppose that the metric was at least C^2.

An inmediate consequence of Theorem 20 above is the following.

Corollary 15. Let (M, g) be a Riemannian manifold. Suppose that in some system of coordinates ϕ the expression of the metric g_ϕ is C^r for some $r > 1$. We have:

(1) Suppose that in some system of harmonic coordinates φ the expression R_{ab} of the Ricci tensor is smooth. Then the expression g_{ij} of the metric in the φ coordinates is also smooth.

(2) In particular, if $R_{ab} = 0$ in some harmonic coordinates φ, then g_{ij} is smooth in the φ coordinates.

From this we can prove the test case for metrics less regular than C^2.

Corollary 16. (Test case for non-regular metrics) Let (M, g) be a Riemannian manifold. Suppose that in some system of coordinates ϕ the expression of the metric $g_{ij}(\phi)$ is C^r for some $r > 1$ and that the curvature tensor $R_{abcd}(\phi)$ vanishes (as a distribution) in the ϕ coordinates. Then we can find another system of coordinates ϕ^* so that the metric $g_{ij}(\phi^*)$ expressed in these coordinates is flat, i.e, $g_{ij}(\phi^*) = \delta_{ij}$.

Proof. As $R_{abcd}(\phi) = 0$ and $g_{ij}(\phi)$ is C^r in the ϕ coordinates, then in any harmonic coordinates φ we have that $g_{ij}(\varphi)$ is C^r_*, so we can make sense of the curvature tensor in the φ coordinates also. Denote $R_{abcd}(\varphi)$ for the curvature tensor in the φ coordinates. Then by tensoriality we have $R_{abcd}(\varphi) = 0$, so if $R_{ab}(\varphi)$ denote the components of the Ricci tensor in the φ coordinates, we have $R_{ab}(\varphi) = 0$.

By the Corollary above, we conclude that the metric $g_{ij}(\varphi)$ is smooth in these φ coordinates. Moreover, as $R_{abcd}(\varphi) = 0$, from the cassical version of the test case for regular metrics we see that there exists another coordinate system ϕ^* so that $g_{ij}(\phi^*) = \delta_{ij}$ in the ϕ^* coordinates. This proves the Corollary. $\square$

Remark 48.

(1) Note that the components R_{ij} of the Ricci tensor depend on all the metric g and not only on the component g_{ij}, so the Ricci tensor is a differential operator acting on g, but it is not linear. However, the ricci tensor depens algebraically on the derivatives of the metric up to order 2, and if we look at its expression, we have

$$\begin{aligned}
&Ric(g) = (R_{ij})_{i,j=1}^n(g) = f_1[g(x)]D^2g + f_2[g(x), D^1g(x)]D^1g + f_3[g(x), D^1g(x)]g \\
&= B_1(x)D^2g + B_2(x)D^1g + B_3(x)g \\
&B_1(x) := f_1[g(x)] \quad ; \quad B_2(x) := f_2[g(x), D^1g(x)] \quad ; \quad B_3(x) := f_3[g(x), D^1g(x)]
\end{aligned}$$

for some $n^2 \times n^2$ matrixes $B_i(x)$ whose entries are polinomials depending on the derivatives of g up to order 1, in fact $B_1(x)$ only depends on g. Note that g is regarded as an $n^2 \times 1$ column here. In general, to establish regularity results for tensors which are non-linear differential operators on g, we shall consider the linearizations of the corresponding tensors. Let us consider now the Ricci tensor.

Given an open set $\Omega \subset \mathbb{R}^n$, as the set of metrics in Ω, denoted by $\mathcal{M}(\Omega)$, is an open set in the vectorial space of simmetric (2,0)-tensors in Ω, denoted $\mathcal{ST}_2(\Omega)$, then for every $g \in \mathcal{M}(\Omega)$, the tangent space $T_g\mathcal{M}(\Omega)$ is canonically identified as $\mathcal{ST}_2(\Omega)$. Given the Ricci tensor Ric, we can consider its differential at g denoted d_gRic, and we have

$$Ric : \mathcal{M}(\Omega) \to \mathcal{ST}_2(\Omega) \ ; \ d_gRic : \mathcal{ST}_2(\Omega) \to \mathcal{ST}_2(\Omega) : h \to \frac{d}{dt}\Big|_0 Ric(g + th)$$

From the expression of Ric we have $T_g(h) := d_gRic(h) = f_1[g(x)]D^2(h) + T_{1,g}(h)$, where

$$T_{1,g}(h) = A_1[g(x), D^1g(x)]D^1h + A_2[g(x), D^1g(x)]h$$

is a linear function on h which depens on the derivatives of h up to order 1. The coefficients of $A_i(g(x), D^1g(x))$, which are $n^2 \times n^2$ matrixes, depend algebraically on the derivatives of g up to order 1.

That said, to establish regularity of the Ricci tensor, we consider the principal symbol of the now linear operator $T_g(h) = d_gRic(h)$, which is $P_2(x, \xi) := B_1(x)\xi^2$, where ξ^2 stands for products of two of the components of $\xi \in \mathbb{R}^n$. The important thing here is that we can regard the Ricci tensor as a linear differential operator of order 2 acting on g, by considering $B_i(x)$ as fixed matrix functions depending only on x and not on g.

Moreover if $g \in C^r_*$ for $r > 1$ then these matrix coefficients $B_i(x)$ of the Ricci tensor are C^{r-1}_*. And the crucial point is that the principal symbol of the Ricci tensor regarded as a linear operator via this trick is again $B_1(x)\xi^2$. We conclude that *the ricci tensor and its linearization at g have the same principal symbol* . So if we see that $T_g(h) = d_gRic(h)$ is a linear elliptic (or overdetermined elliptic) operator, then also the Ricci tensor regardad as a linear operator will be elliptic (overdetermined elliptic).

Besides, the same argument applies for the Weyl tensor. The only difference is that the matrix coefficients have other dimensions.

(2) Note that the principal symbol of the Ricci tensor Ric is a diagonal matrix, i.e, the higher order term of the expression of R_{ij} only depends on g_{ij}, and the other components of g appear only as lower order terms. This is why in establishing regularity for the Ricci tensor we can work as with single equations, and there is no need of elliptic regularity results for matrix differential operators. We shall see later that this is not the case for the Weyl tensor.

(3) It can be proved (see [21], Section 2.3, page 76 for example) that the Ricci tensor is never an

elliptic differential operator, i.e, its symbol has non-trivial kernel. This shows how important it is to work in harmonic coordinates in order to make the expression of the ricci tensor an elliptic equation. This explains also Example 6 where we saw that it can happen that $Ric = 0$ but g is not smooth.

7.4. Ellipticity of the Weyl Tensor in n-Harmonic Coordinates

.

Now we shall make some computations (yet more) to see that, working in special coordinates, the expression of the Weyl tensor becomes an overdetermined elliptic operator acting on g. For the Weyl tensor these coordinates will not be in general harmonic. This time we will require that in some coordinates the determinant of g, denoted by $|g|$, satisfies $|g| = 1$.

This will simplify a lot the expression of the Weyl tensor and will allow us to prove ellipticity. Of course this assumption is not valid for every metric g, but it is valid for every conformal class of metrics $[g]$. Therefore we must work in some special coordinates (special in the sense that are good to establish regularity results) and we require that the special property that the coordinates have is conformally invariant. We saw before that being an n-harmonic function is a conformally invariant condition, so we shall choose n-harmonic coordinates and then we can suppose that $|g| = 1$.

That said, let us go for the calculations. First note that, as done before with the ricci tensor, if we regard the metric $g = (g_{ij})_{ij}$ as a column of size n^2, the (non-linear) differential operator $W(g)$ admits the expression

$$
\begin{aligned}
&W(g) = (W_{abcd}(g))_{a,b,c,d=1}^{n}(g) = f_1[g(x)]D^2g + f_2[g(x), D^1g(x)]D^1g + f_3[g(x), D^1g(x)]g \\
&= B_1(x)D^2g + B_2(x)D^1g + B_3(x)g \\
&B_1(x) := f_1[g(x)] \quad ; \quad B_2(x) := f_2[g(x), D^1g(x)] \quad ; \quad B_3(x) := f_3[g(x), D^1g(x)]
\end{aligned}
$$

this time $B_i(x)$ are $n^4 \times n^2$ matrixes, and the Weyl tensor is regarded as a $n^4 \times 1$ column. Therefore the principal symbol of the Weyl tensor regarded as a linear operator is the same as the principal symbol of its linearization at g, commonly called its differential d_gW. If we consider the scalar valued differential operators $W_{abcd}(g)$, then the coefficients are $1 \times n^2$ rows acting on g.

It is convenient to set some notation.

Definition 45. Fix some coordinates (Ω, x) on a Riemannian manifold (M, g). For a tensor $T(g)$ (expressed in the x coordinates) regarded as a differential operator acting on the set of metrics $g \in \mathcal{M}(\Omega)$, we denote d_gT as the differential (or linearization) of T at the fixed metric g. Note that d_gT is a linear differential operator acting on simmetric $(2,0)$ tensors $s \in \mathcal{ST}_2(\Omega)$ (as before, s is considered a $n^2 \times 1$ column valued function), and $d_gT(s)$ is a tensor with the same simmetries as T (since the set of tensors having some simmetries is a vectorial space, and then it coincides with its tangent space).

Also, we denote $\sigma_g(T)(x, \xi)$ as the principal symbol of the linearization of T at the fixed metric g. Note that $\sigma_g(T)(x, \xi)$ acts on simmetric bilinear forms h identified as $n^2 \times 1$ columns, and $\sigma_g(T)(x, \xi)(h)$ is a column of the same dimensions that T.

In particular, $\sigma_g(W_{abcd})(x, \xi)$ is an $1 \times n^2$ row-valued polinomial in ξ of degree 2. This rows as commented above, depend on x via the metric $g(x)$ (and not on its derivatives). The same applies for $\sigma_g(R_{ab})(x, \xi)$.

Proposition 34. Let (x, Ω) some system of coordinates on the Riemannian manifold (M, g). Suppose that $g \in C^r_*$ in these coordinates. Then we have

$$
\begin{aligned}
&\xi^a\xi^b\sigma_g(W_{abcd})(x, \xi) \\
&= -\frac{n-3}{2(n-2)}\left[-2|\xi|^2\sigma_g(R_{bc})(x, \xi) + \frac{n-2}{n-1}\xi_b\xi_c\sigma_g(R)(x, \xi) + \frac{1}{n-1}|\xi|^2 g_{bc}\sigma_g(R)(x, \xi)\right]
\end{aligned}
\tag{162}
$$

Proof. Remind the first contracted and second contracted Bianchi Identities from 32 and 34. In Einstein notation they have the form

$$g^{dm}\nabla_m R_{abcd} = \nabla^d R_{abcd} = \nabla_a R_{bc} - \nabla_b R_{ac} \;\; ; \;\; g^{ma}\nabla_m R_{ab} = \nabla^a R_{ab} = \frac{1}{2}\nabla_b R \tag{163}$$

These identities require 3 derivatives on the metric, but here $g \in C^r_*$ so they are not valid. However the Bianchi identities imply the corresponding identities for the symbols. If we consider the components of the tensors above as differential operators acting on metrics which are identically the same, then its linealizations at g must have the same principal symbol.

Note also that for a differential operator $T(g)$ of order 2 and for a smooth function $f(x, g, D^1 g)$ which depends on the lower order derivatives of g, then we have

$$\sigma_g(T(g)f(x, g, D^1 g))(x, \xi) = f(x, g, D^1 g)\sigma_g(T)(x, \xi)$$

The last assertion can be seen, for example, using the formula which says that for $s \in \mathcal{ST}_2(\Omega)$ we have $d_g(Tf)(s) = \partial_t|_0[T(g + ts)f(x, g + th, D^1 g + tD^1 s)]$, and applying leibnitz rule to see that the second order derivatives of s are multiplied by f. So, taking the symbols in 163 above we have

$$\begin{aligned}
\xi^d \sigma_g(R_{abcd})(x, \xi) &= \xi_a \sigma_g(R_{bc})(x, \xi) - \xi_b \sigma_g(R_{ac})(x, \xi) \\
\xi^a \sigma_g(R_{ab})(x, \xi) &= \frac{1}{2}\xi_b \sigma_g(R)(x, \xi)
\end{aligned}$$

These identities do not require 3 derivatives anymore, but only require that the operators $R_{abcd}(g)$, $R(g)$, and $R_{ab}(g)$ are well defined, and we know this is true (in the distributional sense) for C^r_* metrics if $r > 1$. Therefore these identities hold also for C^r_* metrics. Equivalent identities are

$$\begin{aligned}
&\xi^d \sigma_g[R_{abcd} + g_{bd}R_{ac} - g_{ad}R_{bc}](x, \xi) = 0 \\
&\xi^a \sigma_g[R_{ab}g_{cd} - \frac{1}{2}g_{cd}g_{ab}R](x, \xi) = 0
\end{aligned} \tag{164}$$

Taking into account the expresion of S_{ab} given in 157 we want to see the expression of the Weyl tensor. Below, underlined terms on the same color mean that they are summed and subtracted on the expression, so they cancel, but we want them to be in the expression to use 164.

$$\begin{aligned}
W_{abcd} &= R_{abcd} - S_{ac}g_{bd} + S_{ad}g_{bc} - g_{ac}S_{bd} + g_{ad}S_{bc} \\
&= R_{abcd} + \underline{(R_{ac}g_{bd} - R_{bc}g_{ad})} + \left(\frac{1}{n-2} - \underline{1}\right)(R_{ac}g_{bd} - R_{bc}g_{ad}) \\
&+ \frac{1}{n-2}(R_{bd}g_{ac} - R_{ad}g_{bc}) - \frac{R}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \\
&= (R_{abcd} + R_{ac}g_{bd} - R_{bc}g_{ad}) - \frac{n-3}{n-2}[R_{ac}g_{bd} - \underline{\frac{1}{2}Rg_{ac}g_{bd}}] \\
&+ \frac{1}{n-2}[R_{bd}g_{ac} - \underline{\frac{1}{2}Rg_{bd}g_{ac}} - R_{ad}g_{bc} + \underline{\frac{1}{2}Rg_{ad}g_{bc}}] + \frac{n-3}{n-2}R_{bc}g_{ad} \\
&- \underline{\frac{n-3}{2(n-2)}Rg_{ac}g_{bd}} + \frac{1}{2(n-2)}[\underline{Rg_{bd}g_{ac}} - \underline{Rg_{ad}g_{bc}}] - \frac{R}{(n-1)(n-2)}(g_{ac}g_{bd} - g_{bc}g_{ad})
\end{aligned}$$

Now we take symbols and contract, and we use 164 to get

$$\xi^a\xi^d\sigma_g(W_{abcd})(x,\xi)=\frac{n-3}{n-2}\xi^a\xi^d g_{ad}\sigma_g(R_{bc})(x,\xi)$$
$$+\left[\frac{1}{2(n-2)}+\frac{3-n}{2(n-2)}-\frac{1}{(n-2)(n-1)}\right]\xi^a\xi^d g_{ac}g_{bd}\sigma_g(R)(x,\xi)$$
$$+\left[\frac{-1}{2(n-2)}+\frac{1}{(n-1)(n-2)}\right]\xi^a\xi^d g_{bc}\sigma_g(R)(x,\xi)$$
$$=-\frac{n-3}{2(n-2)}\left[-2|\xi|^2\sigma_g(R_{bc})(x,\xi)+\frac{n-2}{n-1}\xi_b\xi_c\sigma_g(R)(x,\xi)+\frac{1}{n-1}|\xi|^2 g_{bc}\sigma_g(R)(x,\xi)\right]$$

and this concludes the proof. □

Remark 49. Remind now some notations previously introduced, as $\Gamma^l=\Gamma^l_{ab}g^{ab}$ and $\Gamma_k=\Gamma^l g_{kl}$. Suppose that the metric g is C^r_* for $r>1$ in coordinates. Note that from what we saw in (152) we have that in any coordinate system where the metric is C^2 we have

$$\begin{aligned}R_{ab}&=-\frac{1}{2}g^{kl}\partial_{kl}g_{ab}+\frac{1}{2}[g_{la}\partial_b\Gamma^l+g_{lj}\partial_i\Gamma^l]+T_1(g)\\&=-\frac{1}{2}\Delta g_{ab}+\frac{1}{2}[g_{la}g^{lk}\partial_b\Gamma_k+g_{lb}g^{lk}\partial_a\Gamma_b]+T_1(g)\\&=-\frac{1}{2}\Delta g_{ab}+\frac{1}{2}[\partial_b\Gamma_a+\partial_a\Gamma_b]+T_1(g)\end{aligned}\tag{165}$$

This formula also holds for C^r_* metrics in the distributional sense, as mentioned so many times, and will be very useful. We see taking the principal symbol that

$$\sigma_g(R_{ab})(x,\xi)h=-\frac{1}{2}|\xi|^2h_{ab}+\frac{1}{2}[\xi_a\sigma_g(\Gamma_b)(x,\xi)h+\xi_b\sigma_g(\Gamma_a)(x,\xi)h]\tag{166}$$

The following Lemma shows us how the condition $|g|=1$ simplifies the expression of the scalar curvature.

Lemma 34. Let (x,Ω) a system of coordinates in the Riemannian manifold (M,g). Suppose that g is C^r_* for $r>1$, and $|g|=1$ in the x coordinates. Then we have

$$\begin{aligned}&R_{ab}=-\tfrac{1}{2}\Delta g_{ab}+\tfrac{1}{2}(\partial_a\Gamma_b+\partial_b\Gamma_a)+T_1(g)=\partial_l\Gamma^l_{ab}+T_1(g)\\&R=\partial_a\Gamma^a+T_1(g)\quad;\quad\sigma_g(R)(x,\xi)=\xi_a\sigma_g(\Gamma^a)(x,\xi)\end{aligned}\tag{167}$$

in the distributional sense.

Proof. First note that from (14) that

$$\Gamma^k=-\partial_i g^{ki}-\frac{1}{2}g^{ki}|g|^{-1}\partial_i|g|=-\partial_i g^{ki}-\frac{1}{2}\partial_i[log(|g|)]$$

so $\Gamma^k=-\partial_i g^{ki}$ if $|g|=1$. So if $|g|=1$ we have $\Gamma_j=-g_{jk}\partial_l g^{kl}$ and

$$\partial_i\Gamma_j=-g_{jk}\partial_{il}g^{kl}+T_1(g)=g^{kl}\partial_{il}g_{jk}+T_1(g)$$

The last equality follows applying ∂_{il} to the expression $\delta^l_j=g_{jk}g^{lk}$. From (165) we see that this implies

$$\begin{aligned}&R_{ab}=-\frac{1}{2}g^{kl}\partial_{kl}g_{ab}+\frac{1}{2}g^{kl}\partial_{al}g_{bk}+\frac{1}{2}g^{kl}\partial_{bl}g_{ak}+T_1(g)\\&=\tfrac{1}{2}g^{kl}[\partial_{al}g_{bk}+\partial_{bl}g_{ak}-\partial_{kl}g_{ab}]+T_1(g)=\partial_l\Gamma^l_{ab}+T_1(g)\\&R=R_{ab}g^{ab}=g^{ab}\partial_l\Gamma^l_{ab}+T_1(g)=\partial_l[\Gamma^l_{ab}g^{ab}]+T_1(g)=\partial_l\Gamma^l+T_1(g)\end{aligned}$$

Note that all the manipulations done are valid for C^r_* metrics, since only second derivatives of the metric are involved. This concludes the Lemma. $\square$

Substituting formulas (166) and (167) for the symbols of the Ricci tensor and scalar curvature (for metrics with determinant one) into the general expression of $\sigma_g(W_{abcd})$ given in (162) we obtain the following.

Corollary 17. Let (x,Ω) some system of coordinates on the Riemannian manifold (M,g). Suppose that $g\in C^r_*$ and $|g|=1$ in these coordinates. Then the symbol of the linearization of the Weyl tensor satisfies

$$\xi^a\xi^d\sigma_g(W_{abcd})(x,\xi)h = -\frac{n-3}{2(n-2)}\Bigg[|\xi|^4 h_{bc} - |\xi|^2[\xi_b\sigma_g(\Gamma_c)(x,\xi)h + \xi_c\sigma_g(\Gamma_b)(x,\xi)h] \\ +\frac{n-2}{n-1}\xi_b\xi_c\xi_l\sigma_g(\Gamma^l)(x,\xi)h + \frac{1}{n-1}|\xi|^2 g_{bc}\xi_l\sigma_g(\Gamma^l)(x,\xi)h\Bigg] \tag{168}$$

This Corollary in turn yields the main result, which we state now. First we need a remark.

Remark 50. Recall that in n-harmonic coordinates Γ^l has a nice expression given in 15. Now we shall make a minor modification from the formula given in 15. We compute

$$\Gamma^l = (2-n)\frac{g^{la}g^{lb}}{g^{ll}}\Gamma^l_{ab} = \frac{2-n}{2}\frac{g^{la}g^{lb}}{g^{ll}}g^{lr}[\partial_a g_{br}+\partial_b g_{ar}-\partial_r g_{ab}] = \frac{2-n}{2g^{ll}}[A+B+C]$$
$$A = g^{la}g^{lb}g^{lr}\partial_a g_{br} = -g^{la}g^{lb}g_{br}\partial_a g^{lr} = -g^{la}\partial_a g^{ll}$$
$$B = g^{la}g^{lb}g^{lr}\partial_b g_{ar} = -g^{lb}\partial_g g^{ll} = A$$
$$C = -g^{lb}g^{lr}g^{la}\partial_r g_{ab} = g^{lr}g^{lb}g_{ab}\partial_r g^{la} = g^{lr}\partial_r g^{ll} = -A = -B$$

Therefore we conclude that

$$\Gamma^l = \frac{n-2}{2}\frac{C}{g^{ll}} = \frac{2-n}{2}\frac{g^{lr}g^{lb}g^{la}}{g^{ll}}\partial_r g_{ab} \tag{169}$$

This formula will be crucial to the the ellipticity of the weyl tensor in n-harmonic coordinates.

Proposition 35. Let (Ω,x) be a coordinate system of a Riemannian manifold (M,g) of dimension $n\geq 4$. Suppose that the metric g is C^r_* in the x coordinates and satisfies $|g|=1$. Suppose furthermore that the coordinates x are n-harmonic. Then the expression $W_{abcd}(g)$ of the Weyl tensor in the x coordinates is such that the principal part of its linearization $\sigma_g(W)(x,\xi)$ is injective.

Therefore, if we regard W as a linear operator acting on g (via the trick mentioned above), then $W(g)$ is an overdetermined elliptic operator in the x-coordinates.

Proof. As the coordinates are n-harmonic, the formula (169) given in the remark above is true, and it implies that for $s\in\mathcal{ST}_2(\Omega)$, and h a bilinear form we have

$$\Gamma^l(g) = \frac{2-n}{2}\frac{g^{lr}g^{la}g^{lb}}{g^{ll}}\partial_r g_{ab} \quad ; \quad d_g\Gamma^l(s) = \frac{2-n}{2}\frac{g^{lr}g^{la}g^{lb}}{g^{ll}}\partial_r s_{ab}$$
$$\sigma_g(\Gamma^l)(x,\xi)h = \frac{2-n}{2}\frac{g^{lr}g^{la}g^{lb}}{g^{ll}}\xi_r h_{ab} = \frac{2-n}{2}\frac{h^{ll}}{g^{ll}}\xi^l$$
$$\sigma_g(\Gamma_a)(x,\xi)h = \sigma_g(g_{la}\Gamma^l)(x,\xi)h = g_{la}\sigma_g(\Gamma^l)(x,\xi)h = g_{la}\frac{2-n}{2}\frac{h^{ll}}{g^{ll}}\xi^l$$

Coming back to 168 we have

$$\begin{aligned}\xi^a\xi^d\sigma_g(W_{abcd})(x,\xi)h = -\frac{n-3}{2(n-2)}\Bigg[&|\xi|^4 h_{bc} - |\xi|^2[\xi_b g_{lc} + \xi_c g_{lb}]\sigma_g(\Gamma^l)(x,\xi)h \\ +\frac{n-2}{n-1}\xi_b\xi_c\xi_l\sigma_g(\Gamma^l)(x,\xi)h + \frac{1}{n-1}|\xi|^2 g_{bc}\xi_l\sigma_g(\Gamma^l)(x,\xi)h\Bigg] &:= -\frac{n-3}{2(n-2)}(Q(x,\xi)h)_{bc}\end{aligned}\tag{170}$$

Being $Q(x,\xi)$ the $n^2\times n^2$ matrix symbol acting on a bilinear simmetric form h, regarded as a $n^2\times 1$ column, by the formula

$$\begin{aligned}(Q(x,\xi)h)_{ab} &= |\xi|^4 h_{ab} - |\xi|^2[\xi_a g_{bm} + \xi_b g_{am}]\sigma_g(\Gamma^m)(x,\xi)h \\ &+\frac{n-2}{n-1}\xi_a\xi_b\xi_m\sigma_g(\Gamma^m)(x,\xi)h + \frac{1}{n-1}g_{ab}|\xi|^2\xi_m\sigma_g(\Gamma^m)(x,\xi)h\end{aligned}\tag{171}$$

Let us see that $Q(x,\xi)$ is injective, i.e, if $Q(x,\xi)h=0$ then $h=0$. Computations will be easier by raising the index and consider $Q(x,\xi)h^{ab}$.

$$\begin{aligned}&(Q(x,\xi)h)_{lk} = |\xi|^4 h_{lk} - |\xi|^2[\xi_l g_{km} + \xi_k g_{lm}]\frac{2-n}{2}\frac{h^{mm}}{g^{mm}}\xi^m \\ &+\frac{n-2}{n-1}\xi_l\xi_k\xi_m\frac{2-n}{2}\frac{h^{mm}}{g^{mm}}\xi^m + \frac{1}{n-1}g_{lk}|\xi|^2\xi_m\frac{2-n}{2}\frac{h^{mm}}{g^{mm}}\xi^m \\ &(Q(x,\xi)h)^{ab} = (Q(x,\xi)h)_{lk}g^{la}g^{kb} = |\xi|^4 h^{ab} - |\xi|^2[\xi^a\delta^b_m + \xi^b\delta^a_m]\frac{2-n}{2}\frac{h^{mm}}{g^{mm}}\xi^m \\ &-\frac{(n-2)^2}{2(n-1)}\xi^a\xi^b\xi_m\frac{h^{mm}}{g^{mm}}\xi^m + \frac{2-n}{2(n-1)}g^{ab}|\xi|^2\xi_m\frac{h^{mm}}{g^{mm}}\xi^m \\ &(Q(x,\xi)h)^{aa} = |\xi|^4 h^{aa} + (n-2)|\xi|^2(\xi^a)^2\frac{h^{aa}}{g^{aa}} \\ &-\frac{(n-2)^2}{2(n-1)}(\xi^a)^2\xi_m\frac{h^{mm}}{g^{mm}}\xi^m + \frac{2-n}{2(n-1)}g^{aa}|\xi|^2\xi_m\frac{h^{mm}}{g^{mm}}\xi^m \\ &= \left[g^{aa}|\xi|^2 + (n-2)(\xi^a)^2\right]\left[|\xi|^2\frac{h^{aa}}{g^{aa}} - \frac{n-2}{2(n-1)}\xi_m\xi^m\frac{h^{mm}}{g^{mm}}\right]\end{aligned}$$

Suppose $\xi\neq 0$ and $Q(x,\xi)h=0$, which is equivalent to $(Q(x,\xi)h)^{ab}=0$ for all $a,b=1,\dots,n$. In the factorization of $(Q(x,\xi)h)^{aa}$ the first factor is stricty positive if $\xi\neq 0$, so we must have

$$\frac{h^{aa}}{g^{aa}} = \frac{n-2}{2(n-1)}|\xi|^{-2}\xi_m\xi^m\frac{h^{mm}}{g^{mm}} := \lambda_{x,\xi}\tag{172}$$

With λ independent of the index a. So inserting this in 172 we have

$$\lambda = \frac{n-2}{2(n-1)}|\xi|^{-2}\xi_m\xi^m\lambda = \frac{n-2}{2(n-1)}\lambda$$

And therefore $\lambda=0$, so $h^{aa}=0$ for all $a=1,\dots,n$. This implies that $0=(Q(x,\xi)h)^{ab}=|\xi|^4h^{ab}$ for all a,b, so $h=0$. So $Q(x,\xi)$ is an invertible $n^2\times n^2$ matrix for all $x\in\Omega$ and all $\xi\neq 0$. This implies that $\sigma_g(W)(x,\xi)$ is inyective.

Indeed, if $\sigma_g(W_{abcd})(x,\xi)h=0$ for all a,b,c,d, then in particular

$$\xi^a\xi^d\sigma_g(W_{abcd})(x,\xi)h = -\frac{n-3}{2(n-2)}(Q(x,\xi)h)_{bc} = 0$$

for all b,c. As $Q(x,\xi)$ is inyective and $n\neq 3$, we see that $h=0$. This proves the Proposition. $\square$

Remark 51.

(1) In the prove above both the n-harmonic coordinates and the condition $|g| = 1$ have been crucial. We have used the fact that the symbol $\sigma_g(\Gamma^m)$ has a very simple expression in these coordinates to conclude that the symbol of the Weyl tensor is inyective, and the condition $|g| = 1$ has been used also to obtain a simple expression of the Weyl tensor. We show now that these conditions are really needed for the ellipticity of $W(g)$.

First, the Weyl tensor W regardad as an operator acting on metrics is not elliptic in any coordinates, not even in n-harmonic coordinates. This is easy to see. Take a C^3 (and not C^4) positive function u defined in an open set $\Omega \subset \mathbb{R}^n$ and consider the conformally flat metric given by $g = uId$. As the metric is conformally flat, we know by classical arguments that $W = 0$. If the Weyl tensor was elliptic, the typical boothstrap argument combined with classical elliptic regularity results would imply that g is smooth, but this is not true by the choice of u.

Note that as g is diagonal, the identity coordinates in Ω are n-harmonic for g, so necessarily it must fail that $|g| = 1$, and this clearly fails since $|g| = u^n$ is not C^4.

The operator $W(g)$ is not elliptic either if we only ask $|g| = 1$ whitout the condition that the coordinates are n-harmonic. Consider the metric

$$g := |d\varphi|^{\frac{-2}{n}} d\varphi^t \cdot Id \cdot d\varphi$$

with $\varphi \in C^4$. By construction $|g| = 1$, and by classical arguments $W(g) = 0$ but g is not smooth as φ is not smooth.

(2) Suppose for a moment that some metric g can be expressed in harmonic coordinates φ and moreover the pull-back g_φ satisfies $|g_\varphi| = 1$. Then we would have $\Gamma^l = 0$ for all l, and then the proof of the ellipticity of the Weyl tensor W_φ in the φ coordinates would be trivial. However it is not clear how to find such a pull-back of g, even allowing conformal change of the metric g.

Indeed, take a system φ of harmonic coordinates for g. Then the coordinates of the pull-back g_φ are harmonic, but we do not necessarily have $|g_\varphi| = 1$. This situation cannot be fixed by a conformal change: if we consider the conformal metric

$$\tilde{g} := |g_\varphi|^{-\frac{1}{n}} g_\varphi$$

then of course $|\tilde{g}| = 1$, but the φ coordinates are not harmonic for $\tilde{g}$ anymore.

The problem above is fixed if instead of harmonic we suppose that φ is an n-harmonic coordinate for g, since in this case the φ coordinates keep being n-harmonic for $\tilde{g}$.

If we try the inverse order, it does not work either. Indeed, let g be a metric expressed in arbitrary coordinates, and make the conformal change $\tilde{g} := |g|^{-\frac{1}{n}} g$ so that $|\tilde{g}| = 1$. If we consider a system φ of harmonic coordinates for $\tilde{g}$, then obviously $\tilde{g}_\varphi$ is expressed in harmonic coordinates but $|\tilde{g}_\varphi| = |d\varphi^t \tilde{g} d\varphi| = |d\varphi|^2 \neq 1$ in general. So this does not work.

Now we are ready to prove conformal flatness for low regular metrics.

Theorem 21. Let (M, g) a Riemannian manifold of dimension $n \geq 4$, and suppose that in some system of coordinates near $x_0 \in M$ the expression of the metric g is C^r for some $r > 1$. Suppose that the expression of the Weyl tensor W^d_{abc} satisfies that $W^d_{abc} \in C^s_*$ for some $s > r - 2$ in some system φ of n-harmonic coordinates. Then the metric $\tilde{g} := |g_\varphi|^{-\frac{1}{n}} g_\varphi \in C^{s+2}_*$ in the φ coordinates.

Proof. We know by Theorem 18 that n-harmonic coordinates exists near x_0 and moreover the expression of the metric is C^r_* in any system of n-harmonic coordinates. So let $g_{ab} \in C^r_*$ and $W^d_{abc}(g) \in C^s_*$ be the metric and Weyl tensor in a fixed system $(\varphi)^{-1} = x$ of n-harmonic coordinates. Write $\tilde{g}_{ab} := |g_\varphi|^{-\frac{1}{n}} g_{ab}$. We have $\tilde{g}_{ab} \in C^r_*$, by the algebra properties of Zigmund spaces, and the fact that they are closed for composition for $r > 1$. Besides we obviously have $|\tilde{g}| = 1$.

By conformal invariance for the $(3,1)$-Weyl tensor we have

$$W_{abc}{}^d(\tilde{g}) = W_{abc}{}^d(g) := f \tag{173}$$

The expression 173 above can be regarded as a second order linear differential operator acting on $\tilde{g}$ with $f \in C^s_*$. As we mentioned above, the coefficients of the differential operator $W_{abc}{}^d(\tilde{g})$ depend algebraically on the derivatives up to order 1 of $\tilde{g}$, and in fact the second order coefficients of $W(\tilde{g})$ only depend on $\tilde{g}$. Besides, the principal symbol of the differential operator $W_{abc}{}^d(\tilde{g})$ coincides with the principal symbol of its linearization $\sigma_{\tilde{g}}(W_{abc}{}^d(\tilde{g}))(x,\xi)$.

Now, by the calculation done in 35, we know that $\sigma_{\tilde{g}}(W_{abcd})(x,\xi)$ is injective. From the relation $\tilde{g}_{ld} W_{abc}{}^l = W_{abcd}$ we get

$$\tilde{g}^{ld} \sigma_{\tilde{g}}(W_{abc}{}^l)(x,\xi) = \sigma_{\tilde{g}}(W_{abcd})(x,\xi)$$

So if $\sigma_{\tilde{g}}(W_{abc}{}^d)(x,\xi)h = 0$ for some bilinear form h and all a,b,c,d, then also $\sigma_{\tilde{g}}(W_{abcd})(x,\xi)h = 0$ for all a,b,c,d, so $h = 0$. This argument obviously generalizes to see that the ellipticity of a tensor T is equivalent to the ellipticity of any tensor obtained from T by raising or lowering the index.

Therefore we have a system of the following type

$$W(x,D)\tilde{g} - T_1(\hat{g}) = \sum_{|\alpha|=2} A_\alpha(x) D^\alpha \tilde{g} = f - T_1(\tilde{g}) = \hat{f}$$

where $A_\alpha(x)$ are $n^4 \times n^2$ matrixes with entries in C^r_* (since they depend only on $\tilde{g}$ and not on its derivatives), $f = (W_{abc}{}^d(g)) \in C^s_*$ for $s > r-2$ is an $n^4 \times 1$ column, and $T_1(\tilde{g})$ is also an $n^4 \times 1$ column which depends algebraically on the derivatives up to order 1 of the metric $\tilde{g}$, so $T_1(\tilde{g}) \in C^{r-1}_*$ and $\hat{f} \in C^\sigma_*$ for $\sigma = \min\{r-1, s\}$. Also $\tilde{g} \in C^{2-r+\varepsilon}_*$ for $\varepsilon = 2r - 2 > 0$.

As the linear differential operator $W(x,D)$ is overdetermined elliptic in the x coordinates and it is defined in some open set Ω, we want to apply Proposition 12. Note that as $r > 1$, $-r < r-2 < s$, so $\sigma > r-2 > -r$ also. As $-r < \sigma < r$ we get $\tilde{g} \in C^{\sigma+2}_*$. If $\sigma = s$ we are done.

If $s > r-1$, then $\sigma = r-1$, and $\hat{f} \in C^{r-1}_*$ with $-r < r-1 < r$. Then we obtain $\tilde{g} \in C^{r+1}_*$, so $A_\alpha(x) \in C^{r+1}_*$. We have $-r-1 < r-1 < s$ and $\tilde{g} \in C^{2-(r+1)+\varepsilon+2}_*$. If besides $s \leq r+1$ then apply again Proposition 12 to get $\tilde{g} \in C^{s+2}_*$. If on the contrary $s > r+1$ we obtain $\tilde{g} \in C^{r+3}_*$, so $A_\alpha(x) \in C^{r+3}_*$. If we have $s \leq r+3$ we are done. On the contrary, if we still have $s > r+3$ we repeat this argument to get $A_\alpha(x) \in C^{r+5}_*$.

Iterating this argument k times we get $A_\alpha \in C^{r+2k-1}_*$ so for k large we will have $s \leq r + 2k - 1$, and then $\tilde{g} = |g|^{-\frac{1}{n}} g \in C^{s+2}_*$ expressed in any system of n-harmonic coordinates φ. This completes the proof. □

Remark 52. The proof also works if we consider the $(4,0)$-Weyl tensor W_{abcd} instead of $W_{abc}{}^d$. Take the n-harmonic coordinates $x = \varphi^{-1}$ and suppose that $W_{abcd}(g) \in C^s_*$ for $s > r-2$ in the x coordinates. We work in these coordinates. In this case $W_{abcd}(\tilde{g}) = |g|^{-\frac{1}{n}} W_{abcd}(g)$. By hipothesis $W_{abcd}(g) \in C^s_*$ for $s > r-2$ and also $|g|^{-\frac{1}{n}} \in C^r_*$. We have two cases. If $s > 0$ then $f := |g|^{-\frac{1}{n}} W_{abcd}(g) \in C^{\min\{s,r\}}$ by the algebra properties of the Zigmund spaces with positive exponent. If $s < 0$, then $|s| = -s < 2 - r < r$, so $C^r_* C^s_* \subset C^s_*$ by Lemma 15. In any case, and with notations as above, we obtain that also $\hat{f} \in C^\sigma_*$ for $\sigma = \min\{r-1, s\}$ in the first step, and the rest of the proof is identical.

However note that using the conformal invariance for the $(3,1)$-tensor we avoid appealing to the technichal results about Zigmund spaces.

Theorem 22. Let (M,g) be a Riemannian manifold of dimension $n \geq 4$. Suppose that the metric is C^r for some $r > 1$ in some system of local coordinates ϕ near a point $x_0 \in M$. If the Weyl tensor satisfies $W = 0$ near x_0 in the ϕ coordinates, then the metric g is conformally flat in some system of n-harmonic coordinates φ_1 near x_0, i.e, there exists a positive function λ which is C^r_* in the φ_1 coordinates and such that the metric satisfies $g_{ij} = \lambda\delta_{ij}$ in the φ_1 coordinates.

Remark 53. The converse is also true: if $g_{ij} = \lambda\delta_{ij}$ in some coordinates x for some function $\lambda \in C^r_*$, then the Weyl tensor of g must vanish on this coordinates, and then it also vanish in any other C^r_* coordinates.

Proof. Take any system of n-harmonic coordinates η and express g in the η coordinates, denote g_η for this expression, and note that by the properties of p-harmonic coordinates we have $g_\eta \in C^r_*$, so we can still define the Weyl tensor in the η coordinates, denote W_η for this expression. Now, since the expression of the Weyl tensor W_ϕ vanishes near x_0 we must necessarily have that also $W_\eta = 0$ by tensoriality. Now we apply Theorem 21 above to conclude that the conformal metric $\tilde{g} := |g_\eta|^{-\frac{1}{n}} g_\eta$ is smooth in the η coordinates. Recall that η is an arbitrary sistem of n-harmonic coordinates.

As $\tilde{g}$ is smooth and besides $W(\tilde{g}) = W_\eta = 0$, we can use the classical Weyl-Schouten Theorem to conclude that there exists another system of smooth coordinates φ such that the expresion $\tilde{g}_\varphi$ in the φ coordinates has the form $\tilde{g}_\varphi = c_\varphi Id$ for some smooth function c_φ in the φ coordinates.

The fact that the φ coordinates are smooth is because the metric $\tilde{g}$ is smooth, and the φ coordinates are n-harmonic for $\tilde{g}$ by Proposition 25, so the φ coordinates gain one derivative respect to the regularity of the metric $\tilde{g}$, as proved in Theorem 18. However, this also can be derived directly from the classical proof of the Weyl-Schouten Theorem.

That said, by the definition of the metric $\tilde{g}$ we see that

$$
\begin{aligned}
&c_\varphi Id = \tilde{g}_\varphi = \varphi^*(|g_\eta|^{-\frac{1}{n}} g_\eta) = (|g_\eta|^{-\frac{1}{n}} \circ \varphi) g_{\eta\circ\varphi} \qquad \text{and thus} \\
&(\eta \circ \varphi)^* g = g_{\eta\circ\varphi} = c_\varphi(|g_\eta|^{\frac{1}{n}} \circ \varphi) Id = \lambda Id
\end{aligned}
$$

with $\lambda := c_\varphi(|g_\eta|^{\frac{1}{n}} \circ \varphi)$ a C^r_* function since both φ and c_φ are smooth, and $|g_\eta|^{\frac{1}{n}} \in C^r_*$.

Finally note that the coordinates $\varphi_1 := \eta \circ \varphi$ have to be n-harmonic by Proposition 25. This concludes the Theorem. $\square$

This Theorem, as we announced in the introduction, characterizes when a C^r metric for $r > 1$ is locally conformally flat in dimension $n \geq 4$.

Remark 54.

1) It remains to characterize conformal flatness for low regular metrics for the case $n = 3$, if we require that $g \in C^r$ for $r > 2$ (this seems necessary to define the Cotton tensor as mentioned above). To do this we can proceed with the Cotton tensor as we did with the Weyl, and prove that a 3-manifold with C^r metric for $r > 2$ is locally conformally flat if and only if its Cotton tensor (defined in the weak sense) vanishes. The procedure is totally analogous. See [15] for details.

2) As mentioned in the introduction, it would be of interest to characterize conformal flatness for measurable metrics in order to solve the Beltrami system. If we want to obtain results for measurable metrics this approach does not seem to work for two reasons

The first reason is that the Weyl and Cotton tensors appear not to be well defined unless the Cristoffel symbols are distributions regular enough to allow pointwise multiplication of distributions to be well defined.

The second reason is that the techniques used here rely on elliptic regularity results for second order pseudodifferential operators with C^r_* symbols, and the results avaliable on the literature require $r > 1$ to work. This may be just a technical reason and not an actual obstruction.

Therefore, the problem of characterize conformal flatness for measurable metrics remains open, at least as far as we know, and seems to require another approach.

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