Ricci curvature for metric-measure spaces via optimal transport

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Universidad Autónoma de Madrid

Marcos Solera Diana
Tutores: Luis Guijarro y Daniel Faraco

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Introduction

The first chapter of this essay is devoted to the study of a metric in the space of abstract metric spaces (modulo isometries). The metric we will consider was defined by Mikhail Gromov in 1981 and uses the Hausdorff distance between subsets of a metric space. It is known as the Gromov-Hausdorff distance. We will mainly be concerned about the topology induced and convergence issues. The topology turns out to be relatively weak, and a good characterisation of precompact sets appears rapidly.

The next chapter focuses on proving the Bishop-Gromov inequality, named after Richard L. Bishop and Gromov. This inequality is a comparison theorem in Riemannian geometry relating curvature and volume. It is closely related to Myers' theorem. We will run over all necessary concepts of Riemannian geometry and develop some interesting results on the way.

Using the Bishop-Gromov inequality, we then get that the space of Riemannian manifolds of a given dimension with Ricci curvature bounded below and diameter bounded above is precompact in the Gromov-Hausdorff topology. Learning what kind of spaces appear in the closure would give a way to study smooth Riemannian manifolds, using compactness theorems. Furthermore, since the Gromov-Hausdorff limit of length spaces is again a length space, it would also open a window to define a notion of a length space having 'Ricci curvature bounded below', special cases of which would be Gromov-Hausdorff limits of manifolds with lower Ricci curvature bounds. Note that there is already a good notion of a metric space having 'sectional curvature bounded below by K' or 'sectional curvature bounded above by K', due to Alexandrov and dating back to 1951.

In this work we study an approach given by Lott and Villani in [16]. They use optimal transport to give a notion for a measured length space having Ricci curvature bounded below. Their definition allows one to obtain non-trivial consequences and in the case of Riemannian manifolds it coincides with classical notions. Their work also has the goal of extending results about optimal transport from the setting of smooth Riemannian manifolds to the setting of length spaces.

Contents

Chapter 1

Space of Metric Spaces

The idea of using a 'global' approach to metric spaces, instead of studying one metric space at a time, is the first step towards the Gromov-Hausdorff distance. This 'global' approach is a usual way to work in mathematics.

The Hausdorff distance is a good example of this. Indeed, it turns the set of all compact convex sets in $\mathbb{R}^n$ into a metric space, allowing us to apply 'analytic techniques' to convex sets, like using maxima and minima [1].

Our objective now is to extend this approach by introducing a distance between abstract metric spaces, which will be considered up to an isometry. However, our main concern is to study convergence and precompactness so the distance itself will not be essential, what will matter is the topology induced.

1.1 Gromov-Hausdorff Distance

The Gromov-Hausdorff distance is closely related to the usual Hausdorff distance.

1.1.1 Hausdorff distance

We recall the definition of Hausdorff distance. Denote by $U_r(S)$ the r-neighbourhood of a set S in a metric space, that is

$$U_r(S) = \{x \ : \ \operatorname{dist}(x, S) < r\} = \cup_{x \in S} B_r(x).$$

Definition 1.1.1 Let A and B be subsets of a metric space. The *Hausdorff distance* between A and B, is

$$d_H(A, B) = \inf\{r > 0 \ : \ A \subset U_r(B) \text{ and } B \subset U_r(A)\}.$$

A better grasp of the definition is obtained from the following reformulations:

- $$d_H(A, B) = \max\{\sup_{a \in A} \operatorname{dist}(a, B), \ \sup_{b \in B} \operatorname{dist}(b, A)\}.$$
- $d_H(A, B) \leq r$ if and only if $\operatorname{dist}(a, B) \leq r$ for all $a \in A$ and $\operatorname{dist}(b, A) \leq r$ for all $b \in B$.

[1]The space of convex sets is boundedly compact.

1.1.2 Gromov-Hausdorff distance

The idea behind the Gromov-Hausdorff distance is that the following requirements are satisfied:

1. The distance between subspaces in the same metric space is no greater than the Hausdorff distance between them.

2. The distance between isometric spaces is zero.

The Gromov-Hausdorff distance is the maximum distance satisfying these two requirements.

Definition 1.1.2 Let X and Y be metric spaces. For an $r > 0$, the *Gromov-Hausdorff distance* between them, denoted by $d_{GH}(X, Y)$, satisfies $d_{GH}(X, Y) < r$ if and only if there exist a metric space Z and subspaces X' and Y' of it which are isometric to X and Y respectively and such that $d_H(X', Y') < r$. In other words, $d_{GH}(X, Y)$ is the infimum of positive r for which the above Z, X' and Y' exist.

It follows that the Gromov-Hausdorff distance between isometric spaces is zero. We will prove that, in fact, d_{GH} is a metric on the space of the isometry classes of compact metric spaces.

Remark 1.1.3 Note that, when we take the ambient space Z, the subsets X' and Y' inherit its metric, i.e. they are considered with the restriction of the metric of Z and not with the induced intrinsic metric. For example, if X is a sphere with its standard Riemannian metric, we cannot take $Z = \mathbb{R}^3$ and $X' = S^2 \subset \mathbb{R}^3$ because X and X' would not be isometric.

Remark 1.1.4 The definition considers all metric spaces Z that contain subspaces isometric to X and Y. However, it is possible to reduce this class to disjoint unions of X and Y. Indeed, given some $r > d_{GH}(X, Y)$ take X', Y' and Z as in the definition so that $d_H(X', Y') < r$ and fix isometries $f : X \to X'$ and $g : Y \to Y'$. Now, for $x \in X$ and $y \in Y$ define $d(x, y) = d_Z(f(x), g(y))$, and take d to coincide with d_X and d_Y on X and Y, respectively. Then d is a semi-metric on $X \cup Y$ with $d_H(X, Y) < r$. To obtain a metric, define $d(x, y) = d_Z(f(x), g(y)) + \delta$ instead, where δ is any positive constant. Then $d_H(X, Y) < r + \delta$.

With this we may define the Gromov-Hausdorff distance between two metric spaces (X, d_X) and (Y, d_Y) as the infimum of $r > 0$ such that there exists a semi-metric d on the disjoint union $X \cup Y$ such that the restrictions of d to X and Y coincide with d_X and d_Y respectively and $d_H(X, Y) < r$ in the space $(X \cup Y, d)$. In other words, $d_{GH}(X, Y) = \inf\{d_H(X, Y)\}$, where the infimum is taken over all semi-metrics on $X \cup Y$ extending the metrics of X and Y.

Let's prove that d_{GH} is in fact a metric. In first place we have the following proposition.

Proposition 1.1.5 *d_{GH} satisfies the triangle inequality:*

$$d_{GH}(X_1, X_3) \leq d_{GH}(X_1, X_2) + d_{GH}(X_2, X_3)$$

for any metric spaces X_1, X_2, X_3.

Proof Following the remark, let d_{12} and d_{23} be metrics on $X_1 \cup X_2$ and $X_2 \cup X_3$, respectively, that extend the metrics of X_1, X_2 and X_3. We will now build a metric on $X_1 \cup X_3$. For $x_1 \in X_1$ and $x_3 \in X_3$, define

$$d_{13}(x_1, x_3) = \inf_{x_2 \in X_2} \{d_{12}(x_1, x_2) + d_{23}(x_2, x_3)\}.$$

On X_1 and X_3 we take d_{13} to coincide with the metrics of X_1 and X_3, respectively. It then follows that d_{13} satisfies the triangle inequality and is therefore a semi-metric on $X_1 \cup X_3$. Indeed, let x_1, $x_1' \in X_1$ and $x_3 \in X_3$. Given $\varepsilon > 0$ take $x_2 \in X_2$ such that

$$d_{13}(x_1', x_3) + \varepsilon \geq d_{12}(x_1', x_2) + d_{23}(x_2, x_3).$$

Then

$$\begin{aligned} d_{13}(x_1, x_3) &\leq d_{12}(x_1, x_2) + d_{23}(x_2, x_3) \\ &\leq d_{12}(x_1, x_1') + d_{12}(x_1', x_2) + d_{23}(x_2, x_3) \\ &= d_{13}(x_1, x_1') + d_{12}(x_1', x_2) + d_{23}(x_2, x_3) \\ &\leq d_{13}(x_1, x_1') + d_{12}(x_1', x_3) + \varepsilon. \end{aligned}$$

But $\varepsilon > 0$ is arbitrary thus

$$d_{13}(x_1, x_3) \leq d_{13}(x_1, x_1') + d_{13}(x_1', x_3).$$

For other combinations of points in X_1 and X_3 the triangle inequality follows similarly.

The definition of d_{13} gives that

$$d_H(X_1, X_3) \leq d_H(X_1, X_2) + d_H(X_2, X_3),$$

where $d_H(X_i, X_j)$ is taken with respect to the metric d_{ij} $(i,\ j = 1, 2, 3)$. Indeed, let $x_1 \in X_1$, then $d_{12}(x_1, X_2) \leq d_H(X_1, X_2)$ so, given $\varepsilon > 0$, we may take $x_2 \in X_2$ such that $d_{12}(x_1, x_2) \leq d_H(X_1, X_2) + \varepsilon$. Similarly, there exists $x_3 \in X_3$ such that $d_{23}(x_2, x_3) \leq d_H(X_2, X_3) + \varepsilon$. Consequently,

$$d_{13}(x_1, X_3) \leq d_{13}(x_1, x_3) \leq d_{12}(x_1, x_2) + d_{23}(x_2, x_3) \leq d_H(X_1, X_2) + d_H(X_2, X_3) + 2\varepsilon.$$

But $\varepsilon > 0$ and $x_1 \in X_1$ are arbitrary thus

$$d_{13}(x_1, X_3) \leq d_H(X_1, X_2) + d_H(X_2, X_3)$$

for all $x_1 \in X_1$. The same argument yields

$$d_{13}(x_3, X_1) \leq d_H(X_1, X_2) + d_H(X_2, X_3),$$

for all $x_3 \in X_3$. These two inequalities together give

$$d_H(X_1, X_3) \leq d_H(X_1, X_2) + d_H(X_2, X_3).$$

Taking the infimum over all metrics d_{12} and d_{23} it follows from the previous remark that

$$d_{GH}(X_1, X_3) \leq d_{GH}(X_1, X_2) + d_{GH}(X_2, X_3).$$

■

1.1.3 Reformulations

The definition of Gromov-Hausdorff distance that we have given requires constructing a new metric space Z and verifying the triangle inequality. This may lead to hard work even in simple cases. We will now give equivalent definitions which are easier to deal with. The idea is to compute or estimate $d_{GH}(X, Y)$ by comparing the distances within X and Y to each other. For this we introduce the notion of correspondence.

Definition 1.1.6 Let X and Y be two sets. A *correspondence* between X and Y is a set $\mathcal{R} \subset X \times Y$ such that for every $x \in X$ there exists at least one $y \in Y$ for which $(x, y) \in \mathcal{R}$, and for every $y \in Y$ there exists at least one $x \in X$ such that $(x, y) \in \mathcal{R}$.

We will use correspondences to 'compare' metric spaces. To measure how well they're doing this comparison we define their distortion, which compares the distance of 'corresponding' pairs of points.

Definition 1.1.7 Let $\mathcal{R}$ be a correspondence between metric spaces X and Y. The *distortion* of $\mathcal{R}$ is defined by

$$\text{dis}\mathcal{R} = \sup\{|d_X(x, x') - d_Y(y, y')| \ : \ (x, y), (x', y') \in \mathcal{R}\}$$

where d_X and d_Y are the metrics of X and Y respectively.

Finding a correspondence with a small distortion implies that the spaces X and Y are close in the Gromov-Hausdorff sense.

Theorem 1.1.8 *For any two metric spaces X and Y,*

$$d_{GH}(X, Y) = \frac{1}{2} \inf_{\mathcal{R}} (\text{dis}\mathcal{R})$$

where the infimum is taken over all correspondences $\mathcal{R}$ between X and Y.

We are therefore saying that the Gromov-Hausdorff distance between X and Y is equal to the infimum of $r > 0$ for which there exists a correspondence between X and Y with $\text{dis}\mathcal{R} < 2r$.

Proof $\boxed{\geq}$ For any $r > d_{GH}(X, Y)$, there exists a correspondence $\mathcal{R}$ with $\text{dis}\mathcal{R} < 2r$. Indeed, if $r > d_{GH}(X, Y)$ we may assume that X and Y are subspaces of some metric space Z and $d_H(X, Y) < r$ in Z. Define the correspondence

$$\mathcal{R} = \{(x, y) \mid x \in X, \ y \in Y, \ d(x, y) < r\}$$

where d is the metric of Z. Since $d_H(X, Y) < r$ we have that $\mathcal{R}$ is a correspondence. Now, given $(x, y) \in \mathcal{R}$ and $(x', y') \in \mathcal{R}$ we have:

1. $d(x, x') \leq d(x, y) + d(y, y') + d(x', y')$
2. $d(y, y') \leq d(x, y) + d(x, x') + d(x', y')$

that together with the definition of $\mathcal{R}$ yields

$$|d(x, x') - d(y, y')| \leq d(x, y) + d(x', y') < 2r.$$

Consequently, $\mathrm{dis}\mathcal{R} \leq 2r$. Since $r > d_{GH}(X,Y)$ is arbitrary we get that $\frac{1}{2}\inf_{\mathcal{R}} \mathrm{dis}\mathcal{R} \leq d_{GH}(X,Y)$.

$\boxed{\leq}$ Let's see that $d_{GH}(X,Y) \leq \frac{1}{2}\mathrm{dis}\mathcal{R}$ for any correspondence $\mathcal{R}$. For this let $\mathcal{R}$ be a correspondence and $r := \frac{1}{2}\mathrm{dis}\mathcal{R}$. We write d_X and d_Y for the metrics of X and Y, respectively.

It suffices to show that there is a semi-metric d on the disjoint union $X \cup Y$ such that $d\restriction_{X\times X} = d_X$, $d\restriction_{Y\times Y} = d_Y$, and $d_H(X,Y) \leq r$ in $(X \cup Y, d)$. For this we take the distance between x and y equal to r whenever $(x,y) \in \mathcal{R}$, and then take the minimal metric d generated by this condition. This metric is given by

$$d(x,y) = \inf\{d_X(x,x') + r + d_Y(y',y) \mid (x',y') \in \mathcal{R}\} \ .$$

Let's verify the triangle inequality for d:

1. Given x_1, $x_2 \in X$ and $y \in Y$, and $\varepsilon > 0$, by definition of d there exist $x' \in X$ and $y' \in Y$ such that $(x',y') \in \mathcal{R}$ and

   $$d_X(x_2,x') + r + d_Y(y,y') \leq d(x_2,y) + \varepsilon \ .$$

   Now, using this and the definition of d (which coincides with d_X and d_Y within X and Y, respectively) we have

   $$\begin{aligned} d(x_1,y) &\leq d_X(x_1,x') + r + d_Y(y',y) \\ &\leq d_X(x_1,x_2) + d(x_2,x') + r + d_Y(y',y) \\ &\leq d(x_1,x_2) + d(x_2,y) + \varepsilon \ . \end{aligned}$$

   Since $\varepsilon > 0$ is arbitrary we get

   $$d(x_1,y) \leq d(x_1,x_2) + d(x_2,y) \ .$$

2. Let x_1, $x_2 \in X$ and $y \in Y$, and $\varepsilon > 0$ again. Take (x_1',y_1'), $(x_2',y_2') \in \mathcal{R}$ such that

   $$d_X(x_1,x_1') + r + d_Y(y_1',y) \leq d(x_1,y) + \varepsilon$$

   and

   $$d_X(x_2,x_2') + r + d_Y(y_2',y) \leq d(x_2,y) + \varepsilon \ .$$

   Then, taking into account that $2r = \mathrm{dis}\mathcal{R} \geq d_X(x_1',x_2') - d_Y(y_1',y_2')$,

   $$\begin{aligned} d(x_1,y) + d(x_2,y) + 2\varepsilon &\geq d(x_1,x_1') + d_Y(y_1',y) + d(x_2,x_2') + d(y_2',y) + 2r \\ &\geq \underbrace{d(x_1,x_1') + d(x_1',x_2') + d(x_2,x_2')}_{\geq d(x_1,x_2)} + \\ &\qquad + \underbrace{d(y_1',y) + d(y,y_2') - d(y_1',y_2')}_{\geq 0} \\ &\geq d(x_1,x_2) \ . \end{aligned}$$

   Since $\varepsilon > 0$ is arbitrary we get

   $$d(x_1,x_2) \leq d(x_1,y) + d(y,x_2) \ .$$

The other cases are analogous.

Finally, $d_H(X,Y) \le r$. Indeed, given $x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathcal{R}$ thus $d(x,y) = r$. Consequently, $d(x,Y) \le r$. The inequality $d(X,y) \le r$ for all $y \in Y$ follows similarly. ■

We will now define another tool which allows us to handle the Gromov-Hausdorff distance. First, we recall the following definitions.

Definition 1.1.9 Let X be a metric space and $\varepsilon > 0$. A set $S \subset X$ is called an *ε-net* if $\operatorname{dist}(x,S) \le \varepsilon$ for every $x \in X$.

Definition 1.1.10 Let X and Y be a metric spaces and $f : X \to Y$ an arbitrary map. The *distortion* of f is defined by

$$\operatorname{dis} f = \sup_{x_1,x_2 \in X} |d_Y(f(x_1), f(x_2)) - d_X(x_1,x_2)|$$

where d_X and d_Y are the metrics of X and Y, respectively.

With these definitions we may now define the concept of ε-isometry and then give a result in terms of ε-isometries which provides a quantity which differs from the Gromov-Hausdorff distance by no more than two times. An estimate of this type is sufficient to study the topology determined by the Gromov-Hausdorff distance.

Definition 1.1.11 Let X and Y be metric spaces and $\varepsilon > 0$. A map $f : X \to Y$ is called an *ε-isometry* if $\operatorname{dis} f \le \varepsilon$ and $f(X)$ is an ε-net in Y.

Note that f does not need to be continuous.

Now, let $f : X \to Y$ be an ε-isometry, we define an *approximate inverse* $f' : Y \to X$ of f as follows. Given $y \in Y$, choose $x \in X$ so that $d_Y(f(x),y)) \le \varepsilon$ and put $f'(y) = x$. Then f' is a 3ε-isometry from Y to X. Moreover, for all $x \in X$, $d_X(x,(f' \circ f)(x)) \le 2\varepsilon$, and for all $y \in Y$, $d_Y(y,(f \circ f')(y)) \le \varepsilon$.

Corollary 1.1.12 *Let X and Y be two metric spaces and $\varepsilon > 0$. Then*

1. *If $d_{GH}(X,Y) < \varepsilon$, there exists a 2ε-isometry from X to Y.*
2. *If there exists an ε-isometry from X to Y, $d_{GH}(X,Y) < 2\varepsilon$.*

Proof 1. Since $\frac{1}{2}\inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R}) = d_{GH}(X,Y) < \varepsilon$ we may take a correspondence $\mathcal{R}$ such that $\operatorname{dis}\mathcal{R} < 2\varepsilon$. For every $x \in X$, choose $f(x) \in Y$ such that $(x,f(x)) \in \mathcal{R}$. This defines a map $f : X \to Y$ and $\operatorname{dis} f \le \operatorname{dis}\mathcal{R} < 2\varepsilon$. It only remains to prove that $f(X)$ is a 2ε-net in Y. Given $y \in Y$, consider $x \in X$ such that $(x,y) \in \mathcal{R}$. Then, since (x,y), $(x,f(x)) \in \mathcal{R}$, we have $d(y,f(x)) \le d(x,x) + \operatorname{dis}\mathcal{R} < 2\varepsilon$. Hence $\operatorname{dist}(y,f(X)) < 2\varepsilon$.

2. Let f be an ε-isometry. Define $\mathcal{R} \subset X \times Y$ by $\mathcal{R} = \{(x,y) \in X \times Y \mid d(y,f(x)) \le \varepsilon\}$. Then $\mathcal{R}$ is a correspondence because $f(X)$ is an ε-net in Y. If (x,y), $(x',y') \in \mathcal{R}$, one has [2]

$$\begin{aligned} |d(y,y') - d(x,x')| &\le |d(f(x),f(x')) - d(x,x')| + d(y,f(x)) + d(y',f(x')) \\ &\le \operatorname{dis} f + \varepsilon + \varepsilon \le 3\varepsilon \ . \end{aligned}$$

[2]Start with

$$d(y,y') \le d(y,f(x)) + d(f(x),f(x')) + d(f(x'),y') \tag{1.1}$$

and

$$d(f(x),f(x')) \le d(y,f(x)) + d(y,y') + d(f(x'),y') \ . \tag{1.2}$$

Consequently, $\mathrm{dis}\mathcal{R} \leq 3\varepsilon$ thus, theorem 1.1.8 implies that $d_{GH}(X,Y) \leq \frac{3}{2}\varepsilon < 2\varepsilon$. ∎

We end this section by completing the proof that the Gromov-Hausdorff distance defines a finite metric on the space of isometry classes of compact metric spaces.

Theorem 1.1.13 *Gromov-Hausdorff distance is non-negative, symmetric and satisfies the triangle inequality; moreover $d_{GH}(X,Y) = 0$ if and only if X and Y are isometric.*

Proof We start by proving that if X and Y are bounded metric spaces then $d_{GH}(X,Y) < \infty$. Let $C = \max\{\mathrm{diam}X, \mathrm{diam}Y\}$ and define the following distance d on the disjoint union $X \cup Y$:

$$d(x,y) = \begin{cases} d_X(x,y) & \text{if } x,y \in X \\ d_Y(x,y) & \text{if } x,y \in Y \\ C & \text{in any other case} \end{cases}$$

The triangle inequality follows easily:

1. x,$y \in X$ and $z \in Y$, then
$$d(x,y) \leq C = d(x,z) \leq d(x,z) + d(z,y) \ .$$

2. $x \in X$ and y, $z \in Y$, then
$$d(x,y) = C = d(x,z) \leq d(x,z) + d(z,y) \ .$$

3. x, $z \in X$ and $y \in Y$, then
$$d(x,y) = C = d(x,z) \leq d(x,z) + d(z,y) \ .$$

Now, since $d_H(X,Y) = C$ when we consider X and Y as subspaces of their disjoint union with the metric d, it follows that $d_{GH}(X,Y) \leq C$.

It only remains to prove that $d_{GH}(X,Y) = 0$ implies that X and Y are isometric. Let X and Y be two compact metric spaces with $d_{GH}(X,Y) = 0$. By Corollary 1.1.12 there exist maps $f_n : X \to Y$ such that $\mathrm{dis} f_n \to 0$ when $n \to \infty$. Now, since a compact metric space is separable, we may take a countable dense set $S \subset X$. Then, using the Cantor diagonal procedure, we extract a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that for every $x \in S$ the sequence $\{f_{n_k}(x)\}$ converges in Y [3] For ease of notation we continue denoting the subsequence by $\{f_n\}$.

Then, (1.1) gives
$$d(y,y') - d(f(x),f(x')) \leq d(y,f(x)) + d(f(x'),y')$$
and (1.2)
$$d(f(x),f(x')) - d(y,y') \leq d(y,f(x)) + d(y',f(x')) \ .$$
Consequently,
$$|d(y',y) - d(f(x),f(x'))| \leq d(y,f(x)) + d(y',f(x'))$$
but $|a| - |b| \leq |a-b|$ so
$$|d(y,y') - d(x,x')| - |d(f(x),f(x')) - d(x,x')| \leq |d(y',y) - d(f(x),f(x'))| \leq d(y,f(x)) + d(y',f(x')) \ .$$

[3]Remember that compact and sequentially compact are equivalent in metric spaces.

Define a map $f : S \to Y$ as the limit of f_n, i.e. $f(x) = \lim_n f_n(x)$ for all $x \in S$. Since

$$|d(f_n(x), f_n(y)) - d(x, y)| \leq \text{dis} f_n \to 0$$

we have that

$$d(f(x), f(y)) = \lim_n d(f_n(x), f_n(y)) = d(x, y)$$

for all x, $y \in S$. In other words, f is a distance-preserving map from S to Y, and it can be extended to a distance preserving map from X to Y in the usual way. Similarly, there is a distance preserving map from Y to X, $g : Y \to X$.

Now, $f \circ g : Y \to Y$ is a distance-preserving map from Y to itself, but Y is compact so $f \circ g$ is bijective [4] and therefore f is surjective. This proves that f is an isometry.

■

1.2 Gromov-Hausdorff Convergence

In this section we consider converging sequences in the Gromov-Hausdorff space of compact metric spaces. Since d_{GH} is a metric, the limit is unique up to an isometry.

Corollary 1.1.12 gives a criterion for convergence in the Gromov-Hausdorff topology. Namely, a sequence $\{X_n\}$ of metric spaces converges to a metric space X if and only if there exist a sequence of numbers $\{\varepsilon_n\}$ and a sequence of maps $f_n : X_n \to X$ (or, $f_n : X \to X_n$) such that f_n is an ε_n-isometry and $\varepsilon_n \to 0$.

The following generalisation of the Arzelà-Ascoli theorem will be useful.

Lemma 1.2.1 *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of compact metric spaces converging to X in the Gromov-Hausdorff topology, with ε_i-isometries $f_i : X_i \to X$. Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of compact metric spaces converging to Y in the Gromov-Hausdorff topology, with ε_i-isometries $g_i : Y_i \to Y$. For each i, let $f_i' : X \to X_i$ be an approximate inverse to f_i. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of maps $\alpha_i : X_i \to Y_i$ that are asymptotically equicontinuous in the sense that for every $\varepsilon > 0$, there are $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon) \in \mathbb{Z}^+$ so that for all $i \geq N$,*

$$d_{X_i}(x_i, x_i') < \delta \quad \Rightarrow \quad d_{Y_i}(\alpha_i(x_i), \alpha_i(x_i')) < \varepsilon.$$

Then after passing to a subsequence, the maps $g_i \circ \alpha_i \circ f_i' : X \to Y$ converge uniformly to a continuous map $\alpha : X \to Y$.

Proof Can be found in ([14], page 66) and ([15], App. A). ■

An important point is that finite spaces form a dense set in the Gromov-Hausdorff space.

Example 1.2.2 Every compact metric space X is a limit of finite spaces. For this, take a sequence $\varepsilon_n \to 0$ of positive numbers and choose a finite ε_n-net S_n in X for every n. Then $S_n \xrightarrow{GH} X$, simply because $d_{GH}(X, S_n) \leq d_H(X, S_n) \leq \varepsilon_n$.

Furthermore, taking appropriate ε-nets one can essentially reduce convergence of arbitrary compact metric spaces to convergence of their finite subsets.

[4] A compact metric space cannot be isometric to a proper subset of itself (Theorem 1.6.14 in [5]).

Definition 1.2.3 Let X and Y be two compact metric spaces, and ε, $\delta > 0$. We say that X and Y are (ε, δ)-approximations of each other if there exist finite collections of points $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$ in X and Y, respectively, such that:

1. The set $\{x_i \mid 1 \leq i \leq N\}$ is an ε-net in X, and $\{y_i \mid 1 \leq i \leq N\}$ is an ε-net in Y.
2. $|d_X(x_i, x_j) - d_Y(y_i, y_j)| < \delta$ for all i, $j \in \{1, \ldots, N\}$.

An *ε-approximation* is an $(\varepsilon, \varepsilon)$-approximation.

We now can give a criterion for convergence in terms of ε-approximations. Namely, $X_n \xrightarrow{GH} X$ if and only if, for any $\varepsilon > 0$, X_n is an ε-approximation of X for all sufficiently large enough n.

Proposition 1.2.4 *Let X and Y be compact metric spaces.*

1. *If Y is an (ε, δ)-approximation of X, then $d_{GH}(X, Y) < 2\varepsilon + \delta$.*
2. *If $d_{GH}(X, Y) < \varepsilon$, then Y is a 5ε-approximation of X.*

Proof 1. Let $X_0 = \{x_i\}_{i=1}^N$ and $Y_0 = \{y_i\}_{i=1}^N$ as in the definition. The second condition means that the correspondence $\{(x_i, y_i) \mid 1 \leq i \leq N\}$ between X_0 and Y_0 has distortion less than δ, so $d_{GH}(X_0, Y_0) < \delta/2$. Since X_0 and Y_0 are ε-nets in X and Y, respectively, we have $d_{GH}(X, X_0) \leq \varepsilon$ and $d_{GH}(Y, Y_0) \leq \varepsilon$. Hence the result follows by the triangle inequality for d_{GH}.

2. Corollary 1.1.12 gives us the existence of a 2ε-isometry $f : X \to Y$. Let $X_0 = \{x_i\}_{i=1}^N$ be an ε-net in X and $y_i = f(x_i)$. Then $|d(x_i, x_j) - d(y_i, y_j)| < 2\varepsilon < 5\varepsilon$ for all i, j. We must now see that $Y_0 = \{y_i\}_{i=1}^N$ is a 5ε-net in Y. Let $y \in Y$, since $f(X)$ is an 2ε-net in Y, there exists $x \in X$ such that $d(y, f(x)) \leq 2\varepsilon$. But X_0 is an ε-net in X so there exists $x_i \in X_0$ such that $d(x, x_i) \leq \varepsilon$. Then

$$\begin{aligned} d(y, y_i) = d(y, f(x_i)) &\leq d(y, f(x)) + d(f(x), f(x_i)) \\ &\leq d(y, f(x)) + \mathrm{dis} f + d(x, x_i) \leq 2\varepsilon + \varepsilon + 2\varepsilon \leq 5\varepsilon \end{aligned}$$

thus $d(y, Y_0) \leq 5\varepsilon$. ■

This result can be given in a more elegant formulation:

Proposition 1.2.5 *For compact metric spaces X and $\{X_n\}_{n=1}^\infty$, $X_n \xrightarrow{GH} X$ if and only if the following holds. For every $\varepsilon > 0$ there exist a finite ε-net S in X and an ε-net S_n in each X_n such that $S_n \xrightarrow{GH} S$.*

Moreover, these ε-nets can be chosen so that, for all sufficiently large n, S_n have the same cardinality as S.

Proof $\boxed{\leftarrow}$ If such ε-nets exist, then X_n is an ε-approximation of X for all sufficiently large n. Then $X_n \xrightarrow{GH} X$ by the previous proposition. Note that this also follows from the triangle inequality:

$$d_{GH}(X_n, X) \leq d_{GH}(X_n, S_n) + d_{GH}(S_n, S) + d_{GH}(S, X) \ .$$

$\boxed{\rightarrow}$ Take a finite $(\varepsilon/2)$-net S in X and construct corresponding nets S_n in X_n. Namely, pick a sequence of ε_n-isometries $f_n : X \to X_n$ where $\varepsilon_n \to 0$ and define $S_n = f_n(S)$. Then $S_n \xrightarrow{GH} S$ and, as in the previous proposition, S_n is an ε-net in X_n for all large enough n. Furthermore, for all sufficiently large n, we have that $\mathrm{dis}(f_n) < \min\{d(x, y) \mid x, y \in S\}$ thus f_n is injective on S for all sufficiently large n and therefore $\#S_n = \#S$. ■

1.2.1 Length spaces

This last proposition has important consequences. It opens a way to prove continuity statements about the Gromov-Hausdorff space. Namely, if some property of spaces X_n can be formulated in terms of finite collections of points, then this property is inherited by the limit space X. As an example we have the property of a metric to be intrinsic, which can be expressed in terms of triples of points . It follows that a limit of compact length spaces is a length space ([5], Theorem 2.4.16). We recall that for us a length space is a metric space (X, d) in which the distance between two points equals the infimum of the lengths of curves joining the points.

1.2.2 Compactness theorem

As we said earlier on, the Gromov-Hausdorff topology is a relatively weak one, so we expect that it has 'many' compact sets. We have seen that a sequence $\{X_n\}$ converging in the Gromov-Haudorff space must contain ε-nets of uniformly bounded cardinality. It follows that, if a class $\mathfrak{X}$ of metric spaces is pre-compact in the Gromov-Hausdorff topology, then for every $\varepsilon > 0$ the size of a minimal ε-net is uniformly bounded over all elements of $\mathfrak{X}$. It turns out that this property of $\mathfrak{X}$, along with the fact that the diameters are uniformly bounded, is sufficient for pre-compactness.

Definition 1.2.6 We say that a class $\mathfrak{X}$ of compact metric spaces is *uniformly totally bounded* if

1. There is a constant D such that $\text{diam}X \leq D$ for all $X \in \mathfrak{X}$.
2. For every $\varepsilon > 0$ there exists a natural number $N = N(\varepsilon)$ such that every $X \in \mathfrak{X}$ contains an ε-net consisting of no more than N points.

Theorem 1.2.7 *A class $\mathfrak{X}$ of compact metric spaces is pre-compact in the Gromov-Hausdorff topology if, and only if, it is uniformly totally bounded.*

Proof Let D and $N(\varepsilon)$ be as in the previous definition. Define $N_1 = N(1)$ and $N_k = N_{k-1} + N(1/k)$ for all $k \geq 2$. Now, in each space X_n we construct a countable dense collection S_n as follows. Consider a $(1/k)$-net of $N(1/k)$ points in X_n for each k and then take S_n to be the union of all. Then S_n is obviously a countable dense collection in X_n, $S_n = \{x_{i,n}\}_{i=1}^{\infty} \subset X_n$, where $\{x_{i,n}\}_{i=N_{k-1}}^{N_k}$ is the $(1/k)$-net taken, and therefore $\{x_{i,n}\}_{i=1}^{N_k}$ is also a $(1/k)$-net in X_n.

Since all the X_n have diameter bounded above by D, the distances $|x_{i,n} - x_{j,n}|$ are $\leq D$, and therefore belong to a compact interval. Consequently, using the Cantor diagonal procedure, we can extract a subsequence of $\{X_n\}$ in which $\{|x_{i,n} - x_{j,n}|\}_{n=1}^{\infty}$ converge for all i, j. To simplify the notation, we assume that they converge without passing to a subsequence. So we are assuming that the distances between points in our countable dense collections S_n converge.

We will now construct the limit space $\overline{X}$ for $\{X_n\}$. This will be done by defining a metric space that contains a countable dense subset with distances between points equal to the limit of the distances between the points of S_n. For this, pick an abstract countable set (for example, the set of positive integers) $X = \{x_i\}_{i=1}^{\infty}$ and define a semi-metric d on X by

$$d(x_i, x_j) = \lim_{n\to\infty} |x_{n,i} - x_{n,j}| \ .$$

Then, a usual quotient construction where points x, y with $d(x,y)=0$ are taken to be equivalent, gives us a metric space X/d. We will denote by $\overline{x}_i$ the equivalence class of x_i in X/d. This quotient space may not be complete, so let $\overline{X}$ be the completion of X/d ([5], Theorem 1.5.10). We will prove that $\{X_n\}$ converges to $\overline{X}$.

The last part consists in proving that $\overline{X}$ is compact and that the $(1/k)$-nets in the X_n converge to $(1/k)$-nets in $\overline{X}$, and then use Proposition 1.2.5. For $k \in \mathbb{N}$, consider the set $S^{(k)} = \{\overline{x}_i \mid 1 \leq i \leq N_k\} \subset \overline{X}$. It is a $(1/k)$-net in $\overline{X}$. Indeed, every set $S_n^{(k)} = \{x_{i,n} \mid 1 \leq i \leq N_k\}$ is a $(1/k)$-net in X_n. Consequently, for every $x_{i,n} \in S_n$ there is a $j \leq N_k$ such that $|x_{i,n} - x_{j,n}| \leq 1/k$. But the N_k do not depend on n so for every fixed $i \in N$, given $n \in \mathbb{N}$, we find a $j \leq N_k$ so that $|x_{i,n} - x_{j,n}| \leq 1/k$. However, there are only a finite number of $j \leq N_k$ so, for each i, there is a $j \leq N_k$ such that $|x_{i,n} - x_{j,n}| \leq 1/k$ for infinitely many indices n. Passing to the limit we obtain that $|\overline{x}_i - \overline{x}_j| \leq 1/k$ for this j. Therefore, $S^{(k)}$ is a $(1/k)$-net in X/d and hence in $\overline{X}$. Since $\overline{X}$ is complete and has a $(1/k)$-net for any $k \in N$, it is compact ([5], Theorem 1.6.5).

Furthermore, the set $S^{(k)}$ is a Gromov-Hausdorff limit of the sets $S_n^{(k)}$ as $n \to \infty$, because these are finite sets consisting of N_k points (some of which may coincide) and the distances converge. Thus for every $k \in N$ we have a $(1/k)$-net in $\overline{X}$ which is a Gromov-Hausdorff limit of some $(1/k)$-nets in the spaces X_n. By Proposition 1.2.5 it follows that $X_n \to \overline{X}$. ■

1.3 Measured Gromov-Hausdorff convergence

For this work it will be necessary to consider not only compact metric spaces, but measured compact metric spaces. This is why we need a notion of convergence in this context.

For our purposes, we will assume that the ε-isometries f and their approximate inverses f' are Borel. Let $P(X)$ denote the space of Borel probability measures on X. We consider $P(X)$ with the weak-$\star$ topology, i.e. $\lim_{i\to\infty} \mu_i = \mu$ if and only if for all $F \in C(X)$, $\lim_{i\to\infty} \int_X F d\mu_i = \int_X F d\mu$.

Definition 1.3.1 Given $\mu \in M(X)$, consider the metric-measure space (X,d,ν). A sequence $\{(X_i,d_i,\nu_i)\}_{i=\infty}^{\infty}$ converges to (X,d,ν) in the measured Gromov-Hausdorff topology if there are ε_i-isometries $f_i : X_i \to X$, with $\lim_{i\to\infty} \varepsilon_i = 0$, so that $\lim_{i\to\infty}(f_i)_\star \nu_i = \nu$ in the weak topology of measures.

In this context we have the following compactness theorem.

Theorem 1.3.2 *Let $C>0$, $D>0$ and $0<m\leq M$ be finite positive constants, and let $\mathcal{F}$ be a family of compact metric-measure spaces, such that for each $(X,d,\nu) \in \mathcal{F}$*

1. *the diameter of (X,d) is bounded above by D;*
2. *the measure ν has a doubling constant bounded above by C* [5] *;*
3. *$m \leq \nu[X] \leq M$.*

Then $\mathcal{F}$ is precompact in the measured Gromov-Hausdorff topology.

[5] $\nu(B_{2r}(x)) \leq C\nu(B_r(x))$ for all $x \in X$ and $r > 0$.

The proof of this theorem follows from our previous compactness theorem and the following propositions.

Proposition 1.3.3 (Prokhorov's theorem in Gromov-Hausdorff converging sequences) *Let $(X_k)_{k\in\mathbb{N}}$ be a sequence of compact metric spaces, converging in the Gromov-Hausdorff topology to some compact metric space X, by means of ε_k-isometries $f_k : X_k \to X$. For each k, let μ_k be a probability measure on X_k. Then, after extraction of a subsequence, $(f_k)_\#\mu_k$ converges in the weak topology to a probability measure μ on X as $k \to \infty$.*

Proposition 1.3.4 (Doubling implies uniform total boundedness) *Let (X, d) be a Polish space [6] with diameter bounded above by D, equipped with a finite (non-zero) C-doubling measure ν [7]. Then for any $\varepsilon > 0$ there is a number $N = N(\varepsilon)$, only depending on D, C and ε, such that X can be covered with N balls of radius ε.*

Proof Without loss of generality, we can assume that $\nu[X] = 1$ [8]. Let $r = \varepsilon/2$, and take n such that $D \leq 2^n r$. Choose an arbitrary point $x_1 \in X$, then a point $x_2 \in X \setminus (B_{2r}(x_1))$, a point x_3 in $X \setminus (B_{2r}(x_1) \cup B_{2r}(x_2))$, and so forth. All the balls $B_r(x_j)$ are disjoint, and by the doubling property each of them has a measure at least C^{-n} [9]. So $X \setminus (B_r(x_0) \cup \cdots \cup B_r(x_k))$ has measure at most $1 - kC^{-n}$, and therefore C^n is an upper bound on the number of points x_j that can be chosen.

Now let $x \in X$. There is at least one index j such that $d(x, x_j) < 2r$ (otherwise x would lie in the complement of the union of all the balls $B_{2r}(x_j)$, and could be added to the family $\{x_j\}$). Consequently, $\{x_j\}$ is a $2r$-net in X, with cardinality at most $N = C^n$. This concludes the proof. ■

[6] A polish space is a separable completely metrizable topological space.

[7] $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all $x \in X$ and $r > 0$.

[8] We could just take $d\tilde{\nu} = \frac{d\nu}{\nu[X]}$.

[9] $\nu(B_{2r}(x)) \leq C\nu(B_r(x))$ thus

$$\nu(B_r(x)) \geq \frac{\nu(B_{2r}(x))}{C} \geq \cdots \geq \frac{\nu(B_{2^n r}(x))}{C^n} = \frac{1}{C^n}$$

since $D \leq 2^n r$ so $B_{2^n r}(x) = X$.

Chapter 2

Bishop-Gromov's theorem

In this section we present the Bishop-Gunther-Gromov inequality relating curvature and volume. We will assume that the results given in the basic Differential Geometry course of the masters degree are known. In fact, we will refer to the notes of this course if necessary.

2.1 Jacobi fields

This first section is devoted to Jacobi fields. We start by proving the uniqueness of Jacobi fields with given initial conditions and an observation relating Jacobi fields and geodesic variations. We will not prove this relation in the case $J(0) \neq 0$ but a similar construction may be done. For ease of notation we will consider geodesics defined on intervals of the form $[0, a]$ but, obviously, this is no restriction.

We recall that a Jacobi field along a geodesic γ is a vector field satisfying the Jacobi equation

$$\frac{D^2}{dt^2} J(t) + R(J(t), \gamma'(t))\gamma'(t) = 0,$$

where D denotes the covariant derivative with respect to the Levi-Civita connection, R the Riemann curvature tensor [1], $\gamma'(t)$ the tangent vector field, and t is the parameter of the geodesic.

Proposition 2.1.1 *A Jacobi field J, along a geodesic $\gamma : [0, a] \to M$ is determined by its initial conditions $J(0)$ and $\frac{DJ}{dt}(0)$.*

Proof Let $E_1(t), \ldots, E_n(t)$ be a parallel orthonormal frame along γ which can be obtained by choosing an orthonormal basis of $T_{\gamma(0)}M$ and then taking the parallel transport along γ.

We may write J as a combination of the vector fields $E_1, \ldots, E_n$,

$$J(t) = \sum_{i=1}^{n} f_i(t) E_i(t) \ .$$

[1] We take

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \ .$$

Denote
$$a_{ij}(t) = \langle R(\gamma'(t), E_i(t))\gamma'(t), E_j(t)\rangle$$
for $i, j = 1, \ldots, n = \dim M$. Then, since the E_i are parallel vector fields we have that $\dfrac{D^2 J}{dt^2} = \sum_{i=1}^{n} f_i''(t)E_i(t)$ and

$$R(\gamma', J)\gamma' = \sum_{j=1}^{n}\langle R(\gamma', J)\gamma', E_j\rangle E_j = \sum_{j=1}^{n}\langle R(\gamma', \sum_{i=1}^{n} f_i(t)E_i(t))\gamma', E_j\rangle E_j =$$
$$= \sum_{ij} f_i\langle R(\gamma', E_i)\gamma', E_j\rangle E_j = \sum_{ij} f_i a_{ij} E_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} f_i a_{ij}\right) E_j.$$

Therefore, the Jacobi equation $\dfrac{D^2 J}{dt^2} + R(\gamma', J)\gamma' = 0$ is equivalent to the system

$$f_j''(t) + \sum_{i=1}^{n} a_{ij}(t) f_i(t) = 0, \quad j = 1, \ldots, n$$

which is a linear system of second order. Thus, given the initial conditions there exists a unique solution. ∎

In fact, given $J'(0)$ and supposing that $J(0) = 0$ we can construct the geodesic variation which has as its variation field the unique Jacobi field satisfying these initial conditions. The same can be done if $J(0) \neq 0$ but it is not necessary for our purposes.

Proposition 2.1.2 *Let $\gamma : [0, a] \to M$ be a geodesic and J a Jacobi field along γ with $J(0) = 0$. Put $\omega = \dfrac{DJ}{dt}(0)$ and $v = \gamma'(0)$. Considering ω as an element of $T_v(T_{\gamma(0)}M)$ construct a curve $v(s)$ in $T_{\gamma(0)}M$ with $v(0) = v$ and $v'(0) = \omega$. Take $f(t, s) = \exp_p(tv(s))$ where $p = \gamma(0)$, and define a Jacobi field $\overline{J}$ by $\overline{J}(t) = \frac{\partial f}{\partial s}(t, 0)$. Then $\overline{J} = J$ on $[0, a]$.*

Proof Thanks to Proposition 4.2.5 on page 73 of the notes [11] we know that $\overline{J}$, defined in this way, is a Jacobi field along γ so, following the previous proposition, we only need to show that the initial conditions coincide.

Now, at $s = 0$ we have

$$\frac{D}{dt}\frac{\partial f}{\partial s} = \frac{D}{dt}\frac{\partial}{\partial s}(\exp_p(tv(s)) = \frac{D}{dt}((d\exp_p)_{tv}(t\omega)) = \frac{D}{dt}(t(d\exp_p)_{tv}(\omega)) =$$
$$= (d\exp_p)_{tv}(\omega) + t\frac{D}{\partial t}((d\exp_p)_{tv}(\omega))$$

which at $t = 0$ gives

$$\frac{D\overline{J}}{dt}(0) = \frac{D}{dt}\frac{\partial f}{\partial s}(0, 0) = (d\exp_p)_0(\omega) = id_{T_pM}(\omega) = \omega\ .$$

∎

Corollary 2.1.3 *Given a geodesic $\gamma : [0, a] \to M$ and a Jacobi field J along γ with $J(0) = 0$ we have that*

$$J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0)), \qquad t \in [0, a].$$

2.2 Conjugate points

Definition Given a geodesic $\gamma : [0, a] \to M$ and $t_0 \in (0, a]$, the point $\gamma(t_0)$ is said to be *conjugate* to $\gamma(0)$ along γ if there exists a non identically cero Jacobi field along γ with $J(0) = 0 = J(t_0)$. The maximum number of such linearly independent Jacobi fields is called the *multiplicity* of the conjugate point $\gamma(t_0)$.

The next proposition relates the conjugate points to p along a geodesic with the critical points of $\exp_p$.

Proposition 2.2.1 *Let $\gamma : [0, a] \to M$ be a geodesic and put $\gamma(0) = p$. For $t_0 \in (0, a]$, the point $q = \gamma(t_0)$ is conjugate to p along γ if and only if $v_0 = t_0\gamma'(0)$ is a critical point of $\exp_p$. In addition, the multiplicity of q as a conjugate point of p is equal to the dimension of the kernel of $(d\exp_p)_{v_0}$.*

Proof $\boxed{\Rightarrow}$ Suppose that q is conjugate to p along γ. Then, there exists a non-identically zero Jacobi field J along γ with $J(0) = 0 = J(t_0)$. Thanks to Corollary 2.1.3, we know that $J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0))$ for $t \in [0, a]$. Then, since J is non-zero we get that $J'(0) \neq 0$ so

$$0 = J(t_0) = (d\exp_p)_{t_0\gamma'(0)}(t_0J'(0))$$

with $J'(0) \neq 0$, that is, $t_0\gamma'(0)$ is a critical point of $\exp_p$.

$\boxed{\Leftarrow}$ Conversely, suppose that $v_0 = t_0\gamma'(0)$ is a critical point of $\exp_p$. Then, there exists $\omega \in T_{v_0}(T_pM)$, $\omega \neq 0$, such that

$$(d\exp_p)_{t_0\gamma'(0)}(t_0\omega) = 0 \ .$$

Now, following the construction of Proposition 2.1, $J(t) = (d\exp_p)_{t\gamma'(0)}(t\omega)$ is a Jacobi field along γ and J is non-zero because $J'(0) = \omega \neq 0$. Then, q is conjugate to p because

$$J(t_0) = (d\exp_p)_{t_0\gamma'(0)}(t_0J'(0)) = 0 \ .$$

For the last statement note that Jacobi fields $J_1, \ldots, J_k$ along γ with $J_i(\gamma(0)) = 0$ are linearly independent if and only if $J_1'(0), \ldots, J_k'(0)$ are linearly independent in T_pM. Then, following the construction above we get that the multiplicity of q is equal to the kernel of $(d\exp_p)_{t_0\gamma'(0)}$. ■

From this it follows that if a point $p = \gamma(0)$ has no conjugate points along the geodesic $\gamma : [0, a] \to M$ then $\exp_p$ is a diffeomorphism on a neighbourhood of each point of the form $t\gamma'(0)$, $t \in [0, a]$.

2.3 Index Form

In proposition 4.2.8 on page 76 of the notes [11] we find the second variation formula:

Proposition 2.3.1 *If V is the variation field associated to a proper variation of a geodesic $\gamma : [0, 1] \to M$ then*

$$E''(0) = -2\int_0^1 \left\langle V \ , \ \frac{D^2V}{dt^2} + R(V, \gamma')\gamma' \right\rangle dt \ .$$

Following the proof we notice that we may rewrite this formula as

$$E''(0) = -2\int_0^1 \left(\langle V \ , \ R(\gamma', V)\gamma' \rangle - \left\langle \frac{DV}{dt} \ , \ \frac{DV}{dt} \right\rangle \right) dt$$

and this serves as motivation for the next definition.
Definition The *index form* of a geodesic γ is

$$I(X,Y) = \int_a^b \left(\left\langle \frac{DX}{dt}, \frac{DY}{dt} \right\rangle - R(\gamma', X, \gamma', Y) \right) dt$$

where X,Y are two vector fields along γ.

Then, for a proper variation of a geodesic γ with variation field V we have

$$E''(0) = 2I(V,V) \ .$$

Note that after an integration by parts computation, we have

$$I(X,Y) = -\int_a^b \left\langle \frac{D^2X}{Dt^2} + R(\gamma', X)\gamma', Y \right\rangle dt + \left[\left\langle \frac{DX}{dt} \ , \ Y \right\rangle \right]_a^b \ . \tag{2.1}$$

Therefore, for a jacobi field J along $\gamma : [a,b] \to M$ we have

$$I(J,J) = \langle J'(b), J(b) \rangle - \langle J'(a), J(a) \rangle \ . \tag{2.2}$$

We also obtain the next lemma.

Lemma 2.3.2 *(Index lemma) Suppose that $p = \gamma(a)$ has no conjugate point along a geodesic γ. If X is a Jacobi field along γ, and Y is a vector field along γ satisfying $Y(a) = X(a)$ and $Y(b) = X(b)$, then $I(X,X) \leq I(Y,Y)$. Further, equality holds if, and only if, $X = Y$.*

The proof of the lemma may be found on [10] (Chapter 10, Lemma 2.2). However, we will now provide a different proof using the next theorem due to Jacobi which we will not prove.

Theorem 2.3.3 *(Jacobi) Let $\gamma : [a,b] \to M$ be a geodesic. Let $p = \gamma(a)$ and $q = \gamma(b)$, then if there are no conjugate points of p along γ there exists $\varepsilon > 0$ so that for any piecewise smooth curve $\overline{\gamma} : [a,b] \to M$ from p to q satisfying $\mathrm{dist}(\gamma(t), \overline{\gamma}(t)) < \varepsilon$, we have $L(\overline{\gamma}) \geq L(\gamma)$, with equality if and only if $\overline{\gamma}$ is a reparameterization of γ.*

Proof The proof uses the fact, mentioned above, that, since p has no conjugate points along γ then $\exp_p$ is a diffeomorphism at each point $t\gamma'(0)$, with $t \in [0, b-a]$, and hence in a neighbourhood of these points. ■

Thus, for a geodesic $\gamma : [a,b] \to M$ such that $\gamma(a)$ has no conjugate points along γ we have that γ is locally length minimizing. Therefore, for any proper variation of γ, we have $E'(0) = 0$ and $E''(0) \geq 0$. In other words, for any vector field X along γ with $X(a) = 0 = X(b)$ we have $I(X,X) \geq 0$. This is the same reasoning as the one used in the proof of Theorem 4.2.9 (Bonnet-Myers) in the notes [11], the difference being that in this case the geodesic is locally minimizing but not necessarily globally minimizing.

We also need two more theorems to prove the last assertion of the lemma. Put $\mathcal{V} = \{X \ : \ X \text{ is a vector field along } \gamma\}$ and

$$\mathcal{V}^0 = \{X \ : \ X \in \mathcal{V} \text{ with } X(a) = 0 = X(b)\} \ .$$

Theorem 2.3.4 *Let $X \in \mathcal{V}$, then X is a Jacobi field along γ if, and only if, $I(X,Y) = 0$ for every $Y \in \mathcal{V}^0$.*

Proof From equation (2.1) we get that if X is a Jacobi field then $I(X,Y) = 0$ for any $Y \in \mathcal{V}^0$.

Conversely, if X is a vector field along γ satisfying $I(X,Y) = 0$ for every $Y \in \mathcal{V}^0$ then taking a smooth function $f : [a,b] \to \mathbb{R}$ with $f(a) = 0 = f(b)$ and $f(t) > 0$ for $t \in (a,b)$ we define

$$Y = f(t)\left(\frac{D^2X}{dt^2} + R(\gamma', X)\gamma'\right) .$$

We have $Y \in \mathcal{V}^0$ and therefore

$$0 = I(X,Y) = -\int_a^b f(t)\left|\frac{D^2X}{dt^2} + R(\gamma', X)\gamma'\right|^2 dt .$$

It follows that X is a Jacobi field as required. ■

Theorem 2.3.5 *Let $\gamma : [a,b] \to M$ be a geodesic, $p = \gamma(a)$ and $q = \gamma(b)$. Then, if p has no conjugate point along γ the index form I is positive definite on $\mathcal{V}^0$.*

Proof We have already discussed that $I(X,X) \geq 0$ for any $X \in \mathcal{V}^0$ so if I is not positive definite on $\mathcal{V}^0$ then there exists $Y \in \mathcal{V}^0$ not identically zero such that $I(Y,Y) = 0$. Then, for any $Z \in \mathcal{V}^0$ and any $\lambda \in \mathbb{R}$, we have

$$0 \leq I(Y - \lambda Z, Y - \lambda Z) = -2\lambda I(Y,Z) + \lambda^2 I(Z,Z) .$$

But this is true for all $Z \in \mathcal{V}^0$ and $\lambda \in \mathbb{R}$ so we must have $I(Y,Z) = 0$ for all $Z \in \mathcal{V}^0$. Therefore, by the previous theorem, Y is a Jacobi field with $Y(a) = 0 = Y(b)$ but $\gamma(b)$ is not a conjugate point of p so we must have $Y = 0$ which is a contradiction. ■

Proof (of the Index lemma) Since X is a Jacobi field and Y is a vector field with the same values at the boundary, it follows from equation (2.1) that $I(X,X) = I(X,Y)$. Now, by our previous argument

$$0 \leq I(X-Y, X-Y) = I(X,X) - 2I(X,Y) + I(Y,Y) = -I(X,X) + I(Y,Y)$$

and we are done. Now, equality holds if, and only if, $I(X-Y, X-Y) = 0$ and the previous theorem yields $X - Y = 0$. ■

2.4 Jacobi Fields on Spaces with Constant Sectional Curvature

Let (M,g) be a Riemannian manifold with constant sectional curvature k. Then the curvature tensor is given by

$$R(X,Y)Z = k(\langle X,Z\rangle Y - \langle Y,Z\rangle X) .$$

This may be found in Chapter 4, Lemma 3.4 of [10].

With this in mind we can easily obtain the Jacobi fields along geodesics in manifolds with constant sectional curvature. Remember that a Jacobi field along γ is called a *normal Jacobi field* if it is perpendicular to γ' along γ. In general, these are the Jacobi fields that we are interested in.[2]

In this context, if γ is a geodesic parametrized by arc-length, then the Jacobi equation for a normal Jacobi field J along γ is

$$\frac{D^2 J}{dt^2} + kJ = 0 \ .$$

Take a parallel orthonormal frame $\{E_i(t)\}$ along γ with $E_1(t) = \gamma'(t)$, and write $J(t) = \sum_{i=2}^{m} J^i(t)E_i(t)$, then the equation of the coefficient $J^i(t)$ is

$$(J^i)''(t) + kJ^i(t) = 0, \quad 2 \leq i \leq n. \tag{2.3}$$

If $J(0) = 0$ the solution of this equation is

$$J^i(t) = \begin{cases} c^i \sin(\sqrt{k}t), & \text{if } k > 0; \\ c^i t, & \text{if } k = 0; \\ c^i \sinh(\sqrt{-k}t), & \text{if } k > 0; \end{cases}$$

where c^i are constants.

2.5 Cut locus

In this section we consider (M, g) a *complete* Riemannian manifold. For $p \in M$ and $v \in T_p(M)$, denote γ_v the unique geodesic satisfying $\gamma_v(0) = p$ and $\gamma_v'(0) = v$, in other words, $\gamma_v(t) = \exp_p(tv)$. Let

$$I_v = \{t \in \mathbb{R} \ : \ \gamma_v \text{ is minimal on } [0, t]\}.$$

Now, we know that any geodesic is locally minimal and that if a geodesic is minimal on an interval I, then it is also minimal on any subinterval $J \subset I$. Therefore, our interval I_v is closed. Let $I_v = [0, \rho(v)]$ where $\rho(v)$ may be infinite. We call $\gamma_v(\rho(v))$ the *cut point* of $\gamma(0) = p$ along γ and denote by $\mathrm{Cut}(p)$ the set of all cut points of p along all geodesics that start from p, and call it the *cut locus* of p.

If $\omega = \lambda v$, then $\rho(v) = \lambda\rho(\omega)$ so we can restrict the study of the map ρ to the unit bundle of M. We will denote by $S_pM \subset T_pM$ the set of unit vectors in T_pM, and by $SM \subset TM$ the set

$$\{(p, v) \in TM \ : \ \langle v, v\rangle_p = 1\} \ .$$

We will denote by θ the unit vectors.

Then, the function $\rho : SM \to \mathbb{R} \cup \{\infty\}$ which we have just defined as:

$$\rho(p, \theta) = \rho(\gamma(0), \gamma_\theta'(0)) = \begin{cases} t_0, & \text{if } \gamma_\theta(t_0) \text{ is the cut point of } \gamma(0) \text{ along } \gamma, \\ \infty, & \text{if the cut point along } \gamma \text{ does not exist.} \end{cases}$$

[2]The Jacobi fields γ' and $t\gamma'$ will not be important in our developments.

is continuous. A proof is found in Chapter 13, Proposition 2.9 of [10]. It can also be proved ([12], Proposition 2.113) that

$$M = \exp_p(U(p)) \cup \mathrm{Cut}(p) \ .$$

Furthermore, in ([10], Chap.13 Proposition 2.2), we find that

Proposition 2.5.1 *If $\gamma(t_0)$ is the cut point of $p = \gamma(0)$ along γ, then one of the following is true:*

- *$\gamma(t_0)$ is the first conjugate point of $p = \gamma(0)$ along γ,*
- *there exists a geodesic $\sigma \neq \gamma$ from p to $\gamma(t_0)$ such that $l(\sigma) = l(\gamma)$.*

  Conversely, if either of the previous is satisfied, then there exists $\bar{t}$ in $(0, t_0]$ such that $\gamma(\bar{t})$ is the cut point of p along γ.

Then, if we define $U(p) = \{tv \mid v \in S_pM,\ 0 \leq t \leq \rho(v)\}$, thanks to this result and previous results concerning the relation between conjugate points and singularities of the exponential map we have that $\exp_p : U(p) \to M \setminus \mathrm{Cut}(p)$ is a diffeomorphism.

Further, since $U(p)$ is an open star-shaped domain in T_pM, we can take $(\exp_p^{-1}, V(p) = M \setminus Cut(p))\}$ as a coordinate chart on $V(p)$. Then, we can fix an orthonormal basis $\{e_i\}$ in $U(p)$, and denote the corresponding coordinate functions on $V(p)$ by $\{u^i\}$.

Definition 2.5.2 The local chart $\{V(p); u^1, \ldots, u^n\}$ is called *normal coordinate system* at p.

2.6 Riemannian volume

Let (M, g) be an n-dimensional Riemannian manifold and $\{(U_k, \varphi_k)\}$ an atlas. We define the *canonical measure* of (M, g), and denote it v_g, as the measure corresponding to the density given in each chart (U_k, φ_k) of our atlas by

$$\mu = \sqrt{\det(g_{ij}) \circ \varphi_k^{-1}} L_n,$$

where $L_n = dx^1 \cdots dx^n$ is the Lebesgue measure in $\mathbb{R}^n$. It can be proved that this is well defined (see for example Chapter 1, section 2 in [10]).

Lemma 2.6.1 *For any $p \in M$, the cut-locus Cut_p has measure zero.*

Proof Note that $\mathrm{Cut}(p) = \exp_p(\partial U(p))$ and, since every ray from the origin in T_pM cuts $\partial U(p)$ once at most, we have that $\partial U(p)$ has measure zero. Therefore, $\mathrm{Cut}(p) = \exp_p(\partial U(p))$ has measure zero. ■

If M is orientable, and we take an atlas compatible with the orientation, then v_g can be given by a volume form. We define the *volume* of a Riemannian manifold as the integral $\int_M v_g$.

For example, if $R \subset M$ is a region whose closure is compact and R is contained in a coordinate neighbourhood $x(U)$ with a parametrization $x : U \to M$, and the boundary of $x^{-1}(R) \subset U$ has measure zero in $\mathbb{R}^n$ we have that the volume $\mathrm{Vol}(R)$ of R is

$$\mathrm{Vol}(R) = \int_{x^{-1}(R)} \sqrt{\det(g_{ij})} dx^1 \cdots dx^n \ .$$

In particular, we have the following case. Consider the chart $(\exp_p^{-1}, V(p) = M \setminus \mathrm{Cut}(p))\}$ given in the previous section and on $U(p)$ (or T_pM) we decompose the Lebesgue measure via polar coordinates

$$dx^1 \dots dx^n = r^{n-1} dr d\theta,$$

where $d\theta$ is the usual surface measure on S^{n-1}. Then, in this chart we may write the Riemannian density as

$$\mu = \lambda(r,\theta) dr d\theta$$

where $\lambda(r,\theta) = \sqrt{G \circ \exp_p}(r,\theta) r^{n-1}$ with $G = \det(g_{ij})$.

Now, taking $\lambda(r,\theta) = 0$ outside $U(p)$ since, by definition,

$$\overline{B_r(p)} = \exp_p(\overline{B_r(0)}) = \exp_p(\overline{B_r(0)} \cap \overline{U(p)})$$

and $\mathrm{Cut}(p)$ is of zero measure in M, we have

$$\mathrm{Vol}(B_r(p)) = \int_{B_r(0)\cap U(p)} \lambda(r,\theta) dr d\theta = \int_{B_r(0)} \lambda(r,\theta) dr d\theta.$$

Then, if M is complete, since $M = \exp_p(U(p)) \cup \mathrm{Cut}(p)$ we have

$$\mathrm{Vol}(M) = \int_{U(p)} (\exp_m)_\star v_g = \int_{U(p)} \lambda(r,\theta) dr d\theta = \int_{S^{n-1}} \int_0^{\rho(\theta)} \lambda(r,\theta) dr d\theta\ .$$

We will now calculate the function $\lambda(r,\theta)$. For this we will fix a tangent vector $\theta \in S_pM$ and consider Jacobi fields along the geodesic $\gamma(t) = \exp_p(t\theta)$.

Proposition 2.6.2 *Fix $\theta \in S_pM$. Let $J_2, \dots, J_n$ be normal Jacobi fields along the geodesic $\gamma(t) = \exp_p(t\theta)$, with $J_i(0) = 0$ and $(J_2'(0), \dots, J_n'(0))$ linearly independent. Then, for $r\theta \in U_p$,*

$$\lambda(r,\theta) = \frac{\det(J_2(r), \dots, J_n(r))}{\det(J_2'(0), \dots, J_n'(0))}$$

where the determinant is taken with respect to an orthonormal frame $E_i(t)$, $i = 2, \dots, n$, in $(\gamma'(t))^\perp$ along the geodesic γ.

Proof Take $e_1 = \theta$ and $e_2, \dots, e_n$ a set of linearly independent vectors in $e_1^\perp \subset T_pM$. Using this basis of T_pM we can define global coordinates $u_1, u_2, \dots, u_n$ on T_pM. With these coordinates on T_pM we have

$$du_1 \cdots du_n = \frac{1}{\det(e_2, \dots, e_m)} dx^1 \cdots dx^n = \frac{r^{n-1}}{\det(e_2, \dots, e_m)} dr d\theta$$

and $e_i = \partial_i \restriction_{u=0}$.

In particular, we take $e_i = J_i'(0)$. Following proposition 2.1.3 we have that

$$J_i(t) = (d\exp_p)_{t\theta}(te_j).$$

Since $r\theta \in U(p)$ we have that $\exp_p$ is a diffeomoprhism on a neighbourhood of $r\theta$ so under $\exp_p$, our coordinate system $u_1, u_2, \dots, u_n$ also gives a coordinate system in a neighbourhood of $\exp_p(r\theta)$. For this coordinate system we have

$$\partial_1 = \frac{d}{dt}(\exp_p((r+t)\theta)) \restriction_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_\theta(r+t)) \restriction_{t=0} = \gamma_\theta'(r)$$

and

$$\partial_i = \frac{d}{dt}(\exp_p(r\theta + te_i) \restriction_{t=0} = (d\exp_p)_{r\theta}(e_i) = \frac{1}{r}J_i(r) \ .$$

It follows that

$$r^2 g_{ij} = r^2\langle \partial_i, \partial_j \rangle = \langle J_i, J_j \rangle$$

for $i, j \geq 2$ along γ. Now, γ is normal so $\langle \partial_1, \partial_1 \rangle = 1$. Furthermore, $\langle e_1, e_i \rangle = 0$ for $i \geq 2$ so by the Gauss lemma we get

$$\langle \partial_1, \partial_i \rangle = \langle (d\exp_p)_{r\theta}(e_1), (d\exp_p)_{r\theta}(e_i) \rangle = 0$$

for $i \geq 2$. Then

$$G = \det(g_{ij}) = r^{-2(n-1)} \det(\langle J_i, J_j \rangle)_{i,j\geq 2} = r^{-2(n-1)} \det(J_2(r), \ldots, J_n(r))^2 \ .$$

Finally,

$$\sqrt{G} du_1 \cdots du_n = \sqrt{G} \frac{r^{n-1}}{\det(J_2'(0), \ldots, J_n'(0))} dr d\theta = \frac{\det(J_2(r), \ldots, J_n(r))}{\det(J_2'(0), \ldots, J_n'(0))} dr d\theta \ .$$

■

2.7 Geometric meaning of the Ricci and scalar curvatures

It can be proved that, with respect to normal coordinates around p, the function $g_{ij}(u^1, \ldots, u^n)$ admits the Taylor expansion at $x = 0$,

$$g_{ij} = \delta_{ij} + \frac{1}{3} R_{iklj}(p) u^k u^l + O(|x^3|) \ .$$

The proof uses Jacobi fields and the fact that they determine the metric as we have seen in the proof of the previous proposition. This result allows us to give some geometric interpretations of the different curvatures. For example, the sectional curvature measures the deviation of the geodesic circle to the standard circle in Euclidean space. But the interpretation that we are interested in relates to the Ricci curvature. It states that the Ricci curvature measures the change of the volume element in a given direction, more precisely, in normal coordinates the volume element has the expansion

$$\sqrt{\det(g_{ij})} = 1 - \frac{1}{6} R_{kl}(p) x^k x^l + O(|x|^3) \ .$$

From this we obtain the expansion for the volume of a small enough geodesic ball:

Theorem 2.7.1 *For r small enough*

$$\mathrm{Vol}(B_r(p)) = \omega_n r^n \left(1 - \frac{\mathrm{Scal}(p)}{6(n+2)} r^2 + O(|r^3|)\right) ,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Since this will only be used for a small detail in the proof of the Bishop-Gunther-Gromov inequality we will not provide a proof. However, proofs may be found in ([13], Theorem 3.1) or in ([12], Theorem 3.98).

2.8 Bishop-Gromov-Gunther inequality

Before proving the Bishop-Gunther-Gromov inequality we need the following lemma which compares volume densities of a Riemannian manifold with Ricci curvature bounded below and of the space form M_k^n of Riemannian manifolds of constant sectional curvature. M_k^n may be seen as the simply connected Riemannian manifold of constant curvature k. If $k > 0$ we have a sphere, if $k = 0$ the euclidean space and for $k < 0$ we get an hyperbolic space.

We will denote by $\lambda_k(r)$ the density function μ for the space form M_k^n, in this way $\lambda_k(r)$ is independent of p and of θ.

Lemma 2.8.1 *If (M, g) is a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)k$, [3] *then for any fixed $\theta \in S_pM$ and any r with $r\theta \in U(p)$, we have*

$$\frac{\lambda'(r,\theta)}{\lambda(r,\theta)} \leq \frac{\lambda_k'(r)}{\lambda_k(r)},$$

where the derivative is taken with respect to r.

If, on the other hand, $K_M \leq k$ [4] then

$$\frac{\lambda'(r,\theta)}{\lambda(r,\theta)} \geq \frac{\lambda_k'(r)}{\lambda_k(r)}\ .$$

Proof $\boxed{1}$ Suppose first that $\mathrm{Ric} \geq (n-1)k$. Given $\theta \in S_pM$ and r such that $r\theta \in U(p)$ consider a parallel orthonormal frame $\{E_i(t)\}$ along $\gamma = \gamma_\theta$ with $E_1(0) = \theta$ (hence, $E_1(t) = \gamma'(t)$). Take, as in the proof of the previous proposition, the Jacobi fields $J_i(t)$ along γ such that $J_i(0) = 0$ and $J_i(r) = E_i(r)$.

Denote $A(t) = (\langle J_i(t), J_j(t)\rangle)_{i,j\geq 2}$ and $D(t) = \det A(t)$. Then, since

$$\{J_1(r) = E_1(r), \ldots, J_n(r) = E_n(r)\}$$

is an orthonormal basis of $T_{\gamma(r)}M$ we have $A(r) = \mathrm{Id}$ and therefore $D(r) = 1$. Now, remembering that $\lambda(r,\theta) = C^{-1}\sqrt{D(r)}$ where $C = \det(J_2'(0), \ldots, J_n'(0))$ we have

$$\frac{\lambda'(r,\theta)}{\lambda(r,\theta)} = \frac{1/2C^{-1}(D(r))^{-1/2}D'(r)}{C^{-1}(D(r))^{1/2}} = \frac{1}{2}\frac{D'(r)}{D(r)} = \frac{1}{2}D'(r)\ .$$

By a well known formula $D'(t) = D(t)\mathrm{Tr}(A^{-1}(t)A'(t))$, so

$$\frac{\lambda'(r,\theta)}{\lambda(r,\theta)} = \frac{1}{2}\det(\mathrm{Id})\,\mathrm{Tr}(\mathrm{Id}A'(t)) = \frac{1}{2}\mathrm{Tr}(A'(t)) = \sum_{i=2}^{n}\langle J_i(r), J_i'(r)\rangle = \sum_{i=2}^{n} I(J_i, J_i) \qquad (2.4)$$

where we have also used (2.2).

We now consider the vector fields $H_i(t) = \dfrac{S_k(t)}{S_k(r)}E_i(t)$, where $S_k(t)$ is the solution of $S_k'' + kS_k = 0$ with conditions $S_k(0) = 0$, $S_k'(0) = 1$ (equation (2.3)). Then H_i and J_i have the same values at the end points and by the index lemma we get

$$I(J_i, J_i) \leq I(H_i, H_i).$$

[3] By $\mathrm{Ric} \geq (n-1)k$ we mean that for all unit tangent vectors v, $\mathrm{Ric}(v,v) \geq (n-1)k$.

[4] We say that $K_M \leq k$ if for all orthonormal tangent vectors x, y we have $K(x,y) \leq k$ where $K(x,y)$ is the sectional curvature of the plane σ_p generated by x and y.

Now, by definition

$$
\begin{aligned}
I(H_i,H_i) &= -\int_0^r \left\langle \frac{D^2H_i}{dt^2} + R(\gamma', H_i)\gamma' \ , \ H_i \right\rangle dt + \left\langle \frac{DH_i}{dt}(r) \ , \ H_i(r) \right\rangle = \\
&= -\int_0^r \left\langle \frac{S_k''(t)}{S_k(r)} E_i + R(\gamma', \frac{S_k(t)}{S_k(r)} E_i)\gamma' \ , \ \frac{S_k(t)}{S_k(r)} E_i \right\rangle dt + \left\langle \frac{DH_i}{dt}(r), H_i(r) \right\rangle = \\
&= -\int_0^r \left\langle \frac{-kS_k(t)}{S_k(r)} E_i + R(\gamma', \frac{S_k(t)}{S_k(r)} E_i)\gamma' \ , \ \frac{S_k(t)}{S_k(r)} E_i \right\rangle dt + \left\langle \frac{DH_i}{dt}(r), H_i(r) \right\rangle = \\
&= \int_0^r \left(\frac{S_k(t)}{S_k(r)} \right)^2 (k - \langle R(\gamma', E_i)\gamma', E_i \rangle) dt + \left\langle \frac{DH_i}{dt}(r), H_i(r) \right\rangle
\end{aligned}
$$

and summing over the $i = 2, \ldots, n$, we get

$$
\sum_{i=2}^n I(H_i,H_i) = \int_0^r \left(\frac{S_k(t)}{S_k(r)} \right)^2 ((n-1)k - \text{Ric}(\gamma', \gamma')) dt + \sum_{i=2}^n \left\langle \frac{DH_i}{dt}(r), H_i(r) \right\rangle \ .
$$

On the other hand, we may do the same construction for the space form M_k^n, i.e. consider a normal geodesic γ^k in M_k^n together with an orthonormal parallel frame $\{E_i^k\}$ along it and let $H_i^k(t) = \frac{S_k(t)}{S_k(r)} E_i^k(t)$. In this case, following our arguments of section 2.4, we get that H_i^k are Jacobi fields so

$$
\frac{\lambda'(r)}{\lambda(r)} = \sum_{i=2}^n \left\langle H_i^k(r), \frac{DH_i^k}{dt}(r) \right\rangle = \sum_{i=2}^n \left\langle H_i(r), \frac{DH_i}{dt}(r) \right\rangle
$$

Putting everything together and using the hypothesis we get

$$
\frac{\lambda'(r,\theta)}{\lambda(r,\theta)} \leq \frac{\lambda'(r)}{\lambda(r)} \ .
$$

$\boxed{2}$ Suppose now that $K_M \leq k$. Denote by J_i^k the vector field along γ^k in M_k^n whose coordinates in the orthonormal frame $\{E_i^k\}$ are the same as the coordinates of J_i in $\{E_i\}$.

Now, by definition

$$
I(J_i,J_i) = \int_0^r \left(\left\langle \frac{DJ_i}{dt} \ , \frac{DJ_i}{dt} \right\rangle - R(\gamma', J_i, \gamma', J_i) \right) dt
$$

and by construction we have $\langle J_i, J_i \rangle = \langle J_i^k, J_i^k \rangle$ and

$$
\left\langle \frac{DJ_i}{dt}(r) \ , \frac{DJ_i}{dt}(r) \right\rangle = \left\langle \frac{DJ_i^k}{dt}(r) \ , \frac{DJ_i^k}{dt}(r) \right\rangle \ .
$$

Finally, since $|\gamma'| = 1$ and J_i is normal, the hypothesis $K_M \leq k$ yields

$$
\frac{R(\gamma', J_i, \gamma', J_i)}{\langle J_i, J_i \rangle} \leq k
$$

thus

$$
\begin{aligned}
I(J_i,J_i) &= \int_0^r \left(\left\langle \frac{DJ_i}{dt} \ , \frac{DJ_i}{dt} \right\rangle - R(\gamma', J_i, \gamma', J_i) \right) dt \geq \int_0^r \left(\left\langle \frac{DJ_i}{dt} \ , \frac{DJ_i}{dt} \right\rangle - k\langle J_i, J_i \rangle \right) dt = \\
&= \int_0^r \left(\left\langle \frac{DJ_i^k}{dt} \ , \frac{DJ_i^k}{dt} \right\rangle - k\langle J_i^k, J_i^k \rangle \right) dt = I_k(J_i^k, J_i^k) \ .
\end{aligned}
$$

We end as before, the Index lemma yields

$$I(J_i, J_i) \geq I_k(J_i^k, J_i^k) \geq I_k(H_i^k, H_i^k)$$

and summing over $i = 2, \ldots, n$ we get

$$\frac{\lambda'(r,\theta)}{\lambda(r,\theta)} = \sum_{i=2}^{n} I(J_i, J_i) \geq \sum_{i=2}^{n} I(H_i^k, H_i^k) = \frac{\lambda_k'(r)}{\lambda_k(r)}$$

where the equalities follow from equation (2.4). ■

With this we now prove the theorem.

Theorem 2.8.2 *(Bishop-Gromov-Gunther). If (M, g) is a complete manifold with $\operatorname{Ric} \geq (n-1)k$, and $p \in M$ is an arbitrary point then the function*

$$r \mapsto \frac{\operatorname{Vol}(B_r(p))}{\operatorname{Vol}(B_r^k)} \tag{2.5}$$

is a non-increasing function which tends to 1 as r goes to 0, where B_r^k is a geodesic ball of radius r in the space form M_k^n. In particular, $\operatorname{Vol}(B_r(p)) \leq \operatorname{Vol}(B_r^k)$.

If, on the other hand, $K_M \leq k$ and $B_r(p)$ does not cut $\operatorname{Cut}(p)$, then the preceding function is non-decreasing and tends to 1 as r goes to 0. In particular, $\operatorname{Vol}(B_r(p)) \geq \operatorname{Vol}(B_r^k)$.

Proof $\boxed{1}$ Suppose first that $\operatorname{Ric} \geq (n-1)k$. Let $a(t) = \displaystyle\int_{S^{n-1}} \lambda(t,\theta)d\theta$ and $b(t) = \int_{S^{n-1}} \lambda_k(t)d\theta$. Then $\operatorname{Vol}(B_r(p)) = \displaystyle\int_0^r a(t)dt$ and $\operatorname{Vol}(B_r^k) = \displaystyle\int_0^r b(t)dt$. Hence,

$$\frac{d}{dr}\left(\log \frac{\operatorname{Vol}(B_r(p))}{\operatorname{Vol}(B_r^k)}\right) = \frac{a(r)}{\int_0^r a(t)dt} - \frac{b(r)}{\int_0^r b(t)dt} = \frac{\int_0^r (a(r)b(t) - a(t)b(r))dt}{\int_0^r a(t)dt \int_0^r b(t)dt} \ .$$

To prove that the function in the statement is non-increasing, we will prove that its logarithm is non-increasing. For this, it is enough to see that $a(r)b(t) - a(t)b(r) \leq 0$ for $t \leq r$, in other words, that $\dfrac{a(r)}{b(r)} \leq \dfrac{a(t)}{b(t)}$ for $t \leq r$. We will now prove that $\dfrac{a(t)}{b(t)}$ is non-increasing. Since $\lambda_k(r)$ is independent of θ we have,

$$\frac{a(t)}{b(t)} = \frac{\int_{S^{n-1}} \lambda(t,\theta)d\theta}{\int_{S^{n-1}} \lambda_k(t)d\theta} = \frac{1}{4\pi}\int_{S^{n-1}} \frac{\lambda(t,\theta)}{\lambda_k(t)}d\theta \ .$$

Therefore, it is enough to prove that the function $\dfrac{\lambda(t,\theta)}{\lambda_k(\theta)}$ is non-increasing in t for any fixed θ but this follows immediately from the previous lemma. Indeed,

$$\frac{d}{dt}\left(\log \frac{\lambda(t,\theta)}{\lambda_k(t)}\right) = \frac{\lambda'(t,\theta)}{\lambda(t,\theta)} - \frac{\lambda_k'(t)}{\lambda_k(t)} \leq 0 \ .$$

Note first that if $r\theta \in U(p)$ then $t\theta \in U(p)$ for $0 \leq t \leq r$. On the other hand, if $t\theta \notin U(p)$ then we have established that $\lambda(t,\theta) = 0$ so the function $t \mapsto \dfrac{\lambda(t,\theta)}{\lambda_k(t)} \geq 0$ is still non-increasing.

2 Suppose now that $K_M \geq k$. The proof is analogous, with the only difference being that we must restrict the value of r to ensure that $B_r(p)$ does not cut $\mathrm{Cut}(p)$ so that $t \mapsto \dfrac{\lambda(t,\theta)}{\lambda_k(t)}$ is non-decreasing.

In both cases, the fact that the function tends to 1 as $r \to 0^+$ follows from Theorem 2.7.1. ■

If we go over the proof of the lemma, we realise that for $B_r(p)$ disjoint of $\mathrm{Cut}(p)$,

$$\mathrm{Vol}(B_r(p)) = \mathrm{Vol}(B_r^k)$$

if, and only if $B_r(p)$ is isometric to B_r^k.

Indeed, if $\mathrm{Vol}(B_r(p)) = \mathrm{Vol}(B_r^k)$ then $\mathrm{Vol}(B_t(p)) = \mathrm{Vol}(B_t^k)$ for all $t \in [0,r]$. But then, following the proof of the theorem we must have that

$$\frac{\lambda'(t,\theta)}{\lambda(t,\theta)} = \frac{\lambda_k'(t)}{\lambda_k(t)}$$

for any $\theta \in S_pM$ and $0 \leq t \leq r$. If we now go back to the proof of the lemma we get that $I(J_i,J_i) = I(H_i,H_i)$ and by the Index Lemma this implies that $J_i = H_i$. Now, as in the proof of Proposition 2.6.2 we have

$$t^2 g_{ij} = \langle J_i, J_j \rangle = \langle H_i^k, H_j^k \rangle = t^2 g_{ij}^k$$

and then $g_{ij}(t,\theta) = g_{ij}^k(t)$ for $0 < t \leq r$, $\theta \in S_pM$ and also for $t = 0$ by continuity.

If we identify the ball $B_r(0) \subset T_pM$ and the ball $B_r^k(0) \subset T_{\overline{p}}M_k^n$ for some $\overline{p} \in M_k^n$ it can be proved that the isometry would be given by $\exp_{\overline{p}} \circ \exp_p^{-1} : B_r(p) \to B_r^k(\overline{p})$.

Similarly,

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r^k)} = \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}(B_R^k)}$$

for $r < R$ if, and only if, $B_R(p)$ is isometric to B_R^k.

With this remark we may now prove a nice consequence of the theorem.

Theorem 2.8.3 *(S.Y. Cheng) Let (M,g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)k$ *for some $k > 0$, and* $\mathrm{diam}(M,g) = \frac{\pi}{\sqrt{k}}$*, then M is isometric to the standard sphere of radius $\frac{1}{\sqrt{k}}$.*

Proof (Shiohama) For simplicity assume $k = 1$. Then since $\mathrm{diam}(M,g) = \pi$ we have that $\mathrm{Vol}(M) = \mathrm{Vol}(B_\pi(p))$.

By the Bishop-Gunther-Gromov theorem, for any $p \in M$,

$$\frac{\mathrm{Vol}(B_{\pi/2}(p))}{\mathrm{Vol}(M)} = \frac{\mathrm{Vol}(B_{\pi/2}(p))}{\mathrm{Vol}(B_\pi(p))} \geq \frac{\mathrm{Vol}(B_{\pi/2}^1(p))}{\mathrm{Vol}(B_\pi^1)} = \frac{1}{2}\ .$$

Now, the manifold is complete with diameter π so we may take p,$q \in M$ so that $\mathrm{dist}(p,q) = \pi$. Then, by the above inequality we have

$$\mathrm{Vol}(B_{\pi/2}(p)) \geq \frac{1}{2}\mathrm{Vol}(M), \quad \mathrm{Vol}(B_{\pi/2}(q)) \geq \frac{1}{2}\mathrm{Vol}(M)\ .$$

Since $d(p,q) = \pi$ we have $B_{\pi/2}(p) \cap B_{\pi/2}(q) = \emptyset$, this yields $\text{Vol}(B_{\pi/2}(p)) = \frac{1}{2}\text{Vol}(M)$ and $\text{Vol}(B_{\pi/2}(q)) = \frac{1}{2}\text{Vol}(M)$. Then

$$\frac{\text{Vol}(B_{\pi/2}(p))}{\text{Vol}(B_{\pi}(p))} = \frac{\text{Vol}(B^1_{\pi/2}(p))}{\text{Vol}(B^1_{\pi})} = \frac{1}{2}, \quad \frac{\text{Vol}(B_{\pi/2}(q))}{\text{Vol}(B_{\pi}(q))} = \frac{\text{Vol}(B^1_{\pi/2}(p))}{\text{Vol}(B^1_{\pi})} = \frac{1}{2}.$$

Then, by the previous comments we have that $B_{\pi/2}(p)$ and $B_{\pi/2}(q)$ are both isometric to half sphere so M is isometric to S^m. ■

There are many other important applications, we will now state one of them. This result can be proved in a few lines using the Bishop-Gunther-Gromov inequality. It gives a lower bound for the volume growth.

Theorem 2.8.4 *(Calabi-Yau) Let (M,g) be a complete non-compact Riemannian manifold with* $\text{Ric} \geq 0$. *Then there exists a positive constant c depending only on p and the dimension n so that*

$$\text{Vol}(B_r(p)) \geq cr$$

for any $r > 2$.

2.9 Gromov's precompactness theorem

We now prove one of the main theorems in this essay.

Theorem 2.9.1 *For any $n \in \mathbb{N}$, $k \in \mathbb{R}$ and $D > 0$, the class of all n-dimensional Riemannian manifolds with diameter $\leq D$ and Ricci curvature $\geq (n-1)k$ is pre-compact in the Gromov-Hausdorff topology.*

Proof Choose a maximal set of points $x_1, \ldots, x_N \in M$ with $d(x_i, x_j) \geq \varepsilon$. Then $B_\varepsilon(x_i)$, $i = 1, \ldots, N$, covers M, so $N(\varepsilon) \leq N$. As before, $N(\varepsilon)$ denotes the minimum number of points we need in order to build an ε-net in X. Since (M,g) is equal to any geodesic ball of radius D, by the Gromov-Bishop theorem

$$\frac{\text{Vol}(M)}{\text{Vol}(B_{\varepsilon/2}(x_i))} \leq \frac{\int_0^D (\sinh(\sqrt{|k|}t))^{n-1} dt}{\int_0^{\varepsilon/2} (\sinh(\sqrt{|k|}t))^{n-1} dt} \leq C\varepsilon^{-n}$$

for all $\varepsilon \leq D$ and some $C = C(k,D)$. Therefore, we get an estimate of type $N(\varepsilon) \leq C(k,D)\varepsilon^{-n}$, with an explicit function C of k and D. Indeed, $N \cdot \min\{\text{Vol}(B_{\varepsilon/2}(x_i)) \mid i = 1, \ldots, N\} \leq \text{Vol}(M)$, because the balls $B_{\varepsilon/2}(x_i)$ are disjoint. Theorem 1.2.7 then gives the result. ■

The same result is true with the measured Gromov-Hausdorff topology and is obtained similarly using Theorem 1.3.2.

Corollary 2.9.2 *Let $K \in \mathbb{R}$, $n \in (1,\infty]$ and $D \in (0,+\infty)$. Let $\mathcal{M}(n,K,D)$ be the set of Riemannian manifolds (M,g) such that* $\dim(M) \leq n$, $\text{Ric}_M \geq Kg$ *and* $\text{diam}(M) \leq D$, *equipped with their geodesic distance and their volume measure. Then $\mathcal{M}(n,K,D)$ is precompact in the measured Gromov-Hausdorff topology.*

2.9.1 Ricci limits

Let $\mathcal{M}$ denote the set of compact metric spaces (modulo isometry) with the Gromov-Hausdorff topology. The limit points in $\mathcal{M}$ of the subset $\mathcal{M}(n, K, D)$ will be metric spaces of Hausdorff dimension at most n ([15], Theorem A), but they are not generally manifolds. However, one would like to say that in some generalized sense they do have Ricci curvature bounded below by K, this is what will be studied in the last chapter. The structure of such limit points, which are called Ricci limits, was studied by Cheeger and Colding [[6],[7],[8]].

Remark 2.9.3 A limit of n-dimensional spaces may have dimension strictly less than n. For example, for every compact non negatively curved space X rescaled spaces $\{\lambda X\}$ are non negatively curved and converge to a point as $\lambda \to 0$. It is possible to make the dimension drop in the limit keeping both lower and upper curvature bounds. For example, 'thin' flat tori $S^1 \times (\lambda S^1)$ converge to the circle as $\lambda \to 0$.

2.10 Generalised Ricci tensor

The necessity for a modified Ricci tensor appears when we consider a reference measure $\nu(dx) = e^{-\psi(x)}\mathrm{vol}(dx)$ different from the volume measure. Indeed, the Jacobian determinants (regarded as the limit of the relative change in volume) are affected. Then, since Ricci curvature can be defined in terms of the the Jacobian of the exponential map, the change of reference measure makes the Ricci curvature estimates lose their meaning. This means that we must change the definition of Ricci tensor to take the new reference measure into account. However, this may affect the dependence of estimates on the dimension, so we must also introduce an 'effective dimension' N, which may be larger then the dimension n of the manifold. If ψ is constant, i.e., if $\nu = \frac{\mathrm{dvol}_M}{\mathrm{vol}(M)}$, then we take the usual Ricci tensor Ric. For general ψ, a modified Ricci tensor

$$\mathrm{Ric}_\infty = \mathrm{Ric} + \mathrm{Hess}(\psi)$$

was introduced by Bakry and Émery [2]. Moreover, if $N \in (n, \infty)$ then put

$$\mathrm{Ric}_N = \mathrm{Ric} + \mathrm{Hess}(\psi) - \frac{1}{N-n} d\psi \otimes d\psi,$$

where $\dim(M) = n$. In general we have:

Definition 2.10.1 Let (M, g, ν) be a Riemannian manifold with a given reference measure ν. For $N \in [1, \infty]$, define the N-Ricci tensor Ric_N of (M, g, ν) by

$$\mathrm{Ric}_N = \begin{cases} \mathrm{Ric} + \mathrm{Hess}(\psi) & \text{if } N = \infty, \\ \mathrm{Ric} + \mathrm{Hess}(\psi) - \frac{1}{N-n} d\psi \otimes d\psi & \text{if } n < N < \infty, \\ \mathrm{Ric} + \mathrm{Hess}(\psi) - \infty(d\psi \otimes d\psi) & \text{if } N = n, \\ -\infty & \text{if } N < n, \end{cases}$$

where $\nu(dx) = e^{-\psi(x)}\mathrm{vol}(dx)$ and by convention $\infty \cdot 0 = 0$.

If (M, d, ν) satisfies $\mathrm{Ric}_N \geq K$ then M is said to satisfy the $CD(K, N)$ curvature-dimension bound.

We now give the one-dimensional $CD(K, N)$ model spaces.

Example 2.10.2

1. For $K > 0$ and $1 < N < \infty$, consider

$$M = \left(-\sqrt{\frac{N-1}{K}}\frac{\pi}{2}, \sqrt{\frac{N-1}{K}}\frac{\pi}{2} \right) \subset \mathbb{R},$$

equipped with the usual distance on $\mathbb{R}$, and the reference measure

$$\nu(dx) = \cos^{N-1}\left(\sqrt{\frac{K}{N-1}}x \right) dx;$$

then M satisfies $CD(K, N)$.

2. For $K < 0$, $1 \leq N < \infty$, the same is true if $M = \mathbb{R}$ and

$$\nu(dx) = \cosh^{N-1}\left(\sqrt{\frac{|K|}{N-1}}x \right) dx.$$

3. For $N \in [1, \infty)$, $M = (0, +\infty)$ with the reference measure $\nu(dx) = x^{N-1}dx$, gives an example of $CD(0, N)$ space.

4. For any $K \in \mathbb{R}$, take $M = \mathbb{R}$ and equip it with the reference measure

$$\nu(dx) = e^{-\frac{Kx^2}{2}} dx$$

then M satisfies $CD(K, \infty)$.

Chapter 3

Optimal transport

3.1 Introduction

The optimal transportation problem arises, for example, when one considers the problem of moving a pile of sand in order to completely fill up a hole. In this context, we would obviously ask for both the pile and the hole to have the same total volume, so that all the sand is used to fill up the hole. With this in mind, we may model (after normalizing) both the pile and the hole by Borel probability measure spaces (X, μ) and (Y, ν) respectively, such that for any measurable $A \subset X$, $\mu(A)$ measures the volume of sand on A, and for any measurable $B \subset Y$, $\nu(B)$ measures how much sand can be piled on B.

This transportation of sand has a cost, which depends on how it is done. We will model the effort required to move the sand around by a cost function, which we assume to be measurable and non-negative, $c : X \times Y \to \mathbb{R}_+ \cup \{\infty\}$. In some sense, $c(x, y)$ measures the cost of moving one unit of mass from location x to location y.

The question that arises is, how do we realize the transportation at minimal cost?

3.1.1 Kantorovich's problem

Before answering this question we need to give a definition of **transference plan**, which tells us how the sand is being moved. We model transference plans by Borel probability measures π on the product space $X \times Y$. Informally, $d\pi(x, y)$ measures the amount of mass transported from location x to location y. So, for a transference plan $\pi \in P(X \times Y)$ to go along with our model, we must have

$$\pi[A \times Y] = \mu[A], \qquad \pi[X \times B] = \nu[B] \tag{3.1}$$

for all measurable subsets A of X and B of Y. This ensures that all the sand on A is moved to somewhere on Y, while B receives all the sand it can. We denote by $\Pi(\mu, \nu)$ the set of all Borel probability measures on $X \times Y$ satisfying this condition. This set is non-empty, since the tensor product $\mu \otimes \nu$ lies in it, and it can easily be seen to be convex. Furthermore, $\pi \in \Pi(\mu, \nu)$ if and only if it is a non-negative measure on $X \times Y$ such that, for all measurable functions $(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu)$, or equivalently $L^\infty(d\mu) \times L^\infty(d\nu)$,

$$\int_{X\times Y} [\varphi(x) + \psi(y)]d\pi(x, y) = \int_X \varphi d\mu + \int_Y \psi d\nu.$$ [1]

We may use a narrower class of test functions under some topological assumptions:

1. When X and Y are Polish spaces (complete separable metric spaces), and μ, ν are Borel probability measures, it is sufficient to consider $(\varphi, \psi) \in C_b(X) \times C_b(Y)$ only. This follows from the fact that Borel probability measures on Polish spaces are regular, thus we may build the usual bump functions and approximate simple functions by continuous bounded ones.

2. If in addition X and Y are locally compact, then we can require (3.2) for $(\varphi, \psi) \in C_0(X) \times C_0(Y)$.

From now on $P(X)$ will stand for the set of Borel probability measures on X, and $\Pi(\mu, \nu)$ will be the set of all Borel probability measures on $X \times Y$ satisfying (3.1).

Definition 3.1.1 *Kantorovich's optimal transportation problem* consists in minimizing the linear functional

$$\pi \longmapsto I[\pi] = \int_{X\times Y} c(x, y)d\pi(x, y)$$

on $\Pi(\mu, \nu)$. For a given transportation cost π, the quantity $I[\pi]$ is called the **total transportation cost** associated to π.

Any minimizer for this variational problem is called an *optimal transference plan.*

Example 3.1.2 Assume that ν is a Dirac mass: $\nu = \delta_a$. Then there is a unique element in $\Pi(\mu, \nu)$, and

$$\inf I[\pi] = \int_X c(x, a)d\mu(x).$$

[1] Indeed, if $\pi \in \Pi(\mu, \nu)$ then, for any measurable sets $A \subset X$ and $B \subset Y$ we have

$$\begin{aligned}\int_{X\times Y} [\chi_A(x) + \chi_B(y)]d\pi(x, y) &= \int_{X\times Y} \chi_A(x)d\pi(x, y) + \int_{X\times Y} \chi_B(y)d\pi(x, y)\\ &= \int_{X\times Y} \chi_{A\times Y}(x, y)d\pi(x, y) + \int_{X\times Y} \chi_{X\times B}(x, y)d\pi(x, y)\\ &= \pi[A \times Y] + \pi[X \times B] = \mu[A] + \nu[B]\\ &= \int_X \chi_A(x)d\mu(x) + \int_Y \chi_B(y)d\nu(y).\end{aligned}$$

The general case now follows by the density of simple functions in L^1 or L^∞. To prove that π is non-negative we first prove (easily) that π is non-negative on sets of type $A \times B$, and then, since this kind of sets form a basis of open sets in $X \times Y$, we deduce that the same is true for any Borel set.

The reciprocal follows similarly.

3.1.2 Monge's problem.

Although we have started with Kantorovich's problem, this problem is a relaxed version of the original mass transportation problem considered by Monge. Monge considered the same problem but, in order to move the sand around, he only accepted transference plans which took all the mass on a location x to a destination y, i.e. the mass could not be split. We are therefore asking for π to have the special form

$$d\pi(x,y) = d\pi_T(x,y) = d\mu(x)\delta\left[y = T(x)\right],$$

where T is a measurable map $X \to Y$. In other words,

$$\pi = (\mathrm{Id} \times T)\#\mu.$$

This is the probability measure on $X \times Y$ satisfying:

$$\int_{X\times Y} \zeta(x,y)d\pi_T(x,y) = \int_X \zeta(x,T(x))d\mu(x)$$

for every measurable function ζ on $X \times Y$. Consequently,

$$I[\pi_T] = \int_X c(x,T(x))d\mu(x).$$

Condition (3.2) is now

$$\int_X [\varphi(x) + \psi \circ T(x)]d\mu(x) = \int_X \varphi(x)d\mu(x) + \int_Y \psi(y)d\nu(y) \quad\Longrightarrow\quad \int_X (\psi\circ T)d\mu = \int_Y \psi d\nu$$

for all $\psi \in L^1(d\nu)$, or $\psi \in L^\infty(d\nu)$. With this we mean that for all $\psi \in L^1(d\nu)$, the measurable function $\psi \circ T$ should lie in $L^1(d\mu)$, and the preceding integrals should coincide.

This condition for $\pi \in \Pi(\mu,\nu)$ is equivalent to

$$\nu[B] = \mu[T^{-1}(B)], \quad \forall \text{ measurable } B \subset Y.$$

If these equivalent conditions are satisfied then we say that ν is the **push-forward** of μ by T (or that T transports μ onto ν) and we write $\nu = T\#\mu$. Then

Definition 3.1.3 *Monge's optimal transportation problem* consists in minimizing the linear functional

$$T \longmapsto I[T] = \int_X c(x,T(x))d\mu(x)$$

over the set of all measurable maps T such that $T\#\mu = \nu$.

3.1.3 $d^2/2$-concave functions

A function $\varphi : X \to [-\infty,\infty)$ is $\frac{d^2}{2}$-concave if it is not identically $-\infty$ and it can be written in the form

$$\varphi(x) = \inf_{x'\in X}\left(\frac{d(x,x')^2}{2} - \tilde{\varphi}(x')\right)$$

for some function $\tilde{\varphi} : X \to [-\infty,\infty)$. Such functions play an important role in the description of optimal transport on Riemannian manifolds.

3.2 Optimal transport on Riemannian manifolds

Optimal transport with a quadratic cost (i.e. square of the distance) in $\mathbb{R}^n$ has been thoroughly studied, good references are [25] and [26]. In this context, optimal transport is very well understood and optimal transference plans have been characterised in terms of convex functions. Under some extra conditions on the measures, it has been proved that the transference plans are unique and correspond to Monge transports.

Namely, given μ_0, $\mu_1 \in P(\mathbb{R}^n)$ which are compactly supported and absolutely continuous with respect to Lebesgue measure, it was proved by Brenier [4] and Rachev-Rüschendorf [22] that there is a unique optimal transference plan between μ_0 and μ_1, which is a Monge transport. Moreover, there is a convex function φ on $\mathbb{R}^n$ such that for almost all x, the Monge transport is given by the gradient of φ. Therefore, finding a convex function φ such that the push-forward, under the map $\nabla\varphi : \mathbb{R}^n \to \mathbb{R}^n$, sends μ_0 to μ_1 gives us the optimal transport.

However, it is not easy to extend these results to Riemannian manifolds. In [20], McCann achieved the desired extension by noticing that, on $\mathbb{R}^n$, we may write $\nabla\varphi = x - \nabla\phi$, where $\phi(x) = \frac{|x|^2}{2} - \varphi(x)$. McCann then proved that, on a Riemannian manifold (M, g), an optimal transference plan between two compactly supported absolutely continuous measures is a Monge transport F that satisfies $F(m) = \exp_m(-\nabla_m\phi)$ for almost all m. Here ϕ is $\frac{d^2}{2}$-concave, that is, it can be written in the form

$$\phi(m) = \inf_{m' \in M} \left(\frac{d(m, m')^2}{2} - \tilde{\phi}(m') \right)$$

for some function $\tilde{\phi} : M \to [-\infty, \infty)$.

3.3 Wassertian distance

Optimal transport can be used to produce a useful metric on $P(X)$. Indeed, using a quadratic cost function, and given μ_0, $\mu_1 \in P(X)$, we consider the variational problem

$$W_2(\mu_0, \mu_1)^2 = \inf_{\pi \in \Pi(\mu_0, \mu_1)} \int_{X \times X} d(x_0, x_1)^2 d\pi(x_0, x_1).$$

We will now prove that there always exists at least one optimal transference plan so we may replace the infimum by a minimum.

Theorem 3.3.1 *Let X be a Polish space and μ_0, $\mu_1 \in P(X)$ Borel probability measures on X. The minimization problem* $\inf\{I[\pi] \ : \ \pi \in \Pi(\mu_0, \mu_1)\}$ *admits a minimizer.*

Proof Since we may always obtain a transference plan given by the tensor product $\mu_0 \otimes \mu_1$ of the initial measures, $\Pi(\mu_0, \mu_1)$ is non-empty. The key point is that $\Pi(\mu_0, \mu_1)$ is compact for the weak topology of probability measures [2]. Indeed, following Ulam's lemma [3] we get that μ_0 and μ_1 are tight thus, given $\delta > 0$, we may take compact subsets $K \subset X$, $L \subset Y$ such that

$$\mu_0[X \setminus K] \leq \delta, \qquad \mu_1[Y \setminus L] \leq \delta.$$

[2]The topology induced by $C_b(\mathbb{R}^n \times \mathbb{R}^n)$.

[3] **Ulam's lemma** [3]: A probability measure μ on a Polish space X is tight, which means that for any $\varepsilon > 0$ there exists a compact K_ε such that $\mu[X \setminus K_\varepsilon] \leq \varepsilon$.

Then, for any $\pi \in \Pi(\mu_0, \mu_1)$,

$$\pi[(X \times Y) \setminus (K \times L)] \leq \pi[(X \setminus K) \times L] + \pi[X \times (Y \setminus L)] = \mu_0[X \setminus K] + \mu_1[Y \setminus L] \leq 2\delta.$$

This proves that the set $\Pi(\mu_0, \mu_1)$ is tight, and then Prokhorov's theorem [4] gives that $\Pi(\mu_0, \mu_1)$ is precompact with respect to the weak topology . But the conditions which define $\Pi(\mu_0, \mu_1)$ are continuous with respect to the weak topology so $\Pi(\mu_0, \mu_1)$ is weakly closed and therefore compact.

We now prove that there exists a minimizer for I. Let $(\pi_k)_{k\in\mathbb{N}}$ be a sequence of probability measures on $X \times X$ such that $\int c d\pi_k$ converges to the infimum cost. Since we have just proved that $\Pi(\mu_0, \mu_1)$ is weakly compact, we may extract a subsequence, that converges to some $\pi \in \Pi(\mu_0, \mu_1)$. For simplicity of notation we continue denoting this subsequence by $(\pi_k)_{k\in\mathbb{N}}$. Write the cost function $c(x, y) = d(x, y)^2$ as the supremum of a non-decreasing sequence $(c_l)_{l\in\mathbb{N}}$ of bounded continuous functions. By using the monotone convergence theorem, the fact that π is the limit point, the inequality $c_l \leq c$ and the minimizing property of (π_k), we obtain

$$\begin{aligned} \int c(x, y) d\pi(x, y) &= \lim_{l\to\infty} \int c_l(x, y) d\pi(x, y) \\ &= \lim_{l\to\infty} \lim_{k\to\infty} \int c_l(x, y) d\pi_k(x, y) \\ &\leq \liminf_{k\to\infty} \int c(x, y) d\pi_k(x, y) = \inf I. \end{aligned}$$

This shows that π is a minimizer of I. ■

Definition 3.3.2 The quantity W_2 will be called the *Wassertian distance of order* 2 between μ_0 and μ_1.

When X is compact the Wassertian distance defines a metric on $P(X)$ [5] and induces the weak-$\star$ topology on $P(X)$ ([25], Ths. 7.3 and 7.12). We denote $P(X)$ equipped with the metric W_2 by $P_2(X)$. If X is compact then so is $P_2(X)$, thus to each compact metric space X we have assigned another compact metric space $P_2(X)$.

Remark 3.3.3 In general, given (X, d) a Polish metric space, and $p \in [1, \infty)$, for any two probability measures μ_0, μ_1 on X, the Wassertian distance of order p between μ_0 and μ_1 is defined by the formula

$$W_p(\mu_0, \mu_1) = \left(\inf_{\pi\in\Pi(\mu_0,\mu_1)} \int_X d(x, y)^p d\pi(x, y) \right)^{1/p}.$$

So, why are Wassertian distances useful? One reason is that the definition makes them work well in problems where optimal transport is being used. Furthermore, since they

[4]**Prokhorov's theorem** [3]: If X is a Polish space, then a set $\mathcal{P} \subset P(X)$ is precompact for the weak topology if and only if it is tight, i.e. for any $\varepsilon > 0$ there is a compact set K_ε such that $\mu[X \setminus K_\varepsilon] \leq \varepsilon$ for all $\mu \in \mathcal{P}$.

[5]Since X has finite diameter, the infimum is obviously finite. If X is a Polish space, the Wassertian distance defines a metric on $P_2(X)$ which is the subset of $P(X)$ formed by measures with finite second order moments. This restriction is done because W_2 takes finite values on $P_2(X) \times P_2(X)$. In our case $P_2(X) = P(X)$.

are defined as an infimum, they are easy to bound from above. Indeed, any admissible transference plan between μ_0 and μ_1 will give us a bound on the distance between μ_0 and μ_1. In addition, there is an isometric embedding $X \to P_2(X)$ given by $x \to \delta_x$. Indeed, $W_2(\delta_x, \delta_y) = d(x, y)$ (in fact, $W_p(\delta_x, \delta_y) = d(x, y)$ for all $p \in [1, \infty)$). This shows that $\text{diam}(P_2(X)) \geq \text{diam}(X)$. But the reverse inequality follows from the definition of W_2, thus $\text{diam}(P_2(X)) = \text{diam}(X)$.

Chapter 4

Wassertian space

This section is devoted to the study of the Wassertian space $P_2(X)$ associated to a compact length space (X, d). We will show that $P_2(X)$ is again a length space, and characterise its geodesics.

4.1 Displacement interpolations

A transference plan tells us how much mass is moved from one point to another, but it does not specify the path followed for this transportation. This information is given by a dynamical transference plan.

Before defining this notion we give some notation. The space of Lipschitz continuous maps $c : [0, 1] \to X$ with the uniform topology is denoted by $\mathrm{Lip}([0, 1], X)$. Furthermore, for any $k > 0$,

$$\mathrm{Lip}_k([0, 1], X) = \{c \in \mathrm{Lip}([0, 1], X) \mid d(c(t), c(t')) \leq k|t - t'| \text{ for all } t, t' \in [0, 1]\} \ .$$

Then $\mathrm{Lip}_k([0, 1])$ is a compact subset of $\mathrm{Lip}([0, 1], X)$, this follows from a Cantor diagonal procedure and the following proposition:

Proposition 4.1.1 *Let X be a metric space and X' a dense subset of X. Let Y be a complete space and $f : X' \to Y$ a Lipschitz map. Then there exists a unique continuous map $\overline{f} : X \to Y$ such that $\overline{f} \restriction_{X'} = f$. Moreover $\overline{f}$ is Lipschitz and* $\mathrm{dil}\overline{f} = \mathrm{dil} f$.

We denote the set of minimizing, constant speed geodesics on X, $\gamma : [0, 1] \to X$, by Γ. This is a closed subspace [1] of $\mathrm{Lip}_{\mathrm{diam}(X)}([0, 1], X)$, defined by the equation $L(c) = d(c(0), c(1))$, and is therefore compact.

An evaluation map tells us where has the geodesic arrived on time t and the endpoints map gives us the initial and final points of a geodesic.

Definition 4.1.2 For any $t \in [0, 1]$, the map $e_t : \Gamma \to X$ defined by

$$e_t(\gamma) = \gamma(t)$$

is an *evaluation map* and it is continuous. We define the *endpoints map* $E : \Gamma \to X \times X$ by

$$E(\gamma) = (e_0(\gamma), e_1(\gamma)) \ .$$

It follows that E is also continuous.

[1]This follows from the fact that the length of continuous curves is lower semicontinuous ([5], Proposition 2.3.4).

We are now in place to define a dynamical transference plan.

Definition 4.1.3 A *dynamical transference plan* consists of a transference plan π and a Borel measure Π on Γ such that $E_\star \Pi = \pi$. It is said to be optimal if π is.

It seems obvious that the movement of mass in an optimal transference plan must occur along geodesics, since these are the shortest paths. However, there can be more than one geodesic between a pair of points so the transport may be divided among them. Informally, if Π gives mass to a certain geodesic we are saying that this geodesic is being used for the transport and how much of the mass flows through that particular one.

We will now see how these dynamical transference plans give us the geodesics in the Wassertian space $P_2(X)$.

Definition 4.1.4 If Π is an optimal dynamical transference plan then for $t \in [0,1]$, put

$$\mu_t = (e_t)_\star \Pi \ .$$

Then the one-parameter family of measures $\{\mu_t\}_{t\in[0,1]}$ is called a *displacement interpolation.*

Intuitively, μ_t is what has become of the mass of μ_0 after it has travelled from time 0 to time t according to the dynamical transference plan Π.

Lemma 4.1.5 *The map $c : [0,1] \to P_2(X)$ given by $c(t) = \mu_t$ has length $L(c) = W_2(\mu_0, \mu_1)$.*

Proof Given $0 \le t \le t' \le 1$, since

$$\begin{aligned}(p_0)_\star[(e_t, e_{t'})_\star \Pi] &= (p_0 \circ (e_t, e_{t'}))_\star \Pi = (e_t)_\star \Pi = \mu_t \\ (p_1)_\star[(e_t, e_{t'})_\star \Pi] &= (p_1 \circ (e_t, e_{t'}))_\star \Pi = (e_{t'})_\star \Pi = \mu_{t'}\end{aligned}$$

we have that $(e_t, e_{t'})_\star \Pi$ is a particular transference plan from μ_t to $\mu_{t'}$. Therefore,

$$\begin{aligned}W_2(\mu_t, \mu_{t'})^2 &\le \int_{X\times X} d(x_0,x_1)^2 d((e_t, e_{t'})_\star \Pi)(x_0,x_1) = \int_\Gamma d(\gamma(t), \gamma(t'))^2 d\Pi(\gamma) \\ &= \int_\Gamma (t'-t)^2 L(\gamma)^2 d\Pi(\gamma) = (t'-t)^2 \int_\Gamma d(\gamma(0), \gamma(1))^2 d\Pi(\gamma) \\ &= (t'-t)^2 \int_{X\times X} d(x_0,x_1)^2 (dE_\star \Pi)(x_0,x_1) = (t'-t)^2 W_2(\mu_0,\mu_1)^2 \ .\end{aligned}$$

Consequently, $L(c) \le W_2(\mu_0, \mu_1)$ [2], and so $L(c) = W_2(\mu_0, \mu_1)$ [3]. ■

With this in mind, the following proposition proves that the Wassertian space of a compact length space is itself a compact length space.

[2] If γ is a curve in a metric space X, meaning that $\gamma : [0,1] \to X$ is a continuous map, then its length is

$$L(\gamma) = \sup_{J\in\mathbb{N}} \sup_{0=t_0\le t_1 \le \cdots \le t_J = 1} \sum_{j=1}^{J} d(\gamma(t_{j-1}), \gamma(t_j)).$$

[3] Clearly $L(c) \ge W_2(\mu_0, \mu_1)$.

Proposition 4.1.6 *Let (X, d) be a compact length space. Then any two points μ_0, $\mu_1 \in P_2(X)$ can be joined by a displacement interpolation.*

To prove this result we will use the following result ([28], Corollary A.6).

Proposition 4.1.7 *Suppose that X, Y are metrizable by complete separable metrics. Let $f : X \to Y$ be a Borel map such that for each $y \in Y$, $f^{-1}(y)$ is a countable union of compact sets. Then $f(X)$ is Borel and there is a Borel section $f(X) \to X$ of f.*

Proof (of Proposition 4.1.6) The endpoints map is Borel, in fact, we mentioned before that it is continuous. Furthermore, since X is a compact length space every pair of points can be connected by a minimizing geodesic [4], this is equivalent to E being surjective. Now, given $(x_0, x_1) \in X \times X$, the Arzela-Ascoli Theorem [5] tells us that $E^{-1}(x_0, x_1)$ is sequentially compact and hence compact. Then the previous proposition gives us the existence of a Borel map $S : X \times X \to \Gamma$ so that $E \circ S = \mathrm{Id}_{X\times X}$. The map S gives a way of joining points by minimizing geodesics in a measurable manner. Finally, given μ_0, $\mu_1 \in P_2(X)$, let π be an optimal transference plan between μ_0 and μ_1, and take $\Pi = S_\star(\pi)$. The corresponding displacement interpolation joins μ_0 to μ_1. Indeed,

$$(p_0)_\star(E_\star(\Pi)) = (p_0)_\star(E_\star(S_\star(\pi))) = (p_0)_\star((S \circ E)_\star(\pi)) = (p_0)_\star\pi = \mu_0$$

and, similarly, $(p_0)_\star(E_\star(\Pi)) = \mu_1$. ■

Corollary 4.1.8 *If X is a compact length space then $P_2(X)$ is a compact length space.*

Proof We have mentioned before that $P_2(X)$ is compact. Now, given μ_0, $\mu_1 \in P_2(X)$, Proposition 4.1.6 gives a displacement interpolation from μ_0 to μ_1 but, following Lemma 4.1.5, this path has precisely length $W_2(\mu_0, \mu_1)$. ■

Furthermore, every Wassertian geodesic arises from a displacement interpolation.

Proposition 4.1.9 *([16], Proposition 2.10) Let (X, d) be a compact length space and let $\{\mu_t\}_{t\in[0,1]}$ be a geodesic path in $P_2(X)$. Then there exists some optimal dynamical transference plan Π such that $\{\mu_t\}_{t\in[0,1]}$ is the displacement interpolation associated to Π.*

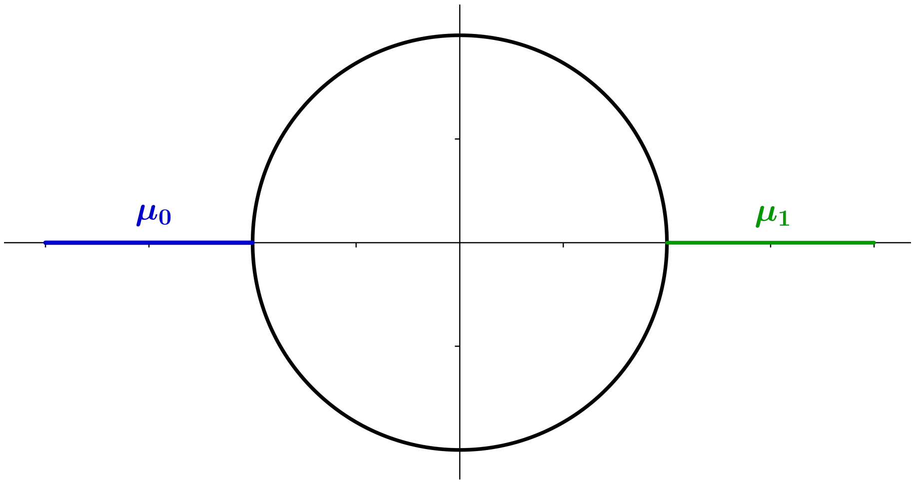
Between two points in $P_2(X)$ there may exist an infinity of Wassertian geodesics. The following is an example of this.

Example 4.1.10 Let $X = A \cup B \cup C$, where A, B and C are subsets of the plane defined as

$$\begin{aligned} A &= \{(x_1, 0) \ : \ -2 \le x_2 \le -1\} \\ B &= \{(x_1, x_2) \ : \ x_1^2 + x_2^2 = 1\} \\ C &= \{(x_1, 0) \ : \ 1 \le x_2 \le 2\} \end{aligned}$$

[4]([5], Theorem 2.5.23) Let (X, d) be a complete locally compact length space. Then this space is strictly intrinsic.

[5]([5], Theorem 2.5.14) In a compact metric space, any sequence of curves with uniformly bounded lengths contains a uniformly converging subsequence.



Take μ_0 the one-dimensional Hausdorff measure of A and μ_1 be the one-dimensional Hausdorff measure of C. Then there is an uncountable number of Wassertian geodesics from μ_0 to μ_1, determined by whether each piece of the mass is taken along the top or bottom semicircumference.

4.1.1 Riemannian case

In the Riemannian case, given μ_0, $\mu_1 \in P_2(M)$ which are absolutely continuous with respect to the Riemannian volume measure dvol_M, there is a unique Wasserstein geodesic c joining μ_0 to μ_1 ([20], Th. 9). In addition, for each $t \in [0,1]$, $c(t)$ is absolutely continuous with respect to dvol_M ([9], Prop. 5.4). Consequently, the space $P_2^{ac}(M)$ of Borel probability measures on M that are absolutely continuous with respect to dvol_M, considered with the metric W_2, is a length space and it is a totally convex subset of $P_2(M)$.

On the other hand, if $\mu_0 = \delta_{m_0}$ and $\mu_1 = \delta_{m_1}$ then some Wassertian geodesics from μ_0 to μ_1 are of the form $\mu_t = \delta_{c(t)}$, where c is a minimizing geodesic from m_0 to m_1.

Chapter 5

Ricci curvature for metric-measure spaces via optimal transport

Finally, in this chapter we will define the notion given by Lott and Villani in [16] of a measured length space X having non-negative N-Ricci curvature, for $N \in [1, \infty)$, or having ∞-Ricci curvature bounded below by K, for $K \in \mathbb{R}$. Lately, the terminology has changed, and it is said that X satisfies the weak $CD(0, N)$ and $CD(K, \infty)$ condition, respectively. The definitions are in terms of the displacement convexity of certain functions on the Wassertian space $P_2(X)$.

For motivation we remind a similar notion which has proved useful in generalising results in Riemannian geometry. Namely, the definition given by Alexandrov of a length space having 'curvature bounded below by K', with K a real number. In [5], Chapter 10, we find a good introduction to this theory. The definition is given in terms of the geodesic triangles in X and, in the case of a Riemannian manifold M with the induced length structure, we recover the Riemannian notion of having sectional curvature bounded below by K. It is important to remark that length spaces with Alexandrov curvature bounded below by K form a closed set with respect to the Gromov-Hausdorff topology on compact metric spaces.

Then, a natural question is whether one can find a notion of 'Ricci curvature bounded below' for length spaces. Some of the attempts to define such a notion have made it clear that it is important to consider a measure on the metric spaces, i.e. the definition must be formulated for metric spaces where a reference non negative measure is also given. This was not obvious, mainly because in the Riemannian case the necessity for the measure is hidden, as a natural reference measure is already given by the volume measure. We will now give some of the reasons which made clear this need for a reference measure.

A first one is the fact that in most of the inequalities where the Ricci curvature appears, the reference measure also appears. We mention the following inequalities, which are given for Riemannian manifolds with Ricci curvature bounded below, as an example.

1. **Brunn-Minkowsky**. Suppose that M has non negative Ricci curvature, and for any A_0, $A_1 \subset M$ compact, let

   $$A_t := \{\gamma(t) \ : \ \gamma \text{ is a constant speed geodesic such that } \gamma_0 \in A_0, \ \gamma_1 \in A_1\}, \ \forall t \in [0,1].$$

   Then it holds

   $$(\mathrm{Vol}(A_t))^{1/n} \geq (1-t)(\mathrm{Vol}(A_0))^{1/n} + t(\mathrm{Vol}(A_1))^{1/n}, \quad \forall t \in [0,1], \tag{5.1}$$

where n is the dimension of M.

2. **Bishop-Gromov** Theorem 2.8.2

A second hint that suggests the need of a reference measure comes from studying stability issues. Suppose that we have a sequence (M_n, g_n) of Riemannian manifolds with Ricci curvature uniformly bounded below by some $K \in \mathbb{R}$. If this sequence converges to another Riemannian manifold (M, g) in the Gromov-Hausdorff topology, does (M, g) have Ricci curvature bounded below by K? The answer is no. However, one can see that when Ricci bounds are not preserved upon the limiting process, then the the volume measures of the (M_n, g_n) do not converge to the volume measure of (M, g). We remark that the same question has a positive answer when considering sectional curvature instead of Ricci curvature.

Furthermore, the definition of 'Ricci curvature bounded below' for length spaces has another element: an $N \in [1, \infty]$ which plays the role of the dimension of X. The necessity for this 'synthetic' dimension can be seen in the Brunn-Minkowski and the Bishop-Gromov inequalities above. Indeed, both of them require the dimension of the manifold to be known, and not just that its Ricci curvature is bounded from below. For example, the Bishop-Gromov inequality says that $r^{-n}\text{vol}(B_r(m))$ is non-increasing in r, where $B_r(m)$ is the r-ball centered at m and n is the dimension of M. If we want to reproduce Bishop-Gromov type inequalities (or derive other useful analytic/geometric consequences) for length spaces, we will therefore need an N doing the job of the dimension. In conclusion, the notion we are looking for will be that of (X, d, ν) having 'N-Ricci curvature bounded below by K'. Since there is no a priori N, this notion will be considered for each $N \in [1, \infty]$.

The following are some of the properties we expect from the definition:

1. **Intrinsicness**. We want the definition to be intrinsic, meaning that it is based only on properties of the space itself. This discards the possibility of saying that (X, d, ν) has 'Ricci curvature bounded below by K' if and only if it is a measured Gromov-Hausdorff limit of Riemannian manifolds with $\text{Ric} \geq Kg$.

2. **Compatibility**. If the metric-measure space is a Riemannian manifold with the canonical volume measure, then the definition must coincide with the usual notion of Ricci curvature bounded below.

3. **Stability**. The curvature bounds are stable under measured Gromov-Hausdorff limits.

4. **Interest**. Geometrical and analytical consequences on a space can be derived from the curvature-dimension condition.

The idea then is to find some property which we know holds for N-dimensional Riemannian manifolds with Ricci curvature bounded below, and turn it into a definition for measured length spaces. One could attempt to use the Bishop-Gromov inequality, at least if $N < \infty$, for example to say that (X, d, ν) has 'non-negative N-Ricci curvature' if and only if for each $x \in \text{supp}(\nu)$, $r^{-N}\nu(B_r(x))$ is non-increasing in r. However, it has been observed ([17], Remark 4.9) that this is not satisfactory.

The definition given in [16] comes from a different field, a branch of applied mathematics: optimal transport. The motivation for this comes from work of Otto-Villani [21] and Cordero-Erausquin-McCann-Schmuckenschläger [9], who showed that optimal transport on a Riemannian manifold is affected by the Ricci tensor. Indeed, they proved the convexity of certain functions on $P(M)$ when M has dimension n and non-negative Ricci curvature. Let's take a closer look at this. Suppose that $A : [0, \infty) \to R$ is a continuous convex function with $A(0) = 0$ such that $\lambda \to \lambda n A(\lambda - n)$ is a convex function on $\mathbb{R}^+$. If $\mu = \rho \frac{d\mathrm{vol}_M}{\mathrm{vol}(M)}$ is an absolutely continuous probability measure then put

$$H_A(\mu) = \int_M A(\rho) \frac{d\mathrm{vol}_M}{\mathrm{vol}(M)}.$$

The statement is that, under the assumption of non-negative Ricci curvature, if μ_0, $\mu_1 \in P(M)$ are absolutely continuous, and $\{\mu_t\}_{t\in[0,1]}$ is the unique Wasserstein geodesic between them, then $H_A(\mu_t)$ is convex in t. In addition, von Renesse and Sturm [23] extended the work of Cordero-Erausquin-McCann-Schmuckenschläger to show that the function H_∞, defined by

$$H_\infty\left(\rho \frac{d\mathrm{vol}_M}{\mathrm{vol}(M)}\right) = \int_M \rho \log \rho \frac{d\mathrm{vol}_M}{\mathrm{vol}(M)},$$

is K-convex along Wasserstein geodesics between absolutely-continuous measures if and only if $\mathrm{Ric} \geq Kg$. Note that the functional $H_\infty\left(\rho \frac{d\mathrm{vol}_M}{\mathrm{vol}(M)}\right)$ is minimized, among absolutely continuous probability measures on M, when $\rho = 1$. Therefore, we can say that H_∞ measures the non-uniformity of μ with respect to $\frac{d\mathrm{vol}_M}{\mathrm{vol}(M)}$.

Example 5.0.11 Take $M = S^2$. Let μ_0 and μ_1 be two small balls of the same radius, centered at the north and south poles respectively. Then $U_\infty(\mu_0) = U_\infty(\mu_1)$. The Wasserstein geodesic from μ_0 to μ_1 takes the ball μ_0 and pushes it down in some way until it becomes μ_1. Along this transformation, say at time $t = 1/2$, the ball has spread out to form a ring. When it spreads, it becomes more uniform with respect to $\frac{d\mathrm{vol}_M}{\mathrm{vol}(M)}$. Consequently, the non-uniformity at an intermediate time is at most that at times $t = 0$ or $t = 1$. This can be seen as a consequence of the convexity of $H_\infty(\mu_t)$ in t, i.e., for $t \in [0, 1]$ we have $H_\infty(\mu_t) \leq H_\infty(\mu_0) = H_\infty(\mu_1)$. One may then see the displacement convexity of H_∞ as an averaged form of the focusing property of positive curvature. However, this example does not indicate why Ricci curvature is the relevant one, instead of some other curvature, but it gives an indication of why curvature is related to displacement convexity.

Then, the idea, used independently by Lott-Villani and Sturm, is to define the property 'N-Ricci curvature bounded below by K', for a measured length space (X, d, ν), in terms of the convexity of certain entropy functionals along optimal transport paths in the space $P(X)$.

5.1 Functionals on the Wassertian space

In this section we will study certain functionals on the Wassertian space $P_2(X)$. For this, we will define the notion of λ-displacement convexity, and some variants. As always, by measured length space we mean a triple (X, d, ν), where (X, d) is a compact length space and ν is a Borel probability measure on X. Given $U : [0, \infty) \to \mathbb{R}$ a convex lower semicontinuous function, we write $U'(\infty) = \lim_{r\to\infty} U'_+(r)$.

Definition 5.1.1 Define

$$P_2(X,\nu) = \{\mu \in P_2(X) \ : \ \mathrm{supp}(\mu) \subset \mathrm{supp}(\nu)\}$$

and $P_2^{\mathrm{ac}}(X,\nu)$ the elements of $P_2(X,\nu)$ that are absolutely continuous with respect to ν.

Definition 5.1.2 Let (X,d,ν) be a measured length space and let U be a continuous convex function on $[0,\infty)$ with $U(0)=0$. We define the functional $U_\nu : P_2(X) \to \mathbb{R}\cup\{\infty\}$ by

$$U_\nu(\mu) = \int_X U(\rho(x))d\nu(x) + U'(\infty)\mu_s(X),$$

where

$$\mu = \rho\nu + \mu_s$$

is the Lebesgue decomposition of μ with respect to ν into an absolutely continuous part $\rho\nu$ and a singular part μ_s.

Note that if $U'(\infty)=\infty$, then finiteness of $U_\nu(\mu)$ implies that μ is absolutely continuous with respect to ν. We will now prove that, as a function of μ, U_ν is minimized at ν.

Lemma 5.1.3 *Let U be a continuous convex function on $[0,\infty)$ with $U(0)=0$.*

$$U_\nu(\mu) \geq U_\nu(\nu) = U(1).$$

If μ is absolutely continuous with respect to ν then this is just Jensen's inequality

$$\int_X U(\rho(x))d\nu(x) \geq U\left(\int_X \rho(x)d\nu(x)\right) .$$

Proof Since U is convex, for any $\alpha \in (0,1)$ we have

$$U(\alpha r + 1 - \alpha) \leq \alpha U(r) + (1-\alpha)U(1),$$

so

$$U(r) - U(1) \geq \frac{U(\alpha r + 1 - \alpha) - U(1)}{\alpha}.$$

We take ρ as in the previous definition. Then

$$\int_X U(\rho)d\nu - U(1) \geq \int_X \frac{U(\alpha\rho + 1 - \alpha) - U(1)}{\alpha\rho - \alpha}(\rho - 1)d\nu \tag{5.2}$$

where we take the integrand of the right-hand-side to vanish whenever $\rho(x)=1$. We separate the right-hand-side of (5.2) according to whether $\rho(x) \leq 1$ or $\rho(x) > 1$. Now, since U is convex, it admits both left and right derivatives and they are non-decreasing. We write $U'(\infty) = \lim_{r\to\infty} U'_+(r) \in \mathbb{R}\cup\{\infty\}$. Moreover, the convexity of U implies that there exists $\varepsilon > 0$ such that U is monotone both in $(1-\varepsilon, 1)$ and $(1, 1+\varepsilon)$. Then, following the monotone convergence theorem, for $\rho \leq 1$ we have

$$\lim_{\alpha\to 0^+} \int_X \frac{U(\alpha\rho + 1 - \alpha) - U(1)}{\alpha\rho - \alpha}(\rho - 1)\chi_{\rho\leq 1}d\nu = U'_-(1)\int_X (\rho - 1)\chi_{\rho\leq 1}d\nu,$$

while for $\rho > 1$ we have

$$\lim_{\alpha\to 0^+}\int_X \frac{U(\alpha\rho+1-\alpha)-U(1)}{\alpha\rho-\alpha}(\rho-1)\chi_{\rho>1}d\nu = U'_+(1)\int_X(\rho-1)\chi_{\rho>1}d\nu.$$

$$\begin{aligned}\int_X U(\rho)d\nu - U(1) &\geq U'_-(1)\int_X(\rho-1)d\nu + (U'_+(1)-U'_-(1))\int_X(\rho-1)\chi_{\rho>1}d\nu\\ &\geq U'_-(1)\int_X(\rho-1)d\nu \geq U'(\infty)\int_X(\rho-1)d\nu\\ &= -U'(\infty)\mu_s(X)\end{aligned}$$

where the last step follows from the fact that $\mu = \rho\nu+\mu_s$ and both μ and ν are probability measures. Since $U_\nu(\nu) = U(1)$, we get the result. ■

We go on with the first notion which will play an important role in the definition of Ricci curvature on an abstract length space.

Definition 5.1.4 Given a compact measured length space (X, d, ν) and a number $\lambda \in \mathbb{R}$, we say that U_ν is

1. λ-*displacement convex* if for all Wassertian geodesics $\{\mu_t\}_{t\in[0,1]}$ with μ_0, $\mu_1 \in P_2(X,\nu)$, we have

$$U_\nu(\mu_t) \leq tU_\nu(\mu_1) + (1-t)U_\nu(\mu_0) - \frac{1}{2}\lambda t(1-t)W_2(\mu_0,\mu_1)^2 \tag{5.3}$$

   for all $t \in [0,1]$.

2. *weakly* λ-*displacement convex* if for all μ_0, $\mu_1 \in P_2(X,\nu)$, there is *some* Wassertian geodesic from μ_0 to μ_1 along which (5.3) is satisfied.

3. (*weakly*) λ-a.c. *displacement convex* if the condition is satisfied when we take μ_0, $\mu_1 \in P_2^{ac}(X,\nu)$.

We write *displacement convex* instead of 0-displacement convex.

In summary, 'weakly' means that we only ask for the condition to hold on some geodesic, instead of all geodesics, and a.c. means that we only require the condition to hold when the two measures are absolutely continuous.

Remark 5.1.5 1. In the previous definition we have that $\mathrm{supp}(\mu_0) \subset \mathrm{supp}(\nu)$ and $\mathrm{supp}(\mu_1) \subset \mathrm{supp}(\nu)$, but we don not know if $\mathrm{supp}(\mu_t) \subset \mathrm{supp}(\nu)$ for $t \in (0,1)$.

2. If U_ν is λ-displacement convex and $\mathrm{supp}(\nu) = X$ then the function $t \to U_\nu(\mu_t)$ is λ-convex on $[0,1]$. Indeed, for all $0 \leq s \leq s' \leq 1$ and $t \in [0,1]$,

$$U_\nu(\mu_{ts'+(1-t)s}) \leq tU_\nu(\mu_{s'}) + (1-t)U_\nu(\mu_s) - \frac{1}{2}\lambda t(1-t)(s'-s)^2W_2(\mu_0,\mu_1)^2. \tag{5.4}$$

   This is not necessarily true if we only assume that U_ν is weakly λ-displacement convex.

Aside from the obvious implications, we also have the following one.

Proposition 5.1.6 *Let U be a continuous convex function on $[0,\infty)$ with $U(0) = 0$. Let (X, d, ν) be a compact measured length space. Then U_ν is weakly λ-displacement convex if and only if it is weakly λ-a.c. displacement convex.*

Proof We want to show that if U_ν is weakly λ-a.c. displacement convex then it is weakly λ-displacement convex. That is, for μ_0, $\mu_1 \in P_2(X, \nu)$, which are not necessarily absolutely continuous with respect to ν, we must show that there is some Wassertian geodesic $\{\mu_t\}_{t\in[0,1]}$ from μ_0 to μ_1 along which

$$U_\nu(\mu_t) \leq tU_\nu(\mu_0) + (1-t)U_\nu(\mu_1) - \frac{1}{2}\lambda t(1-t)W_2(\mu_0, \mu_1)^2. \tag{5.5}$$

We can assume that $U_\nu(\mu_0) < \infty$, as otherwise (5.5) is true for any Wassertian geodesic from μ_0 to μ_1. From ([16], Theorem C.12 in Appendix C), we get that there are sequences $\{\mu_{k,0}\}_{k=1}^\infty$ and $\{\mu_{k,1}\}_{k=1}^\infty$ in $P_2^{ac}(X, \nu)$ such that

$$\begin{cases} \lim_{k\to\infty} \mu_{k,0} = \mu_0 & \lim_{k\to\infty} \mu_{k,1} = \mu_1, \\ \lim_{k\to\infty} U_\nu(\mu_{k,0}) = U_\nu(\mu_0) & \lim_{k\to\infty} U_\nu(\mu_{k,1}) = U_\nu(\mu_1). \end{cases} \tag{5.6}$$

Since $\mu_{k,0}$, $\mu_{k,1} \in P_2^{ac}(X, \nu)$ and U_ν is weakly λ-a.c. displacement convex there exists a minimal geodesic $c_k : [0, 1] \to P_2(X)$ from $\mu_{k,0}$ to $\mu_{k,1}$ such that for all $t \in [0, 1]$,

$$U_\nu(c_k(t)) \leq tU_\nu(\mu_{k,1}) + (1-t)U_\nu(\mu_{k,0}) - \frac{1}{2}\lambda t(1-t)W_2(\mu_{k,0}, \mu_{k,1})^2. \tag{5.7}$$

After taking a subsequence, we may assume that the geodesics $\{c_k\}_{k=1}^\infty$ converge uniformly to a geodesic $c : [0, 1] \to P_2(X)$ from μ_0 to μ_1 ([5], Theorems 2.5.14, 2.5.17) [1]. The lower semicontinuity of U_ν ([16] Theorem B.33(i) in Appendix B) [2], implies that

$$U_\nu(c(t)) \leq \liminf_{k\to\infty} U_\nu(c_k(t)).$$

This, together with (5.6) and (5.7) gives us the proposition. ■

In fact, the previous proof gives the following stronger result.

Lemma 5.1.7 *Let U be a continuous function on $[0,\infty)$ with $U(0) = 0$. Let (X, d, ν) be a compact measured length space. Suppose that for all μ_0, $\mu_1 \in P_2^{ac}(X, \nu)$ with continuous densities, there is some Wassertian geodesic from μ_0 to μ_1 along which (5.3) is satisfied. Then U_ν is weakly λ-displacement convex.*

There is also the following non-trivial implication under sufficient conditions.

[1] **Theorem 2.5.14** (Arzela-Ascoli Theorem). In a compact metric space, any sequence of curves with uniformly bounded lengths contains a uniformly converging subsequence.

Proposition 2.5.17 If shortest paths γ_i in a length space (X, d) converge to a path γ as $i \to \infty$, then γ is also a shortest path.

[2] **Theorem B.33.** Let X be a compact Hausdorff space. Let $U : [0,\infty) \to \mathbb{R}$ be a continuous convex function with $U(0) = 0$. Then $U_\nu(\mu)$ is a lower semicontinuous function of $(\mu, \nu) \in P(X) \times P(X)$. That is, if $\{\mu_k\}_{k=1}^\infty$ and $\{\nu\}_{k=1}^\infty$ are sequences in $P(X)$ with $\lim_{k\to\infty} \mu_k = \mu$ and $\lim_{k\to\infty} \nu_k = \nu$ in the weak-$\star$ topology then

$$U_\nu(\mu) \leq \liminf_{k\to\infty} U_{\nu_k}(\mu_k).$$

Lemma 5.1.8 *(i) Suppose that X has the property that for each minimizing geodesic $c : [0,1] \to P_2(X)$, there is some $\delta_c > 0$ so that the minimizing geodesic between $c(t)$ and $c(t')$ is unique whenever $|t - t'| \leq \delta_c$. Suppose that* $\operatorname{supp}(\nu) = X$*. If U_ν is weakly λ-displacement convex then it is λ-displacement convex.*

(ii) Suppose that $P_2^{ac}(X,\nu)$ is totally convex in $P_2(X)$. Suppose that X has the property that for each minimizing geodesic $c : [0,1] \to P_2^{ac}(X,\nu)$, there is some $\delta_c > 0$ so that the minimizing geodesic between $c(t)$ and $c(t')$ is unique whenever $|t - t'| \leq \delta_c$. Suppose that $\operatorname{supp}(\nu) = X$*. If U_ν is weakly λ-a.c. displacement convex then it is λ-a.c. displacement convex.*

Proof (i) Suppose that U_ν is weakly λ-displacement convex. Given a minimizing geodesic $c : [0,1] \to P_2(X)$, we want to show that U_ν is λ-convex along c.

By definition of weak λ-displacement convexity we have that, for all $0 \leq s \leq s' \leq 1$, there is some geodesic from $c(s)$ to $c(s')$ so that (5.4) is satisfied for all $t \in [0,1]$. But, if $|s - s'| \leq \delta_c$, then this geodesic must be $c \restriction_{[s,s']}$. It follows that the function $s \mapsto U_\nu(c(s))$ is λ-convex on each interval $[s,s']$ with $|s - s'| \leq \delta_c$, and hence on $[0,1]$.

(ii) Apply the same argument with absolutely continuous measures.

■

The following functionals will play an important role.

Definition 5.1.9

$$U_N(r) = \begin{cases} Nr(1 - r^{-1/N}) & \text{if } 1 < N < \infty, \\ r \log r & \text{if } N = \infty \end{cases} \tag{5.8}$$

Definition 5.1.10 We define $H_{N,\nu} : P_2(X) \to [0,\infty]$ as the functional associated to U_N:

1. For $N \in (1,\infty)$,

$$H_{N,\nu} = N - N \int_X \rho^{1-\frac{1}{N}} d\nu, \tag{5.9}$$

   where $\rho\nu$ is the absolutely continuous part in the Lebesgue decomposition of μ with respect to ν.

2. For $N = \infty$, the functional $H_{\infty,\nu}$ is defined as follows: if μ is absolutely continuous with respect to ν, with $\mu = \rho\nu$, then

$$H_{\infty,\nu}(\mu) = \int_X \rho \log \rho d\nu, \tag{5.10}$$

   while if μ is not absolutely continuous with respect to ν then $H_{\infty,\nu}(\mu) = \infty$.

To verify that $H_{N,\nu}$ is indeed the functional associated to U_N, we note that $U_N'(\infty) = N$ and write

$$\begin{aligned} N \int_X \rho \left(1 - \rho^{-\frac{1}{N}}\right) d\nu + N\mu_s(X) &= N \int_X \rho \left(1 - \rho^{-\frac{1}{N}}\right) d\nu + N \left(1 - \int_X \rho d\nu\right) \\ &= N - N \int_X \rho^{1-\frac{1}{N}} d\nu. \end{aligned}$$

Remark 5.1.11 As a function of μ, $H_{N,\nu}(\mu)$ attains a minimum when $\mu = \nu$. Therefore, in some sense, $H_{N,\nu}(\mu)$ is a way of measuring the non-uniformity of μ with respect to ν.

5.2 Weak displacement convexity and measured Gromov-Hausdorff limits

In this section we first show that if a sequence of compact metric spaces converges in the Gromov-Hausdorff topology then their associated Wasserstein spaces also converge in the Gromov-Hausdorff topology. We then prove that weak displacement convexity of U_ν is preserved by measured Gromov-Hausdorff limits. Finally, we define the notion of weak λ-displacement convexity for a family $\mathcal{F}$ of convex functions U.

Proposition 5.2.1 *If $f : (X_1, d_1) \to (X_2, d_2)$ is an ε-Gromov-Hausdorff isometry then $f_\star : P_2(X_1) \to P_2(X_2)$ is an $\tilde{\varepsilon}$-Gromov-Hausdorff isometry, where*

$$\tilde{\varepsilon} = 4\varepsilon + \sqrt{3\varepsilon(2\text{diam}(X_2) + 3\varepsilon)}.$$

Proof Given μ_1, $\mu_1' \in P_2(X_1)$, let π_1 be an optimal transference plan for μ_1 and μ_1'. Then $\pi_2 = (f \times f)_\star \pi_1 \in P_2(X_2 \times X_2)$ is a transference plan between $f_\star\mu_1$ and $f_\star\mu_1'$. Now,

$$\begin{aligned} W_2(f_\star\mu_1, f_\star\mu_1')^2 &\leq \int_{X_2\times X_2} d_2(x_2, y_2)^2 d\pi_2(x_2, y_2) \\ &= \int_{X_1\times X_1} d_2(f(x_1), f(y_1))^2 d\pi_1(x_1, y_1). \end{aligned}$$

Since

$$|d_2(f(x_1), f(y_1))^2 - d_1(x_1, y_1)^2| = |d_2(f(x_1), f(y_1)) - d_1(x_1, y_1)|(d_2(f(x_1), f(y_1)) + d_1(x_1, y_1)),$$

and $|d_2(f(x_1), f(y_1)) - d_1(x_1, y_1)| \leq \varepsilon$ by definition of ε-isometry (Definition 1.1.11), we have

$$|d_2(f(x_1), f(y_1))^2 - d_1(x_1, y_1)^2| \leq \varepsilon(2\text{diam}(X_1) + \varepsilon)$$

and

$$|d_2(f(x_1), f(y_1))^2 - d_1(x_1, y_1)^2| \leq \varepsilon(2\text{diam}(X_2) + \varepsilon).$$

It follows that

$$W_2(f_\star\mu_1, f_\star\mu_1')^2 \leq W_2(\mu_1, \mu_1')^2 + \varepsilon(2\text{diam}(X_1) + \varepsilon) \tag{5.11}$$

and

$$W_2(f_\star\mu_1, f_\star\mu_1')^2 \leq W_2(\mu_1, \mu_1')^2 + \varepsilon(2\text{diam}(X_2) + \varepsilon). \tag{5.12}$$

Then, from (5.12), we get

$$W_2(f_\star\mu_1, f_\star\mu_1') \leq W_2(\mu_1, \mu_1') + \sqrt{\varepsilon(2\text{diam}(X_2) + \varepsilon)}. \tag{5.13}$$

Now, let $f' : (X_2, d_2) \to (X_1, d_1)$ be an approximate inverse of f, which as we know is a 3ε-isometry. We proceed analogously as before taking f', $f_\star\mu_1$ and $f_\star\mu_1'$ in place of f, μ_1 and μ_2, respectively. Then, using the corresponding equation to (5.11) we obtain

$$W_2(f_\star'(f_\star\mu_1), f_\star'(f_\star\mu_1')) \leq W_2(f_\star\mu_1, f_\star\mu_1') + \sqrt{3\varepsilon(2\text{diam}(X_2) + 3\varepsilon)}. \tag{5.14}$$

But $f' \circ f$ is an admissible Monge transport between μ_1 and $(f' \circ f)_\star\mu_1$, or between μ_1' and $(f' \circ f)_\star\mu_1'$, and $d(x_1, (f' \circ f)(x_1)) \leq 2\varepsilon$ for all $x_1 \in X_1$, thus

$$\begin{aligned} W_2((f' \circ f)_\star\mu_1, \mu_1) &\leq \int_{X_1\times X_1} d(x_1, y_1)^2 d(\text{Id} \times (f' \circ f))_\star\mu_1(x_1, y_1) \\ &= \int_{X_1\times X_1} d(x_1, (f' \circ f)(x_1))^2 d\mu_1 \leq 2\varepsilon \end{aligned}$$

and, analogously,

$$W_2((f' \circ f)_\star \mu_1', \mu_1') \leq 2\varepsilon.$$

It follows by (5.14) and the triangle inequality,

$$W_2(\mu_1, \mu_1') \leq W_2(f_\star \mu_1, f_\star \mu_1') + 4\varepsilon + \sqrt{3\varepsilon(2\mathrm{diam}(X_2) + 3\varepsilon)}. \tag{5.15}$$

Equations (5.13) and (5.15) show that

$$|W_2(f_\star \mu_1, f_\star \mu_1') - W_2(\mu_1, \mu_1')| \leq 4\varepsilon + \sqrt{3\varepsilon(2\mathrm{diam}(X_2) + 3\varepsilon)}\ .$$

Since μ_1 and μ_1' are arbitrary, the first condition of Definition 1.1.11 is satisfied.

Finally, given $\mu_2 \in P_2(X_2)$, consider the Monge transport $f \circ f'$ from μ_2 to $(f \circ f')_\star \mu_2$. Then, since $d_2(x_2, (f \circ f')(x_2)) \leq \varepsilon$ for all $x_2 \in X_2$, we have that $W_2(\mu_2, f_\star(f'_\star \mu_2)) \leq \varepsilon$. We have therefore proven that for each $\mu_2 \in P_2(X_2)$ there exists μ_1 (in this case $\mu_1 = f'_\star \mu_2$) such that $W_2(\mu_2, f_\star(\mu_1)) \leq \varepsilon$, i.e. $f_\star(P_2(X_1))$ is an ε-net in $P_2(X_2)$. That is, the second condition is satisfied. ■

Corollary 5.2.2 *If a sequence of a compact metric spaces $\{(X_i, d_i)\}_{i=1}^\infty$ converges in the Gromov-Hausdorff topology to a compact metric spaces (X, d) then $\{P_2(X_i)\}_{i=1}^\infty$ converges in the Gromov-Hausdorff topology to $P_2(X)$.*

We proceed to study the stability of weak displacement convexity under Gromov-Hausdorff limits.

Theorem 5.2.3 *Let $\{X_i, d_i, \nu_i)\}_{i=1}^\infty$ be a sequence of compact measured length spaces so that $\lim_{i\to\infty}(X_i, d_i, \nu_i) = (X, d, \nu_\infty)$ in the Gromov-Hausdorff topology. Let U be a continuous convex function on $[0, \infty)$ with $U(0) = 0$. Given $\lambda \in \mathbb{R}$, suppose that for all i, U_{ν_i} is weakly λ-displacement convex for (X_i, d_i, ν_i). Then U_{ν_∞} is weakly λ-displacement convex for (X, d, ν).*

Proof By Lemma 5.1.7, it is enough to show that for any μ_0, $\mu_1 \in P_2(X)$ with continuous densities with respect to ν_∞, there is some Wassertian geodesic joining them along which inequality (5.3) holds for U_{ν_∞}. Again, we assume that $U_{\nu_\infty}(\mu_0) < \infty$ and $U_{\nu_\infty}(\mu_1) < \infty$, as otherwise any Wassertian geodesic works.

Since μ_0, $\mu_1 \in P_2^{ac}(X, d, \nu_\infty)$, we may write $\mu_0 = \rho_0 \nu_\infty$ and $\mu_1 = \rho_1 \nu_\infty$. From Definition 1.3.1, we obtain the existence of ε_i-isometries $f_i : X_i \to X$ such that $\lim_{i\to\infty} \varepsilon_i = 0$ and $\lim_{i\to\infty}(f_i)_\star \nu_i = \nu_\infty$. We will now approximately-lift the measures μ_0 and μ_1 to X_i. That is, we use f_i to pullback the densities to X_i, then multiply by ν_i and then normalize to get probability measures. For this we first need to do the following observation. Since ρ_0 is continuous,

$$\lim_{i\to\infty} \int_X \rho_0 d(f_i)_\star \nu_i = \int_X \rho_0 d\nu_\infty = \int_X d\mu_0 = 1,$$

thus $\int_X \rho_0 d(f_i)_\star \nu_i > 0$ for i sufficiently large. Similarly, $\int_X \rho_1 d(f_i)_\star \nu_i > 0$ for i sufficiently large. Then, for such i, put

$$\mu_{i,0} = \frac{(f_i^\star \rho_0)\nu_i}{\int_X \rho_0 d(f_i)_\star \nu_i}$$

and

$$\mu_{i,1} = \frac{(f_i^\star \rho_1)\nu_i}{\int_X \rho_1 d(f_i)_\star \nu_i}\ .$$

We have

$$(f_i)_\star\mu_{i,0} = \frac{\rho_0 (f_i)_\star \nu_i}{\int_X \rho_0 d(f_i)_\star \nu_i} \tag{5.16}$$

and

$$(f_i)_\star\mu_{i,1} = \frac{\rho_1 (f_i)_\star \nu_i}{\int_X \rho_1 d(f_i)_\star \nu_i} \ .$$

By hypothesis, U_{ν_i} is weakly λ-displacement convex for (X_i, d_i, ν_i) and all i. It follows that there exist geodesics $c_i : [0,1] \to P_2(X_i)$ with $c_i(0) = \mu_{i,0}$ and $c_i(1) = \mu_{i,1}$ so that for all $t \in [0,1]$, we have

$$U_{\nu_i}(c_i(t)) \leq tU_{\nu_i}(\mu_{i,1}) + (1-t)U_{\nu_i}(\mu_{i,0}) - \frac{1}{2}\lambda t(1-t)W_2(\mu_{i,0}, \mu_{i,1})^2. \tag{5.17}$$

We now want to take a convergent subsequence of these Wassertian geodesics in an appropriate sense to get a Wassertian geodesic in $P(X)$. From Lemma 1.2.1 and Corollary 5.2.2, after passing to a subsequence the maps $(f_i)_\star \circ c_i : [0,1] \to P_2(X)$ converge uniformly to a continuous map $c : [0,1] \to P_2(X)$. Indeed, Corollary 5.2.2 tells us that $\{P_2(X_i)\}$ converges in the Gromov-Hausdorff topology to $P_2(X)$, and in fact it does so with the $\tilde{\varepsilon}_i$-isometries $(f_i)_\star : P_2(X_i) \to P_2(X)$, where

$$\tilde{\varepsilon}_i = 4\varepsilon_i + \sqrt{3\varepsilon_i(2\mathrm{diam}(X_2) + 3\varepsilon_i)} \xrightarrow{i\to\infty} 0.$$

Then, in the notation of Lemma 1.2.1 we have $Y_i = P_2(X_i)$, $Y = P_2(X)$, $g_i = (f_i)_\star$ and $\alpha_i = c_i$ (the X_i and X that appear in the lemma are all equal to $[0,1]$, and the f_i are taken to be the identity). Then, it only remains to prove that the sequence of maps $\{c_i\}_{i=1}^\infty$ are asymptotically equicontinuous. But this follows from the fact that the diameters of the $P_2(X_i)$ are uniformly bounded and, since the c_i are geodesics,

$$W_2(c_i(t), c_i(t')) = |t - t'| W_2(\mu_{i,0}, \mu_{i,1}) \ . \tag{5.18}$$

for any t, $t' \in [0,1]$.

Given $F \in C(X)$, since ρ_0 is also continuous, we have that

$$\lim_{i\to\infty} \int_X F d(f_i)_\star \mu_{i,0} = \lim_{i\to\infty} \int_X F\rho_0 \frac{d(f_i)_\star\nu_i}{\int_X \rho_0 d(f_i)_\star \nu_i} = \int_X F\rho_0 d\nu_\infty. \tag{5.19}$$

where the first equality follows from (5.16). Consequently, $\lim_{i\to\infty}(f_i)_\star\mu_{i,0} = \rho_0\nu_\infty = \mu_0$. Similarly, $\lim_{i\to\infty}(f_i)_\star\mu_{i,1} = \mu_1$. It follows from Corollary 5.2.2 that

$$\lim_{i\to\infty} W_2(\mu_{i,0}, \mu_{i,1}) = W_2(\mu_0, \mu_1). \tag{5.20}$$

Making i tend to ∞ in (5.18) we get that $W_2(c(t), c(t')) = |t-t'|W_2(\mu_0, \mu_1)$. Therefore, c is a Wassertian geodesic. The hard part is to pass to the limit in (5.17) as $i \to \infty$. For the right-hand side the job is easy but for the left we will make use the lower semicontinuity of $(\mu, \nu) \to U_\nu(\mu)$.

Now,

$$U_{\nu_i}(\mu_{i,0}) = \int_{X_i} U\left(\frac{f_i^\star \rho_0}{\int_X \rho_0 d(f_i)_\star \nu_i}\right) d\nu_i = \int_X U\left(\frac{\rho_0}{\int_X \rho_0 d(f_i)_\star \nu_i}\right) d(f_i)_\star \nu_i \ . \tag{5.21}$$

Since

$$\lim_{i\to\infty} U\left(\frac{\rho_0}{\int_X \rho_0 d(f_i)_\star \nu_i}\right) = U(\rho_0) \tag{5.22}$$

uniformly on X, it follows that

$$\lim_{i\to\infty}\int_X U\left(\frac{\rho_0}{\int_X \rho_0 d(f_i)_\star \nu_i}\right) d(f_i)_\star\nu_i = \lim_{i\to\infty}\int_X U(\rho_0)d(f_i)_\star\nu_i = \int_X U(\rho_0)d\nu_\infty. \tag{5.23}$$

Thus $\lim_{i\to\infty} U_{\nu_i}(\mu_{i,0}) = U_{\nu_\infty}(\mu_0)$. Similarly, $\lim_{i\to\infty} U_{\nu_i}(\mu_{i,1}) = U_{\nu_\infty}(\mu_1)$.

It follows from ([16] Theorem B.33(ii) in Appendix B) [3] that

$$U_{(f_i)_\star\nu_i}((f_i)_\star c_i(t)) \le U_{\nu_i}(c_i(t)).$$

Then, for any $t \in [0,1]$, we can combine this with the lower semicontinuity of $(\mu,\nu) \to U_\nu(\mu)$ ([16] Theorem B.33(i) in Appendix B) to obtain

$$U_{\nu_\infty}(c(t)) \le \liminf_{i\to\infty} U_{(f_i)_\star\nu_i}((f_i)_\star c_i(t)) \le \liminf_{i\to\infty} U_{\nu_i}(c_i(t)). \tag{5.24}$$

Combining this with (5.20) and the preceding results, we can take $i \to \infty$ in (5.17) and find

$$U_{\nu_\infty}(c(t)) \le tU_{\nu_\infty}(\mu_1) + (1-t)U_{\nu_\infty}(\mu_0) - \frac{1}{2}\lambda t(1-t)W_2(\mu_0,\mu_1)^2\ . \tag{5.25}$$

This concludes the proof. ■

Definition 5.2.4 Let $\mathcal{F}$ be a family of continuous convex functions U on $[0,\infty)$ with $U(0) = 0$. Given a function $\lambda : \mathcal{F} \to \mathbb{R} \cup \{-\infty\}$, we say that a compact measured length space (X, d, ν) is weakly λ-displacement convex for the family $\mathcal{F}$ if for any μ_0, $\mu_1 \in P_2(X,\nu)$, one can find a Wassertian geodesic $\{\mu_t\}_{t\in[0,1]}$ from μ_0 to μ_1 so that for each $U \in \mathcal{F}$, U_ν satisfies

$$U_\nu(\mu_t) \le tU_\nu(\mu_1) + (1-t)U_\nu(\mu_0) - \frac{1}{2}\lambda(U)t(1-t)W_2(\mu_0,\mu_1)^2 \tag{5.26}$$

for all $t \in [0,1]$.

There is also an obvious definition of weakly λ-a.c. displacement convex for the family $\mathcal{F}$, in which one just requires the condition to hold when μ_0, $\mu_1 \in P_2^{ac}(X,\nu)$. Note that in the previous definition, the same Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ is supposed to work for all of the functions $U \in \mathcal{F}$. Hence if (X,d,ν) is weakly λ-displacement convex for the family $\mathcal{F}$ then it is weakly $\lambda(U)$-displacement convex for each $U \in \mathcal{F}$, but the converse is not *a priori* true.

The proof of Theorem 5.2.3 establishes the following result.

[3]**Theorem B.33.** Let X be a compact Hausdorff space. Let $U : [0,\infty) \to \mathbb{R}$ be a continuous convex function with $U(0) = 0$. Then $U_\nu(\mu)$ is non-increasing under pushforward. That is, if Y is a compact Hausdorff space and $f : X \to Y$ is a Borel map then

$$U_{f_\star\nu}(f_\star\mu) \le U_\nu(\mu).$$

Theorem 5.2.5 *Let $\{(X_i, d_i, \nu_i)\}_{i=1}^{\infty}$ be a sequence of compact measured length spaces with $\lim_{i\to\infty}(X_i, d_i, \nu_i) = (X, d, \nu_\infty)$ in the Gromov-Hausdorff topology. Let $\mathcal{F}$ be a family of continuous convex functions U on $[0, \infty)$ with $U(0) = 0$. Given a function $\lambda : \mathcal{F} \to \mathbb{R} \cup \{-\infty\}$, suppose that each (X_i, d_i, ν_i) is weakly λ-displacement convex for the family $\mathcal{F}$. Then (X, d, ν_∞) is weakly λ-displacement convex for the family $\mathcal{F}$.*

Further, the proof of Proposition 5.1.6 establishes the following result.

Proposition 5.2.6 *Let $\mathcal{F}$ be a family of continuous convex functions U on $[0, \infty)$ with $U(0) = 0$. Given a function $\lambda : \mathcal{F} \to \mathbb{R} \cup \{-\infty\}$, (X, d, ν) is weakly λ-displacement convex for the family $\mathcal{F}$ if and only if it is weakly λ-a.c. displacement convex for the family $\mathcal{F}$.*

5.3 Weak $CD(0, N)$ and $CD(K, \infty)$ conditions

The definition of the notion of a measured length space (X, d, ν) having non-negative N-Ricci curvature, or ∞-Ricci curvature bounded below by $K \in \mathbb{R}$, will be in terms of certain classes $\mathcal{DC}_N$ of convex functions. Right after making the definition we will show that these properties pass to totally convex subsets of X. Then we will prove the property that we are most concerned about. That is, the Ricci curvature definitions are preserved by measured Gromov-Hausdorff limits.

After this we will give the generalisation of some of the results that hold on Riemannian manifolds. For example, that non-negative N-Ricci curvature for $N < \infty$ implies a Bishop-Gromov type inequality.

5.3.1 Displacement convex classes

We define the class of convex functions that we were speaking about. This was introduced by McCann in [19].

Definition 5.3.1 For a continuous convex function $U : [0, \infty) \to \mathbb{R}$ with $U(0) = 0$, define the non-negative function

$$p(r) = rU'_+(r) - U(r), \tag{5.27}$$

with $p(0) = 0$. This definition can be motivated by physics. Namely, suppose that U defines an internal energy for a continuous medium then p can be thought of as pressure. By analogy, if U is C^2-regular on $(0, \infty)$ then we define the 'iterated pressure'

$$p_2(r) = rp'(r) - p(r).$$

Definition 5.3.2 For $N \in [1, \infty)$, we define $\mathcal{DC}_N$ to be the set of all continuous convex functions U on $[0, \infty)$, with $U(0) = 0$, such that the function

$$\phi(\lambda) = \lambda^N U(\lambda^{-N}) \tag{5.28}$$

is convex on $(0, \infty)$.

We define $\mathcal{DC}_\infty$ to be the set of all continuous convex functions U on $[0, \infty)$, with $U(0) = 0$, such that the function

$$\phi(\lambda) = e^{\lambda} U(e^{-\lambda}) \tag{5.29}$$

is convex on $(-\infty, \infty)$.

The convexity of U implies that ϕ is non-increasing in λ because $\frac{U(x)}{x}$ is non-decreasing in x. Below are some useful facts about the classes $\mathcal{DC}_N$.

Lemma 5.3.3 *If $N \leq N'$ then $\mathcal{DC}_{N'} \subset \mathcal{DC}_N$. Consequently, the smallest class of all is $\mathcal{DC}_\infty$, while $\mathcal{DC}_1$ is the largest.*

Proof If $N' < \infty$, let ϕ_N and $\phi_{N'}$ denote the corresponding functions. Then $\phi_N(\lambda) = \phi_{N'}\left(\lambda^{N/N'}\right)$. But $x \to x^{N/N'}$ is concave on $[0,\infty)$ and the composition of a non-increasing convex function and a concave function is convex so the results follows.

If $N' = \infty$ and $N < \infty$ then $\phi_N = \phi_\infty(N \log \lambda)$, and the same argument is valid. ■

Lemma 5.3.4 *For $N \in [1,\infty]$, if U is a continuous convex function on $[0,\infty)$ that is C^2-regular on $(0,\infty)$ with $U(0) = 0$, then the following statements are equivalent*

1. $U \in \mathcal{DC}_N$
2. *The function $r \to p(r)/r^{1-\frac{1}{N}}$ is non-decreasing on $(0,\infty)$.*
3. $p_2 \geq -\frac{p}{N}$.

Proof (i)↔(ii) Suppose first that $N \in [1,\infty)$, and write $r(\lambda) = \lambda^{-N}$. Then

$$\phi'(\lambda) = -Np(r)/r^{1-\frac{1}{N}} \ . \tag{5.30}$$

But ϕ is convex if and only if ϕ' is non-decreasing, and, since the map $\lambda \to \lambda^{-N}$ is non-increasing, this is true if and only if the function $r \mapsto p(r)/r^{1-\frac{1}{N}}$ is non-decreasing.

(i)↔(iii) By computation,

$$\phi''(\lambda) = N^2 r^{\frac{2}{N}-1}\left(p_2(r) + \frac{p(r)}{N}\right) \ . \tag{5.31}$$

Then ϕ is convex if and only if $\phi'' \geq 0$, which is the case if and only if $p_2 + \frac{p}{N}$ is non-negative.

In the case $N = \infty$ we have

$$r(\lambda) = e^{-\lambda}, \quad \phi'(\lambda) = -\frac{p(r)}{r}, \quad \phi''(\lambda) = \frac{p_2(r)}{r}$$

and the same arguments give us the result. ■

A property of the behaviour of functions in $\mathcal{DC}_\infty$ is given in the following lemma.

Lemma 5.3.5 *Given $U \in DC_\infty$, either U is linear or there exist $a, b > 0$ such that $U(r) \geq ar \log r - br$.*

Proof We may recover the function U from ϕ by the formula

$$U(x) = x\phi(\log(1/x)).$$

But ϕ is convex and non-increasing so either ϕ is constant or there are constants $a, b > 0$ such that $\phi(\lambda) \geq -a\lambda - b$ for all $\lambda \in \mathbb{R}$. In the first case, U is linear. In the second case, $U(x) \geq -ax\log(1/x) - bx$. ■

5.3.2 Ricci curvature via weak displacement convexity

We have already defined the notion of a compact measured length space (X, d, ν) being weakly λ-displacement convex for a family of convex functions $\mathcal{F}$. The following definition is formulated in those terms.

Definition 5.3.6 Given $N \in [1, \infty]$, we say that a compact measured length space (X, d, ν) has non-negative N-Ricci curvature (or that it satisfies the weak $CD(0, N)$ condition) if it is weakly displacement convex for the family $\mathcal{DC}_N$.

By Lemma 5.3.3, if $N \le N'$ and X has non-negative N-Ricci curvature then it has non-negative N'-Ricci curvature. In the case $N = \infty$, we can define a more precise notion.

Definition 5.3.7 Given $K \in \mathbb{R}$, define $\lambda : DC_\infty \to \mathbb{R} \cup \{-\infty\}$ by

$$\lambda(U) = \inf_{r>0} K \frac{p(r)}{r} = \begin{cases} K \lim_{r\to 0^+} \frac{p(r)}{r} & \text{if } K > 0, \\ 0 & \text{if } K = 0, \\ K \lim_{r\to\infty} \frac{p(r)}{r} & \text{if } K < 0, \end{cases} \tag{5.32}$$

where p is given by (5.27). We say that a compact measured length space (X, d, ν) has ∞-Ricci curvature bounded below by K (or that it satisfies the weak $CD(K, \infty)$ condition) if it is weakly λ-displacement convex for the family DC_∞.

If $K \le K'$ and (X, d, ν) has ∞-Ricci curvature bounded below by K' then it has ∞-Ricci curvature bounded below by K.

The next proposition shows that our definitions localize on totally convex subsets.

Proposition 5.3.8 *Suppose that a closed set $A \subset X$ is totally convex. Given $\nu \in P_2(X)$ with $\nu(A) > 0$, put $\nu' = \frac{1}{\nu(A)} \nu \restriction_A \in P_2(A)$.*

1. *If (X, d, ν) has non-negative N-Ricci curvature then (A, d, ν') has non-negative N-Ricci curvature.*

2. *If (X, d, ν) has ∞-Ricci curvature bounded below by K then (A, d, ν') has ∞-Ricci curvature bounded below by K.*

Proof Let μ_0, $\mu_1 \in P_2(A)$. The notion of optimal coupling is the same whether one considers them as measures on A or on X. Furthermore, we know that since A is totally convex, a path $[0, 1] \to X$ with endpoints in X' is a geodesic in X' if and only if it is a geodesic in X. Then by proposition 4.1.9, $P_2(A)$ is a totally convex subset of $P_2(X)$, i.e. a path $(\mu_t)_{0\le t\le 1}$ with μ_0, $\mu_1 \in P_2(A)$ is a geodesic in $P_2(A)$ if and only if it is a geodesic in $P_2(X)$. Given $\mu \in P_2(A) \subset P_2(X)$, let $\mu = \rho\nu + \mu_s$ be its Lebesgue decomposition with respect to ν. Then $\mu = \rho'\nu' + \mu_s$ is the Lebesgue decomposition with respect to ν', where $\rho' = \nu(A)\rho \restriction_A$.

$\boxed{1.}$ Take $N < \infty$. We must prove that (A, d, ν') is weakly displacement convex for the family $\mathcal{DC}_N$. Given a continuous convex function $U : [0, \infty) \to \mathbb{R}$ with $U(0) = 0$, define

$$\tilde{U}(r) = \frac{U(\nu(A)r)}{\nu(A)}.$$

Then

$$\tilde{U}'(\infty) = \lim_{r\to\infty} \tilde{U}'_+(r) = \lim_{r\to\infty} U'_+(\nu(A)r) = \lim_{r\to\infty} U'_+(r) = U'(\infty),$$

and $U \in \mathcal{DC}_N$ if and only if $\tilde{U} \in \mathcal{DC}_N$. Indeed, denote by ϕ_U and $\phi_{\tilde{U}}$ the functions corresponding to U and $\tilde{U}$, respectively. Then, given $\lambda_1,\ \lambda_2 \in (0,\infty)$, we consider $\tilde{\lambda}_1 = \nu(A)^{-\frac{1}{N}}\lambda_1$ and $\tilde{\lambda}_2 = \nu(A)^{-\frac{1}{N}}\lambda_2$. The convexity of ϕ_U between $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ then yields the convexity of $\phi_{\tilde{U}}$ between λ_1 and λ_2. Now,

$$\begin{aligned} U_{\nu'}(\mu) &= \int_A U(\rho')d\nu' + U'(\infty)\mu_s(A) \\ &= \frac{1}{\nu(A)}\int_A U(\nu(A)\rho)d\nu + U'(\infty)\mu_s(A) \\ &= \int_X \tilde{U}(\rho)d\nu + \tilde{U}'(\infty)\mu_s(X) = \tilde{U}_\nu(\mu). \end{aligned}$$

Since $P_2(A,\nu') \subset P_2(X,\nu)$ and (X,ν) is weakly displacement convex for $\mathcal{DC}_N$, we conclude that (A,d,ν') is weakly displacement convex for $\mathcal{DC}_N$.

2. Let $\tilde{p}$ denote the pressure of $\tilde{U}$, then

$$\frac{\tilde{p}(r)}{r} = \tilde{U}'_+(r) - \frac{\tilde{U}(r)}{r} = U'_+(\nu(A)r) - \frac{U(\nu(A)r)}{\nu(A)r} = \frac{p(\nu(A)r)}{\nu(A)r}.$$

Then with the notation of Definition 5.3.7, $\lambda(\tilde{U}) = \lambda(U)$. The second statement follows. ■

5.3.3 Basic properties

The next result is of great importance, it states the preservation of N-Ricci curvature bounds under Gromov-Hausdorff limits.

Theorem 5.3.9 *Let $\{(X_i,d_i,\nu_i)\}_{i=1}^{\infty}$ be a sequence of compact measured length spaces with $\lim_{i\to\infty}(X_i,d_i,\nu_i) = (X,d,\nu)$ in the Gromov-Hausdorff topology.*

1. *If each (X_i,d_i,ν_i) has non-negative N-Ricci curvature then (X,d,ν) has non-negative N-Ricci curvature.*

2. *If each (X_i,d_i,ν_i) has ∞-Ricci curvature bounded below by K, for some $K \in \mathbb{R}$, then (X,d,ν) has ∞-Ricci curvature bounded below by K.*

Proof If $N < \infty$ then the theorem follows from Theorem with the family $\mathcal{F} = \mathcal{DC}_N$ and $\lambda = 0$. If $N = \infty$ then it follows from Theorem with the family $\mathcal{F} = DC_\infty$ and λ given by Definition 5.3.7. ■

We then obtain an analogue of the Bishop-Gromov theorem.

Proposition 5.3.10 *([16], Proposition 5.27) Suppose that (X,d,ν) has non-negative N-Ricci curvature, with $N \in [1,\infty)$. Then for all $x \in \operatorname{supp}(\nu)$ and all $0 < r_1 \le r_2$,*

$$\nu(B_{r_2}(x)) \le \left(\frac{r_2}{r_1}\right)^N \nu(B_{r_1}(x)).$$

5.4 Weak $CD(K,N)$ condition

We add this section for completeness, giving the general definition of the weak $CD(K,N)$ notion, i.e. N-Ricci curvature bounded below by K.

5.4.1 Distortion coefficients

Definition 5.4.1 If A and B are two measurable sets in a Riemannian manifold M, and $t \in [0,1]$, a *t-barycenter* of A and B is a point which can be written as $\gamma(t)$, where γ is a minimizing, constant speed geodesic $\gamma : [0,1] \to M$ with $\gamma(0) \in A$ and $\gamma(1) \in B$. The set of t-barycenters between A and B is denoted by $[A,B]_t$.

Definition 5.4.2 Let M be a Riemannian manifold, equipped with a reference measure $e^{-\psi}\mathrm{vol}$ with $\psi \in C(M)$, and let x and y be any two points in M. Then the distortion coefficient $\overline{\beta}_t(x,y)$ between x and y at a time $t \in (0,1)$ is defined as follows:

1. If x and y are joined by a unique geodesic γ, then

$$\overline{\beta}_t(x,t) = \lim_{r\downarrow 0} \frac{\nu\left[[x, B_r(y)]_t\right]}{\nu[B_{tr}(y)]} = \lim_{r\downarrow 0} \frac{\nu\left[[x, B_r(y)]_t\right]}{t^n \nu[B_r(y)]}.$$

2. If x and y are joined by several minimizing geodesics, then

$$\overline{\beta}_t(x,y) = \inf_{\gamma} \limsup_{s\to 1^-} \overline{\beta}_t(x,\gamma_s),$$

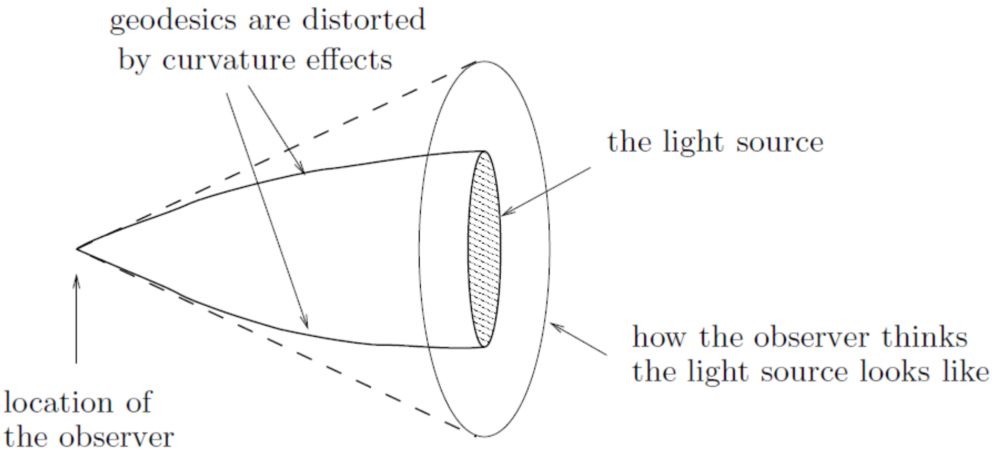
   where the infimum is over all minimizing geodesics joining $\gamma(0) = x$ to $\gamma(1) = y$.

The values of $\overline{\beta}_t(x,y)$ for $t = 0$ and $t = 1$ are defined by

$$\overline{\beta}_1(x,y) \equiv 1; \qquad \overline{\beta}_0 := \liminf_{t\to 0^+} \overline{\beta}_t(x,y).$$

The meaning of these distortion coefficients can be understood from the following example found in [26]. Suppose that you find yourself at a point x, and that you are trying to estimate the volume of an object located at y. Your problem is that the light rays that allow your view of the object are travelling along geodesics, which may not be straight lines. However, in your estimations you are imagining that the light rays are travelling in straight lines, thus you might be overestimating or underestimating its volume. If x and y are joined by a unique geodesic , then the coefficient $\overline{\beta}_0(x,y)$ tells by how much you are overestimating. Therefore, it is less than 1 in negative curvature, and greater than 1 in positive curvature. If there are several geodesics joining x and y then the coefficient is the same, except that you look in the direction where the device looks smallest.

In general, $\beta_t(x,y)$ compares the volume occupied by the light rays emanating from the light source, when they arrive close to $\gamma(t)$, to the volume that they would occupy in a flat space. In the picture, the distortion coefficient would be approximated by the ratio of the volume filled with lines, to the volume whose contour is in dashed line.



The model spaces 2.10.2 give us the reference distortion coefficients.

Definition 5.4.3 Given $K \in \mathbb{R}$, $N \in [1, \infty]$ and $t \in [0, 1]$, and two points x, y in some metric space (X, d), define $\beta_t^{(K,N)}(x, y)$ as follows,

1. If $0 < t \leq 1$ and $1 < N < \infty$ then

$$\beta_t^{(K,N)}(x,y) = \begin{cases} +\infty & \text{if } K > 0 \text{ and } \alpha > pi, \\ \left(\frac{\sin(t\alpha)}{t \sin \alpha}\right)^{N-1} & \text{if } K > 0 \text{ and } \alpha \in [0, \pi], \\ 1 & \text{if } K = 0, \\ \left(\frac{\sinh(t\alpha)}{t \sinh \alpha}\right)^{N-1} & \text{if } K < 0, \end{cases}$$

   where

$$\alpha = \sqrt{\frac{|K|}{N-1}} d(x, y).$$

2. In the two limit cases $N \to 1$ and $N \to \infty$, modify the above expressions as follows:

$$\beta_t^{(K,1)}(x,y) = \begin{cases} +\infty & \text{if } K > 0, \\ 1 & \text{if } K \leq 0, \end{cases}$$

$$B_t^{(K,\infty)}(x,y) = e^{\frac{K}{6}(1-t^2)d(x,y)^2}.$$

3. For $t = 0$ define $\beta_0^{(K,N)}(x, y) = 1$.

The following theorem states the relation between Ricci curvature bounds in terms of distortion coefficients.

Theorem 5.4.4 *([26], Theorem 14.21) Let M be a Riemannian manifold of dimension n, equipped with its volume measure. Then the following two statements are equivalent:*

1. $\mathrm{Ric} \geq K$;
2. $\overline{\beta} \geq \beta^{(K,n)}$.

5.4.2 Definition

The definition will be in terms of the distorted U_ν functional. We first remark that a transference plan π can be disintegrated with respect to its first marginal μ_0 or its second marginal μ_1:

$$d\pi(x_0, x_1) = d\pi(x_1|x_0)d\mu_0(x_0) = d\pi(x_0|x_1)d\mu_1(x_1).$$

Definition 5.4.5 Let (X, d, ν) be a compact measured length space. Let U be a convex function with $U(0) = 0$, let $x \to \pi(dy|x)$ be a family of conditional probabilities on X, indexed by $x \in X$, and let β be a measurable function $X \times X \to (0, +\infty]$. The distorted U_ν functional with distortion coefficient β is defined as follows: For any measure $\mu = \rho\nu$ on X,

$$U^{\beta}_{\pi,\nu}(\mu) = \int_{X\times X} U\left(\frac{\rho(x)}{\beta(x,y)}\right)\beta(x,y)\pi(dy|x)\nu(dx).$$

Definition 5.4.6 Let $K \in \mathbb{R}$ and $N \in [1, \infty]$. A compact measured length-space (X, d, ν) is said to satisfy the weak $CD(K, N)$ condition (or have N-Ricci curvature bounded below by K) if the following is satisfied: Whenever μ_0 and $\mu_1 \in P(X)$ with suppμ_0, supp$\mu_1 \subset$ suppν, there exist a geodesic $\{\mu_t\}_{0\leq t\leq 1}$ joining μ_0 to μ_1 and an associated optimal coupling π of (μ_0, μ_1) such that, for all $U \in \mathcal{DC}_N$ and for all $t \in [0, 1]$,

$$U_\nu(\mu_t) \leq (1-t)U^{\beta^{(K,N)}_{1-t}}_{\pi,\nu}(\mu_0) + tU^{\beta^{(K,N)}_{t}}_{\check{\pi},\nu}(\mu_1).$$

Remark 5.4.7 Some considerations must be taken into account to prove that $U^{\beta^{(K,N)}_{t}}_{\pi,\nu}$ is well defined. This follows from ([26], Application 17.29 and Convention 17.30).

Motivation for this definition is given in ([26], Theorem 17.37), which in our case states that in the Riemannian case this is equivalent to the $CD(K, N)$ condition.

We have the following compactness theorem.

Theorem 5.4.8 *([26], Theorem 29.32) Let $K \in \mathbb{R}$, $N < \infty$, $D < \infty$ and $0 < m \leq M < \infty$. Let $\mathcal{CDD}(K, N, D, m, M)$ be the space of compact measured length spaces (X, d, ν) satisfying the weak $CD(K, N)$ condition, together with* diam$(X, d) \leq D$, *$m \leq \nu[X] \leq M$, and* supp$\nu = X$. *Then $\mathcal{CDD}(K, N, D, m, M)$ is compact in the measured Gromov-Hausdorff topology.*

In ([26], Theorem 29.24) it is proved that this weak $CD(K, N)$ condition is also stable under measured Gromov-Hausdorff limits. In ([26], Theorems 30.7 and 30.11) Brunn-Minkowski and Bishop-Gromov type inequalities are given for weak $CD(K, N)$ spaces. There is also the following Bonnet-Myers diameter bound for weak $CD(K, N)$ spaces:

Proposition 5.4.9 *([26], Theorem 29.11) If (X, d, ν) is a weak $CD(K, N)$ space with $K > 0$ and $N < \infty$, then*

$$\text{diam}(\text{supp}\nu) \leq \pi\sqrt{\frac{N-1}{K}}\ .$$

5.5 Smooth metric-measure spaces

In the Riemannian case the definition of 'non-negative N-Ricci curvature' coincides with the usual notion. By smooth measured length space we will mean a smooth n-dimensional Riemannian manifold M along with a smooth probability measure $\nu = e^{-\psi}\text{dvol}_M$. Recall the definition of the generalised Ricci tensor from Section 2.10.

Theorem 5.5.1 *([16], Theorems 7.3 and 7.42) Given $N \in [1, \infty]$, the smooth measured length space (M, g, ν) has non-negative N-Ricci curvature if and only if $\text{Ric}_N \geq 0$.*

In the special case when ψ is constant, and so $\nu = \frac{\text{dvol}_M}{\text{vol}(M)}$, the theorem proves that we recover the usual notion of non-negative Ricci curvature from our length space definition as soon as $N \geq n$.

Another problem we had was to characterize the limit points that appear from the Gromov precompactness theorem. In general the limit points can be very singular. However, if we only consider limit points which happen to be smooth measured length spaces, it becomes easier.

Corollary 5.5.2 *([16], Corollary 7.45) If $(B, g_B, e^{-\psi}\text{dvol}_B)$ is a measured Gromov-Hausdorff limit of Riemannian manifolds with non-negative Ricci curvature and dimension at most N then $\text{Ric}_N(B) \geq 0$.*

Although a full characterisation has not been found, there is a partial converse.

Proposition 5.5.3 *([16], Corollary 7.45)*

1. *Suppose that N is an integer. If $(B, g_B, e^{-\psi}\text{dvol}_B)$ has $\text{Ric}_N(B) \geq 0$ with $N \geq \dim(B) + 2$ then $(B, g_B, e^{-\psi}\text{dvol}_B)$ is a measured Gromov-Hausdorff limit of Riemannian manifolds with non-negative Ricci curvature and dimension N.*

2. *Suppose that $N = \infty$. If $(B, g_B, e^{-\psi}\text{dvol}_B)$ has $\text{Ric}_\infty(B) \geq 0$ then $(B, g_B, e^{-\psi}\text{dvol}_B)$ is a measured Gromov-Hausdorff limit of Riemannian manifolds M_i with $\text{Ric}(M_i) \geq -\frac{1}{i} g_{M_i}$.*

5.5.1 Examples

The following is an example of a measured length space which is not a manifold having non-negative n-Ricci curvature.

Example 5.5.4 Let M be a smooth compact n-dimensional Riemannian manifold with non-negative Ricci curvature, and let G be a compact Lie group acting isometrically on M. Then let $X = M/G$ and let $q : M \to X$ be the quotient map. Equip X with the quotient distance $d(x, y) = \inf\{d_M(x', y') \ : \ q(x') = x, q(y') = y\}$, and with the measure $\nu = q_\star \text{vol}_M$. The resulting space (X, d, ν) has non-negative n-Ricci curvature, and is not a manifold in general.

Example 5.5.5 ([26], Example 29.17) Let $X = \Pi_{i=1}^{\infty} T_i$, where $T_i = \mathbb{R}/(\varepsilon_i \mathbb{Z})$ is equipped with the usual distance d_i and the normalized Lebesgue measure λ_i, and $\varepsilon_i = 2\text{diam}(T_i)$ is some positive number. If $\sum \varepsilon_i^2 < +\infty$ then the product distance $d = \sqrt{\sum d_i^2}$ turns X into a compact metric space. Taking the product measure $\nu = \Pi \lambda_i$ on X turns (X, d, ν) into a $CD(0, \infty)$ space.

5.6 Analytic consequences

Having Ricci curvature bounded below on Riemannian manifolds allows for several analytic implications, such as eigenvalue inequalities, Sobolev inequalities and local Poincaré inequalities. In [16] some of these inequalities are proved in the generalized setting of weak $CD(K,N)$ spaces. The existence of these Poincaré inequalities is vital since it allows for the theory of differential equations to be developed on weak $CD(K,N)$ spaces. It is worth mentioning that the Finnish school led by Pekka Koskela and J.Heinonen took a different approach to metric spaces with 'controlled geometry', and considered metric spaces which support Poincare inequalities and doubling measures as their starting point, instead of trying to reproduce Ricci lower bounds in more general spaces. It is very interesting to understand the relation between both approaches.

An example of this is the log Sobolev inequality. If a smooth measured length space $(M, g, e^{-\psi}\mathrm{dvol}_M)$ has $\mathrm{Ric}_\infty \geq Kg$, with $K > 0$, then for all $f \in C^\infty(M)$ with $\int_M f^2 e^{-\psi}\mathrm{dvol}M = 1$, it is proved in [2] that

$$\int_M f^2 \log(f^2) e^{-\psi}\mathrm{dvol}_M \leq \frac{2}{K}\int_M |\nabla f|^2 e^{-\psi}\mathrm{dvol}_M.$$

Then the usual log Sobolev inequality on $\mathbb{R}^n$ comes from taking

$$d\nu = (4\pi)^{-\frac{n}{2}} e^{-|x|^2} d^n x,$$

giving

$$\int_{\mathbb{R}^n} f^2 \log(f^2) e^{-|x|^2} d^n x \leq \int_{\mathbb{R}^n} |\nabla f|^2 e^{-|x|^2} d^n x$$

whenever $(4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f^2 e^{-|x|^2} d^n x = 1$.

On measured length spaces we have the following log Sobolev inequality.

Theorem 5.6.1 *([16], Corollary 6.12). Suppose that a compact measured length space (X, d, ν) satisfies the $CD(K,\infty)$ condition with $K > 0$. Suppose that $f \in \mathrm{Lip}(X)$ satisfies $\int_X f^2 d\nu = 1$. Then*

$$\int_X f^2 \log(f^2) d\nu \leq \frac{2}{K}\int_X |\nabla f|^2 d\nu. \tag{5.33}$$

We can then obtain a Poincaré inequality by taking $h \in \mathrm{Lip}(X)$ with $\int_X h d\nu = 0$ and put $f^2 = 1 + \varepsilon h$. Taking ε small and expanding the two sides of (5.33) in ε gives:

Corollary 5.6.2 *Suppose that a compact measured length space (X, d, ν) satisfies the $CD(K,\infty)$ condition for $K > 0$. Then for all $h \in \mathrm{Lip}(X)$ with $\int_X h d\nu = 0$, we have*

$$\int_X h^2 d\nu \leq \frac{1}{K}\int_X |\nabla h|^2 d\nu.$$

The log Sobolev inequality does not depend on the dimension. However, if one has N-Ricci curvature bounded below by $K > 0$ with N finite then one gets a Sobolev inequality, which does depend on N.

Proposition 5.6.3 *([18]) Given $N \in (2,\infty)$ and $K > 0$, suppose that (X, d, ν) has N-Ricci curvature bounded below by K. Then for any non-negative Lipschitz function $f \in \mathrm{Lip}(X)$ with $\int_X f^{\frac{2N}{N-2}} d\nu = 1$, one has*

$$1 - \left(\int_X f d\nu\right)^{\frac{2}{N+2}} \leq \frac{6}{KN}\left(\frac{N-1}{N-2}\right)^2 \int_X |\nabla f|^2 d\nu.$$

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