HYPERELASTICITY AS A Γ-LIMIT OF PERIDYNAMICS WHEN THE HORIZON GOES TO ZERO

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Abstract. Peridynamics is a nonlocal model in Continuum Mechanics, and in particular Elasticity, introduced by Silling (2000). The nonlocality is reflected in the fact that points at a finite distance exert a force upon each other. If, however, those points are more distant than a characteristic length called *horizon*, it is customary to assume that they do not interact. We work in the variational approach of time-independent deformations, according to which, their energy is expressed as a double integral that does not involve gradients. We prove that the Γ-limit of this model, as the horizon tends to zero, is the classical model of hyperelasticity. We pay special attention to how the passage from the density of the non-local model to its local counterpart takes place.

1. Introduction

Hyperelasticity is the central model in the variational approach of Solid Mechanics. According to it, the body under deformation is represented in its reference configuration by a bounded domain $\Omega \subset \mathbb{R}^n$. Here, n is the space dimension, which for actual physical problems is taken to be 2 or 3. We do not consider the time dependence, that is to say, the dynamical process by which deformations evolve, but focus on the equilibrium solutions. In this context, a *deformation* of the body is represented by a map $\mathbf{u} : \Omega \to \mathbb{R}^d$. The integer d usually coincides with n, but since we also want to treat the antiplane case $d = 1$, we prefer to carry out the proofs for general n and d. Thus, $\mathbb{R}^d$ is the space where the deformed configuration lies. The theory of hyperelasticity asserts that there exists a *stored energy* function $W : \Omega \times \mathbb{R}^{d\times n} \to \mathbb{R} \cup \{\infty\}$ such that the elastic energy of the deformation $\mathbf{u}$ is calculated through

$$I(\mathbf{u}) := \int_\Omega W(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}. \tag{1}$$

Here $D\mathbf{u}(\mathbf{x})$ is the gradient of $\mathbf{u}$ at $\mathbf{x}$, and $\mathbf{u}$ is required to be Sobolev $W^{1,p}$ for some $p \geq 1$. The dependence of W on $\mathbf{x}$ expresses the inhomogeneity of the material. A central question in hyperelasticity is the existence of minimizers of $\mathbf{u}$ in $W^{1,p}(\Omega, \mathbb{R}^d)$ satisfying certain given boundary conditions. The search for *minimizers*, as opposed to equilibrium solutions, is partly because the latter are the most stable deformations, and partly because the existence theory for global minimizers is more understood than that for critical points.

In the nonlinear setting, which is the field of interest in the present work, there are, roughly speaking, two sets of assumptions on W guaranteeing the existence of minimizers of I. The first one assumes that W is *quasiconvex* (see the definition in Section 4) and has a coercivity and growth conditions of the form

$$\frac{1}{C}|\mathbf{F}|^p - C \leq W(\mathbf{x}, \mathbf{F}) \leq C\,(1 + |\mathbf{F}|^p)\,, \qquad \text{a.e. } \mathbf{x} \in \Omega, \quad \text{all } \mathbf{F} \in \mathbb{R}^{d\times n}, \tag{2}$$

for some $1 < p < \infty$ and $C > 0$. An alternative possibility, not studied in this paper, is to assume that W is *polyconvex*, and has the coercivity condition

$$\frac{1}{C}\sum_{i=1}^{\min\{n,d\}} |\tau_i(\mathbf{F})|^{p_i} - C \leq W(\mathbf{x}, \mathbf{F}), \qquad \text{a.e. } \mathbf{x} \in \Omega, \quad \text{all } \mathbf{F} \in \mathbb{R}^{d\times n},$$

where $\tau_i(\mathbf{F})$ is the vector composed by all minors (subdeterminants) of order i of $\mathbf{F}$, and $p_i \geq 1$ is a suitable exponent.

Hyperelasticity is, of course, a classic subject. One can find expositions of the theory, as well as precise statements of results mentioned above in, e.g., the books [3, 11, 13, 27], and the review papers [5, 6].

2010 *Mathematics Subject Classification.* 74B20,74G65,35Q74.

Key words and phrases. Peridynamics, Γ-convergence, hyperelasticity.

An alternative model in Solid Mechanics was introduced by Silling [30] with the name *peridynamics*, and its main difference with classical continuum mechanics relies in its non-locality, which is reflected in the fact that points separated by a positive distance exert a force upon each other. To make a comparsion with the hyperelastic model explained above, we again restrict to time-independent deformations. Instead of W, now there is a $w : \Omega \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, the *pairwise potential function*, that measures the interaction between particle $\mathbf{x} \in \Omega$ and particle $\mathbf{x}' \in \Omega$ in both the reference and the deformed configurations. For a given deformation $\mathbf{u} : \Omega \to \mathbb{R}^d$, the interaction between those two particles is calculated through $w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))$, so that the *macroelastic* energy of the deformation is given by

$$\int_\Omega \int_\Omega w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}. \tag{3}$$

Expression (3) reflects the two main features of the peridynamic theory: the non-locality and the absence of gradients. Since $\mathbf{u}$ is not assumed to be weakly differentiable, it is often supposed that it lies in the Lebesgue space L^p for some $1 \le p \le \infty$. After [30], further developments and expositions of the peridynamic theory are presented in [24, 31, 33].

Although it is not relevant for the development of this paper, we ought to mention that the function w must satisfy certain mathematical properties in order to meet some physical requirements such as objectivity or balance laws (see [33]). Moreover, as nearby particles interact with a stronger force than distant ones, it is natural to assume that the function $w(\mathbf{x}, \cdot, \tilde{\mathbf{y}})$ blows up at $\mathbf{0}$, for each $\mathbf{x} \in \Omega$ and $\tilde{\mathbf{y}} \in \mathbb{R}^d$. It is also natural to assume that distant particles do not interact at all, so, if we normalize w so that its minimum value is 0, we impose $w(\cdot, \tilde{\mathbf{x}}, \cdot) \equiv 0$ if $|\tilde{\mathbf{x}}| \ge \delta$. This number $\delta > 0$ is called the *horizon*. It introduces a length scale in the model, and plays a capital role in the numerical simulations since it makes the computation of the double integral (3) tractable. The way we make the parameter δ appear in the model is to fix w, and replace (3) by

$$\int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}. \tag{4}$$

General conditions on w for the existence of minimizers of (4) were studied in Bellido & Mora-Corral [7] (although without the dependence of w on $\mathbf{x}$), extending other existence results based on minimization such as [1, 14, 23].

It is the object of this paper to study the relationship between the nonlocal model of peridynamics and the local model of hyperelasticity. Having a look at (1) and (4), it is natural to ask whether I can somehow be obtained from (4) by a passage to the limit when $\delta \to 0$. The assumptions that we make in this paper on the pairwise potential w in order to be able to perform that limit passage are, on the one hand, compatible with the existence theory of [7] and extended to the inhomogeneous case, and, on the other, modelled so as to cover the paradigmatic case

$$w(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \frac{|\tilde{\mathbf{y}}|^p}{|\tilde{\mathbf{x}}|^\alpha} \tag{5}$$

for some $1 < p < \infty$, and $0 \le \alpha < n + p$. The function in (5) is a non-local analogue of a singular p-Laplacian (see, e.g., [2] for a study of nonlocal p-Laplacian operators). In particular, our assumptions allow for a nonconvex w.

Calling $\beta := p - \alpha$, we find that the right scaling of (4), so as to pass to the limit, is

$$I_\delta(\mathbf{u}) := \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}, \qquad \mathbf{u} \in L^p(\Omega, \mathbb{R}^d). \tag{6}$$

From the function w, we are able to construct a quasiconvex function $W : \Omega \times \mathbb{R}^{d \times n} \to \mathbb{R}$ such that, when we define

$$I(\mathbf{u}) := \int_\Omega W(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \qquad \mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d), \tag{7}$$

we have that the family of functionals I_δ converges to I as $\delta \to 0$. Moreover, this W satisfies the growth conditions (2), so it meets the general assumptions for the existence theory in hyperelasticity. Understanding this passage from the integrand w in (6) to W in (7) was one of our main motivations.

To establish this limit process on solid grounds, we need to assume that w is close to a homogeneous function near $(\mathbf{0},\mathbf{0})$; to be precise, we need the existence of the limit

$$w^\circ(\mathbf{x},\tilde{\mathbf{x}},\tilde{\mathbf{y}}) := \lim_{t\to 0}\frac{1}{t^\beta} w(\mathbf{x},t\tilde{\mathbf{x}},t\tilde{\mathbf{y}}), \tag{8}$$

for some $\beta\in\mathbb{R}$. For example, for the w of (5), we have $w^\circ = w$ and $\beta = p-\alpha$. This w° thus constructed satisfies that $w^\circ(\mathbf{x},\cdot,\cdot)$ is homogeneous of degree β, for each $\mathbf{x}\in\Omega$. Then, we define the density $\bar{w}:\Omega\times\mathbb{R}^{d\times n}\to\mathbb{R}$ by setting

$$\bar{w}(\mathbf{x},\mathbf{F}) := \int_{\mathbb{S}^{n-1}} w^\circ(\mathbf{x},\mathbf{z},\mathbf{F}\mathbf{z})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}). \tag{9}$$

We find that the limit of $I_\delta(\mathbf{u})$, for smooth $\mathbf{u}$, is

$$\bar{I}(\mathbf{u}) := \int_\Omega \bar{w}(\mathbf{x},D\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}, \tag{10}$$

and we define the W that appears in (7) as the quasiconvexification of $\bar{w}$.

This limit process is in the sense of Γ-convergence, which is considered to be the right concept of limit for variational problems (see, e.g., [9]), since it implies in particular that minimizers of I_δ converge to minimizers of I, as well as their energies.

We remark the change of functional space in the definition of I_δ and I, which is not uncommon in Γ-convergence: I_δ is defined in $L^p(\Omega,\mathbb{R}^d)$, I is defined in $W^{1,p}(\Omega,\mathbb{R}^d)$, and the convergence of the deformations is meant in the strong (norm) topology of $L^p(\Omega,\mathbb{R}^d)$.

We consider both Dirichlet and Neumann boundary conditions, with the caveat that *Dirichlet* conditions in peridynamics prescribe the value of the deformation in a set of positive measure, namely, the set of points in Ω at a distance less than δ from $\partial\Omega$ (see [15, 22]). Neumann conditions are not imposed explicitly in the admissible set, since they are a consequence of the minimization process itself. Instead, to avoid the trivial nonuniqueness given by the translation invariance of I_δ and I, we impose that the integral of $\mathbf{u}$ is zero (see [7, 14, 15, 22] for a physical interpretation of nonlocal Neumann conditions).

In the setting of peridynamics, the works [17, 32, 14, 25, 26] also consider a limit passage as the horizon tends to zero, but with quite different approaches. Authors [17] study, in the context of small displacements, the formal limit of a linear model of peridynamics with time dependence to the classical Navier equation of linear elasticity, in the absence of boundary conditions. In [32], they prove, for any fixed smooth deformation, the convergence of the stresses and their spatial divergences for quite general nonlinear models, again without boundary conditions. In [14], they study equilibrium states of linear peridynamics materials, and obtain in the limit the Navier equation. Paper [25] considers the peridynamic evolution of small antiplane deformations with a potential that allows for softening and eventual fracture; he proves that the limit coincides with the classic Griffith's model for fracture evolution. He also observes that the work [21], set in the framework of the Mumford–Shah functional, essentially provides the relevant Γ-limit tools. The recent analysis [26] also uses Γ-convergence to pass from a nonlinear energy to a linear one, both in the context of peridynamics. In a different physical context, namely, the nonlocal diffusion problems, the passage from nonlocal to local for a parabolic equation with a nonlocal p-Laplacian has been presented in the monograph [2], collecting results from several papers. The main features of our approach are the consideration of non-convex potentials in vectorial problems, the possibility of having not necessarily small deformations, the use of Γ-convergence and the inclusion of the boundary conditions. We remark that, as the functions w and W need not be convex, nonuniqueness of minimizers, both for I_δ and for I, is admissible.

We describe a sketch of the proof. Typically, a Γ-convergence result consists of three parts: compactness, lower bound, and upper bound. The compactness part asserts that a family $\{\mathbf{u}_\delta\}_\delta$ of deformations with bounded energy $I_\delta(\mathbf{u}_\delta)$ admits a convergent subsequence in $L^p(\Omega,\mathbb{R}^d)$ as $\delta\to 0$. Moreover, any limit point $\mathbf{u}$ lies in $W^{1,p}(\Omega,\mathbb{R}^d)$, and if $\mathbf{u}_\delta$ satisfies the nonlocal Dirichlet boundary conditions, then $\mathbf{u}$ satisfies the usual (local) boundary conditions in the sense of traces; analogously for Neumann conditions. The proof of this part relies on a compactness criterion in L^p by Bourgain, Brezis & Mironescu [8] based on the boundedness of double integral functionals of the kind (3). Their study arose in connection with a nonlocal characterization of $W^{1,p}$. This criterion was later generalized by Ponce [28].

The lower bound consists in proving that if $\mathbf{u}_\delta \to \mathbf{u}$ as before then

$$I(\mathbf{u}) \leq \liminf_{\delta \to 0} I_\delta(\mathbf{u}_\delta). \tag{11}$$

The proof of this inequality is initially based on Ponce [29], in a follow-up of the results of [8, 28], but with the added difficulty of the inhomogeneity of the material. The strategy is first to compute the limit of $I_\delta(\mathbf{u})$ when $\mathbf{u}$ is a smooth function. In this case, we obtain that the limit of $I_\delta(\mathbf{u})$ turns out to be $\bar{I}(\mathbf{u})$, as defined in (10). Unlike (5), the function $w(\mathbf{x}, \tilde{\mathbf{x}}, \cdot)$ is not assumed to be convex, but it satisfies a certain property of convexity, which allows us to use Jensen's inequality to suitable mollifications of $\mathbf{u}_\delta$ and $\mathbf{u}$, so as to conclude (11).

The upper bound inequality requires the construction of a *recovery sequence*: for each $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$ we must find $\mathbf{u}_\delta \in L^p(\Omega, \mathbb{R}^d)$ satisfying the nonlocal boundary conditions such that $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ and $I_\delta(\mathbf{u}_\delta) \to I(\mathbf{u})$. The main difficulty to define $\mathbf{u}_\delta$ is the satisfaction of the nonlocal Dirichlet condition, so we explain this case. The starting point is the relaxation theorem of Dacorogna [12, 13], which asserts that the functional I of (7) is the lower semicontinuous envelope of $\bar{I}$ of (10) in the strong L^p topology, when W is taken to be the quasiconvexification of $\bar{w}$. Thus, there exists $\mathbf{v}_\delta \in W^{1,p}(\Omega, \mathbb{R}^d)$ such that $\mathbf{v}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ and $\bar{I}(\mathbf{v}_\delta) \to I(\mathbf{u})$. Then, we modify $\mathbf{v}_\delta$ near the boundary so as to satisfy the nonlocal Dirichlet conditions. The estimates for the energy of the modified version of $\mathbf{v}_\delta$ rely on the decomposition lemma of Fonseca, Müller & Pedregal [20] and go beyond the usual boundary estimates common in Γ-convergence for local boundary conditions (see, e.g., [9, Sect. 4.2]).

We finish this introduction with an outline of the paper. In Section 2, we state the main result of the paper, namely, the Γ-limit $I_\delta \to I$ as $\delta \to 0$. Care is taken to explain the nonlocal boundary conditions. Section 3 introduces the general notation. In Section 4, we write down and comment on the assumptions of w. From w, we construct three auxiliary functions: w° (see (8)), $\bar{w}$ (see (9)), and its quasiconvexification $Q\bar{w}$, which in this Introduction has been called W. We also explain the bounds and growth conditions that these functions satisfy. In Section 5, we collect some preliminary results that will be used in the paper, including the compactness criterion of Bourgain, Brezis & Mironescu [8]. Section 6 proves the compactness part of the Γ-convergence result. In Section 7, we compute the pointwise limit of $I_\delta(\mathbf{u})$ for a smooth deformation $\mathbf{u}$. Finally, Sections 8 and 9 prove, respectively, the lower bound, and upper bound parts of the Γ-convergence.

2. Statement of the main theorem

In this section, we state the main result of the paper, namely, the Γ-convergence of I_δ to I as $\delta \to 0$ in $L^p(\Omega, \mathbb{R}^d)$. We start with the explanation of the two natural choices of boundary conditions: Dirichlet and Neumann. As usual in peridynamics (see, e.g., [7, 15, 22, 23]), a Dirichlet condition consists in prescribing the value of $\mathbf{u}$ in a neigbourhood of the boundary. The simplest choice, which we follow in this paper, is to fix a $\mathbf{u}_0 : \Omega \to \mathbb{R}^d$ and prescribe $\mathbf{u} = \mathbf{u}_0$ in Ω^δ (see definition (16) below). For technical reasons, we assume that $\mathbf{u}_0$ is Lipschitz. As usual in variational problems (see, e.g., [4, Sect. 6.2]), Neumann conditions are not imposed in the formulation of the problem, but rather they occur naturally as a consequence of the minimization process. Since the functionals I_δ and I are invariant under tranlastions, to avoid the trivial non-uniqueness of minimizers, and also to prevent the sequence $\{\mathbf{u}_\delta\}_\delta$ from becoming unbounded, we impose the condition $\int_\Omega \mathbf{u} = \mathbf{0}$. On that account, for each $\delta > 0$, we define

$$\mathcal{A}_\delta := \left\{ \mathbf{u} \in L^p(\Omega, \mathbb{R}^d) : \mathbf{u} = \mathbf{u}_0 \text{ in } \Omega^\delta \right\}, \qquad \mathcal{A}_N := \left\{ \mathbf{u} \in L^p(\Omega, \mathbb{R}^d) : \int_\Omega \mathbf{u} \, \mathrm{d}\mathbf{x} = \mathbf{0} \right\},$$

and

$$\mathcal{B} := \left\{ \mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d) : \mathbf{u} = \mathbf{u}_0 \text{ on } \partial\Omega \right\}, \qquad \mathcal{B}_N := \left\{ \mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d) : \int_\Omega \mathbf{u} \, \mathrm{d}\mathbf{x} = \mathbf{0} \right\},$$

where the equality $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$ is understood in the sense of traces. Thus, $\mathcal{A}_\delta$ and $\mathcal{A}_N$ are the admissible sets for I_δ with, respectively, Dirichlet and Neumann boundary conditions, whereas $\mathcal{B}$ and $\mathcal{B}_N$ are the admissible sets for I with, respectively, Dirichlet and Neumann boundary conditions. We refer the reader to [7, 14, 15, 22] for a physical description of the nonlocal boundary conditions, as well as for an interpretation in terms of flux through the boundary.

The assumptions on the density w must be clearly written so as to rigorously prove our main result. Because those hypotheses require some precise statements, we have deferred them until Section 4.

Theorem 1. *Let Ω be a bounded domain of $\mathbb{R}^n$ with a Lipschitz boundary. Fix $\mathbf{u}_0 \in W^{1,\infty}(\Omega, \mathbb{R}^d)$ and let $w : \Omega \times \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ satisfy (W1)–(W5) below. Then the following statements hold:*

D1) For each $\delta > 0$, let $\mathbf{u}_\delta \in \mathcal{A}_\delta$ satisfy

$$\sup_\delta I_\delta(\mathbf{u}_\delta) < \infty. \tag{12}$$

Then there exists $\mathbf{u} \in \mathcal{B}$ such that, for a subsequence, $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$.

D2) For each $\delta > 0$, let $\mathbf{u}_\delta \in \mathcal{A}_\delta$ and $\mathbf{u} \in \mathcal{B}$ satisfy $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$. Then

$$I(\mathbf{u}) \leq \liminf_{\delta \to 0} I_\delta(\mathbf{u}_\delta). \tag{13}$$

D3) For each $\mathbf{u} \in \mathcal{B}$ and $\delta > 0$, there exists $\mathbf{u}_\delta \in \mathcal{A}_\delta$ such that $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ and

$$\limsup_{\delta \to 0} I_\delta(\mathbf{u}_\delta) \leq I(\mathbf{u}). \tag{14}$$

N1) For each $\delta > 0$, let $\mathbf{u}_\delta \in \mathcal{A}_N$ satisfy (12). Then there exists $\mathbf{u} \in \mathcal{B}_N$ such that, for a subsequence, $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$.

N2) For each $\delta > 0$, let $\mathbf{u}_\delta \in \mathcal{A}_N$ and $\mathbf{u} \in \mathcal{B}_N$ satisfy $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$. Then (13) holds.

N3) For each $\mathbf{u} \in \mathcal{B}_N$ and $\delta > 0$, there exists $\mathbf{u}_\delta \in \mathcal{A}_N$ such that $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ and (14) holds.

Theorem 1 is a truly Γ-convergence result, but we have avoided an explicit statement as such, since this would entail to extend the definitions of I_δ and I outside their natural domains ($\mathcal{A}_\delta$, $\mathcal{A}_N$, $\mathcal{B}$ or $\mathcal{B}_N$, as explained before) by infinity, thus making, in our opinion, the conclusion of Theorem 1 less transparent. Statements *D1–D3)* constitute the Γ-convergence result of I_δ to I in the $L^p(\Omega, \mathbb{R}^d)$ topology with Dirichlet conditions, while statements *N1–N3)* are the analogue result for Neumann boundary conditions. As usual, *D1)* and *N1)* is the compactness part of the proof, *D2)* and *N2)* the lower bound inequality, and *D3)* and *N3)* the upper bound inequality.

For the sake of clarity, we have decided to define the functionals I_δ and I, and, consequently, to state the Γ-convergence result with no mention to external forces. It is a well-known fact (see, e.g., [9, Rk. 2.2 and Sect. 4]) that Γ-convergence is stable under continuous perturbations. In our case, this implies that if we add to I_δ and I a functional that is continuous in the $L^p(\Omega, \mathbb{R}^d)$ topology, the Γ-convergence result remains true. It turns out that external body and surface forces have typically the form

$$\int_\Omega F(\mathbf{x}, \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \tag{15}$$

for some potential function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$, which, in fact, is usually taken to be linear (see, e.g., [1, 7, 23, 30, 31, 33]). If the function F is such that the functional (15) is continuous in $L^p(\Omega, \mathbb{R}^d)$ (see, e.g., [19, Cor. 6.51] for a necessary and sufficient condition), then the Γ-convergence result still holds when (15) is added to I_δ and I.

3. Notation

In this section, we set some general notation, most of which is standard.

A *domain* is a non-empty, open and connected set. Let Ω be a bounded domain of $\mathbb{R}^n$ representing the reference configuration of the body. Its boundary is written $\partial\Omega$, and $\bar{\Omega}$, its closure. The set $\tilde{\Omega}$ stands for the difference $\Omega - \Omega$, so that some neighbourhood of the origin is contained in $\tilde{\Omega}$. Given $\mathbf{x} \in \Omega$ and $\delta > 0$, we denote by $B(\mathbf{x}, \delta)$ the open ball centred at $\mathbf{0}$ of radius δ.

The expression *a.e.* for *almost everywhere* or *almost every* refers, unless otherwise stated, to the Lebesgue measure in $\mathbb{R}^n$, which is denoted by $\mathcal{L}^n$. The $(n-1)$-dimensional Hausdorff measure is denoted by $\mathcal{H}^{n-1}$. The set of unit vectors in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1}$, and its $\mathcal{H}^{n-1}$-measure by σ_{n-1}.

We write $\mathbf{x}$ for the coordinates in the reference configuration Ω, and $\mathbf{y}$ in the deformed configuration $\mathbb{R}^d$. We write $\tilde{\mathbf{x}}$ for coordinates in $\tilde{\Omega}$, while $\tilde{\mathbf{y}}$ is reserved for coordinates of functions whose argument is a difference between two points in the deformed configuration. Thus, the natural notation for the variables of w is $(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. Vector and matrix quantities are written in boldface.

For $1 \leq p \leq \infty$, the Lebesgue L^p and the Sobolev $W^{1,p}$ spaces are defined in the usual way. So are the functions of class C^k for some integer k or infinity. We will always indicate the domain and target sets, as in, for example, $L^p(\Omega,\mathbb{R}^d)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^p(\Omega)$. The Lebesgue space $L^1(\mathbb{S}^{n-1})$ is with respect to the measure $\mathcal{H}^{n-1}$.

Convergence for numbers or for functions is indicated by $\to$. Convergence of functions is always understood in the norm topology of the relevant space. We will not deal with weak convergence.

For each $r>0$, we define

$$\Omega_r := \{\mathbf{x}\in\Omega : \operatorname{dist}(\mathbf{x},\partial\Omega) > r\}, \qquad \Omega^r := \{\mathbf{x}\in\Omega : \operatorname{dist}(\mathbf{x},\partial\Omega) < r\}, \tag{16}$$

where $\operatorname{dist}(\mathbf{x},\partial\Omega)$ equals the distance from the point $\mathbf{x}$ to the set $\partial\Omega$.

As usual in Γ-convergence problems indexed by a continuous parameter (see, e.g., [9, Sect. 2]), in the computation of the Γ-limit of I_δ, we will fix a sequence of positive numbers tending to zero, and denote it by $\{\delta\}_\delta$. The letter δ is reserved for a member of the fixed sequence, so expressions like "for every $\delta>0$" mean "for every member δ of the sequence", and $\{\mathbf{u}_\delta\}_\delta$ denotes the sequence of $\mathbf{u}_\delta$ labelled by the sequence of δ. Subsequences of δ will not be relabelled. All convergences involving δ are understood as the sequence $\{\delta\}_\delta$ goes to zero, abbreviated to $\delta\to 0$. For example, in the expression $\mathbf{u}_\delta\to\mathbf{u}$ it is understood that the convergence holds as $\delta\to 0$.

4. Assumptions on the pairwise potential

We next present and comment on the assumptions of the pairwise potential function w. We also describe the properties that the functions w°, $\bar{w}$ and $Q\bar{w}$ inherit from those conditions.

Let $1<p<\infty$, and $\alpha<n+p$, and put $\beta := p-\alpha$. The need of the restriction $\alpha<n+p$ is because otherwise, for w as in (5), the only functions $\mathbf{u}$ with finite energy (3) or (4) would be constant. Notice, as a consequence, that $n+\beta>0$.

The assumptions for $w:\Omega\times\tilde{\Omega}\times\mathbb{R}^d\to\mathbb{R}$ are as follows.

(W1) Measurability. w is $\mathcal{L}^n(\Omega\times\tilde{\Omega})\times\mathcal{B}(\mathbb{R}^d)$-measurable.

(W2) Convexity. There exists $r_0>0$ such that for a.e. $\mathbf{x}\in\Omega$, all $0<r<r_0$ and all $\delta>0$, the function

$$\int_{\Omega_r\cap B(\mathbf{x},\delta)} w(\mathbf{x},\mathbf{x}'-\mathbf{x},\cdot)\,\mathrm{d}\mathbf{x}' \tag{17}$$

is convex in $\mathbb{R}^d$.

(W3) Coercivity. There exist $\psi_0:\tilde{\Omega}\to[0,\infty]$, and $\tilde{\psi}_0\in L^1(\mathbb{S}^{n-1})$ with $\tilde{\psi}_0\geq 0$ such that

$$\limsup_{t\to 0^+}\operatorname*{ess\,sup}_{\mathbf{z}\in\mathbb{S}^{n-1}}\left[\frac{1}{t^\beta}\psi_0(t\mathbf{z})-\tilde{\psi}_0(\mathbf{z})\right]\leq 0$$

and, for some constant $c_0>0$,

$$\max\{0, c_0\frac{|\tilde{\mathbf{y}}|^p}{|\tilde{\mathbf{x}}|^\alpha}-\psi_0(\tilde{\mathbf{x}})\}\leq w(\mathbf{x},\tilde{\mathbf{x}},\tilde{\mathbf{y}}), \qquad \text{for a.e. } \mathbf{x}\in\Omega,\quad \text{all } \tilde{\mathbf{x}}\in\tilde{\Omega}\setminus\{\mathbf{0}\},\quad \text{all } \tilde{\mathbf{y}}\in\mathbb{R}^d.$$

(W4) Growth. There exist $\psi_1:\tilde{\Omega}\to[0,\infty]$, and $\tilde{\psi}_1, h\in L^1(\mathbb{S}^{n-1})$ with $\tilde{\psi}_1, h\geq 0$ such that

$$\limsup_{t\to 0^+}\operatorname*{ess\,sup}_{\mathbf{z}\in\mathbb{S}^{n-1}}\left[\frac{1}{t^\beta}\psi_1(t\mathbf{z})-\tilde{\psi}_1(\mathbf{z})\right]\leq 0$$

and

$$w(\mathbf{x},\tilde{\mathbf{x}},\tilde{\mathbf{y}})\leq h\left(\frac{\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|}\right)\frac{|\tilde{\mathbf{y}}|^p}{|\tilde{\mathbf{x}}|^\alpha}+\psi_1(\tilde{\mathbf{x}}), \qquad \text{for a.e. } \mathbf{x}\in\Omega,\quad \text{all } \tilde{\mathbf{x}}\in\tilde{\Omega}\setminus\{\mathbf{0}\},\quad \text{all } \tilde{\mathbf{y}}\in\mathbb{R}^d.$$

Moreover, there exists a decreasing function $\sigma:[0,\infty)\to[0,\infty)$ such that

$$\lim_{t\to 0^+}\sigma(t)=\sigma(0)=0$$

and for a.e. $\mathbf{x}_1,\mathbf{x}_2\in\Omega$, all $\tilde{\mathbf{x}}\in\tilde{\Omega}\setminus\{\mathbf{0}\}$ and all $\tilde{\mathbf{y}}\in\mathbb{R}^d$,

$$|w(\mathbf{x}_2,\tilde{\mathbf{x}},\tilde{\mathbf{y}})-w(\mathbf{x}_1,\tilde{\mathbf{x}},\tilde{\mathbf{y}})|\leq\sigma(|\mathbf{x}_2-\mathbf{x}_1|)\left[h\left(\frac{\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|}\right)\frac{|\tilde{\mathbf{y}}|^p}{|\tilde{\mathbf{x}}|^\alpha}+\psi_1(\tilde{\mathbf{x}})\right]$$

(W5) Blow-up at zero. The function $w^\circ : \Omega \times (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times \mathbb{R}^d \to \mathbb{R}$ defined through

$$w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) := \lim_{t \to 0^+} \frac{1}{t^\beta} w(\mathbf{x}, t\tilde{\mathbf{x}}, t\tilde{\mathbf{y}}) \tag{18}$$

satisfies that, for each compact $K \subset \mathbb{R}^d$, w° is uniformly continuous in $\Omega \times \mathbb{S}^{n-1} \times K$, and

$$\lim_{t \to 0^+} \operatorname*{ess\,sup}_{\mathbf{x} \in \Omega} \sup_{\tilde{\mathbf{x}} \in \mathbb{S}^{n-1}} \sup_{\tilde{\mathbf{y}} \in K} \left| \frac{1}{t^\beta} w(\mathbf{x}, t\tilde{\mathbf{x}}, t\tilde{\mathbf{y}}) - w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \right| = 0.$$

In the following paragraphs, we make some comments on the assumptions above.

Notice that, since we are interested in the behaviour of (4) as $\delta \to 0$, the function w can be defined arbitrarily (for example, as zero) in $\Omega \times (\tilde{\Omega} \setminus B(\mathbf{0}, \delta_0)) \times \mathbb{R}^d$ for a fixed $\delta_0 > 0$.

Assumption (W1) means that w is (Lebesgue) measurable in the variables $(\mathbf{x}, \tilde{\mathbf{x}})$ and Borel measurable in the variables $\tilde{\mathbf{y}}$. This hypothesis is standard when dealing with composition operators, and implies that the function $(\mathbf{x}, \mathbf{x}') \mapsto w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))$ is measurable whenever $\mathbf{u} : \Omega \to \mathbb{R}^d$ is measurable.

Assumption (W2) requires a longer explanation. In a recent work, Elbau [16] has found a condition on w that, under natural growth assumptions, it is necessary and sufficient for the lower semicontinuity of functionals with double integrals more general than (3). When this condition is recast for functionals of the form of (3) (see, if necessary, [7]), it reads as follows:

(C) For a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{v} \in L^p(\Omega, \mathbb{R}^d)$, the function

$$\mathbf{y} \mapsto \int_\Omega w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{v}(\mathbf{x}') - \mathbf{y})\, \mathrm{d}\mathbf{x}'$$

is convex in $\mathbb{R}^d$.

Since we are concerned in the analysis of (4) rather than (3), the natural assumption for the existence of minimizers would be

(C_δ) For a.e. $\mathbf{x} \in \Omega$, all $\mathbf{v} \in L^p(\Omega, \mathbb{R}^d)$ and all $\delta > 0$, the function

$$\mathbf{y} \mapsto \int_{\Omega \cap B(\mathbf{x}, \delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{v}(\mathbf{x}') - \mathbf{y})\, \mathrm{d}\mathbf{x}' \tag{19}$$

is convex in $\mathbb{R}^d$.

For technical reasons, we need that (19) holds not only for Ω but also for all Ω_r; in this case, and taking $\mathbf{v} = \mathbf{0}$, the new condition reads as: for a.e. $\mathbf{x} \in \Omega$, all $0 < r < r_0$ and all $\delta > 0$, the function

$$\mathbf{y} \mapsto \int_{\Omega_r \cap B(\mathbf{x}, \delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, -\mathbf{y})\, \mathrm{d}\mathbf{x}' \tag{20}$$

is convex. Finally, since convexity is invariant under affine changes of variables, the function (20) is convex if and only if so is the function (17). This discussion shows that condition (W2) is neither stronger nor weaker than condition (C_δ), which is the natural one for the existence of minimizers. Nevertheless, they are obviously related, and, more importantly, both are simultaneously satisfied for a large class of functions w. In fact, we believe that one could apply some approximation result in order to be dispensed with the need of the introduction of Ω_r in (17). In full truth, we ought to say that none of the nonlocal convexity conditions (W2), (C) or (C_δ) is properly understood; see [7, 16] for a brief discussion.

Properties (W3) and (W4) are coercivity and growth conditions modelled on the paradigmatic example (5), with some perturbation controled by the functions ψ_0, $\tilde{\psi}_0$, ψ_1, $\tilde{\psi}_1$ and h. The scaling $1/t^\beta$ is the natural one in view of (18). Of course, ess sup means the essential supremum with respect to the measure $\mathcal{H}^{n-1}$. In (W4) we also require a continuity property of w in the variable $\mathbf{x}$.

Assumption (W5) constructs the function w°, which satisfies that $w^\circ(\mathbf{x}, \cdot, \cdot)$ is homogeneous of order β. This homogeneity is important in order to pass to the pointwise limit in $I_\delta(\mathbf{u})$ for a smooth $\mathbf{u}$ (see Section 7). This represents a generalization of [29], where his analysis concerns homogeneous functions of degree 0; we also cover the dependence of w and w° on $\mathbf{x}$. In essence, condition (W5) permits the use of functions w such that $w(\mathbf{x}, \cdot, \cdot)$ is close to a homogenous function near $(\mathbf{0}, \mathbf{0})$.

Interesting examples of functions w can be constructed as follows. First, we note that the sum of functions satisfying (W1)–(W5) also satisfies (W1)–(W5). One such function is $w(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = f_0(\mathbf{x})\, f(\tilde{\mathbf{x}})\, g(\tilde{\mathbf{y}})$, where $f_0 : \Omega \to \mathbb{R}$, $f : \tilde{\Omega} \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ satisfy the following properties:

(1) f_0 is uniformly continuous, f is measurable and g is Borel measurable.
(2) g is convex.
(3) There exists $c_0 > 0$ such that

$$c_0 \leq f_0(\mathbf{x}), \quad \frac{c_0}{|\tilde{\mathbf{x}}|^\alpha} \leq f(\tilde{\mathbf{x}}) \text{ and } c_0|\tilde{\mathbf{y}}|^p \leq g(\tilde{\mathbf{y}}), \qquad \mathbf{x} \in \Omega, \quad \tilde{\mathbf{x}} \in \tilde{\Omega} \setminus \{\mathbf{0}\}, \quad \tilde{\mathbf{y}} \in \mathbb{R}^d.$$

(4) There exist $c_1 > 0$ and $h \in L^1(\mathbb{S}^{n-1})$ with $h \geq 0$ such that

$$f(\tilde{\mathbf{x}}) \leq h\left(\frac{\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|}\right) \frac{1}{|\tilde{\mathbf{x}}|^\alpha} \text{ and } g(\tilde{\mathbf{y}}) \leq c_1|\tilde{\mathbf{y}}|^p, \qquad \mathbf{x} \in \Omega, \quad \tilde{\mathbf{x}} \in \tilde{\Omega} \setminus \{\mathbf{0}\}, \quad \tilde{\mathbf{y}} \in \mathbb{R}^d.$$

(5) The functions $f^\circ : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ and $g^\circ : \mathbb{R}^d \to \mathbb{R}$ defined as

$$f^\circ(\tilde{\mathbf{x}}) := \lim_{t \to 0^+} t^\alpha f(t\tilde{\mathbf{x}}), \qquad g^\circ(\tilde{\mathbf{y}}) := \lim_{t \to 0^+} \frac{1}{t^p} g(t\tilde{\mathbf{y}})$$

are continuous and satisfy that for each compact $K \subset \mathbb{R}^d$,

$$\lim_{t \to 0^+} \sup_{\tilde{\mathbf{x}} \in \mathbb{S}^{n-1}} |t^\alpha f(t\tilde{\mathbf{x}}) - f^\circ(\tilde{\mathbf{x}})| = 0, \qquad \lim_{t \to 0^+} \sup_{\tilde{\mathbf{y}} \in K} \left| \frac{1}{t^p} g(t\tilde{\mathbf{y}}) - g^\circ(\tilde{\mathbf{y}}) \right| = 0.$$

Given a function w satisfying (W1)–(W5), apart from the function w° of (18), we also construct the functions $\bar{w}$ and $Q\bar{w}$, whose definitions and properties are described as follows.

Lemma 2. *The function w° satisfies*

$$w^\circ(\mathbf{x}, \lambda\tilde{\mathbf{x}}, \lambda\tilde{\mathbf{y}}) = \lambda^\beta w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}), \qquad \mathbf{x} \in \Omega, \quad \lambda > 0, \quad \tilde{\mathbf{x}} \in \tilde{\Omega} \setminus \{\mathbf{0}\}, \quad \tilde{\mathbf{y}} \in \mathbb{R}^d. \tag{21}$$

Moreover, $w^\circ \geq 0$, and there exists $h_1 \in L^1(\mathbb{S}^{n-1})$ with

$$c_0|\tilde{\mathbf{y}}|^p - \tilde{\psi}_0(\tilde{\mathbf{x}}) \leq w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \leq h_1(\tilde{\mathbf{x}}) (1 + |\tilde{\mathbf{y}}|^p), \qquad \mathbf{x} \in \Omega, \quad \tilde{\mathbf{x}} \in \mathbb{S}^{n-1}, \quad \tilde{\mathbf{y}} \in \mathbb{R}^d. \tag{22}$$

Proof. Property (21) is immediate from (18), and so is $w^\circ \geq 0$ since $w \geq 0$.

To prove (22), we argue as follows. For a.e. $\mathbf{x} \in \Omega$ and all $t > 0$, $\tilde{\mathbf{x}} \in \mathbb{S}^{n-1}$ and $\tilde{\mathbf{y}} \in \mathbb{R}^d$, we have from (W4) that

$$\frac{1}{t^\beta} w(\mathbf{x}, t\tilde{\mathbf{x}}, t\tilde{\mathbf{y}}) \leq h(\tilde{\mathbf{x}})|\tilde{\mathbf{y}}|^p + \frac{1}{t^\beta} \psi_1(t\tilde{\mathbf{x}}),$$

so

$$w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \leq h(\tilde{\mathbf{x}})|\tilde{\mathbf{y}}|^p + \tilde{\psi}_1(\tilde{\mathbf{x}}) \leq \max\{h(\tilde{\mathbf{x}}), \tilde{\psi}_1(\tilde{\mathbf{x}})\} (|\tilde{\mathbf{y}}|^p + 1).$$

Because h and $\tilde{\psi}_1$ are in $L^1(\mathbb{S}^{n-1})$, so is $\max\{h, \tilde{\psi}_1\}$.

Analogously, using (W3) instead, one can show the first inequality of (22). □

Another function that plays a capital role in the result is the density $\bar{w} : \Omega \times \mathbb{R}^{d\times n} \to \mathbb{R}$ defined by

$$\bar{w}(\mathbf{x}, \mathbf{F}) := \int_{\mathbb{S}^{n-1}} w^\circ(\mathbf{x}, \mathbf{z}, \mathbf{F}\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}). \tag{23}$$

Lemma 3. *For all $\mathbf{x} \in \Omega$, the function $\bar{w}(\mathbf{x}, \cdot)$ is continuous, $\bar{w} \geq 0$, and there exist $c_1, C_0 > 0$ such that*

$$C_0 (|\mathbf{F}|^p - 1) \leq \bar{w}(\mathbf{x}, \mathbf{F}) \leq c_1 (1 + |\mathbf{F}|^p), \qquad \mathbf{x} \in \Omega, \quad \mathbf{F} \in \mathbb{R}^{d\times n}.$$

Proof. Let $\{\mathbf{F}_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{R}^{d\times n}$ tending to some $\mathbf{F} \in \mathbb{R}^{d\times n}$. Fix $\mathbf{x} \in \Omega$. Then $w^\circ(\mathbf{x}, \mathbf{z}, \mathbf{F}_j\mathbf{z}) \to w^\circ(\mathbf{x}, \mathbf{z}, \mathbf{F}\mathbf{z})$ as $j \to \infty$. Moreover, by Lemma 2,

$$0 \leq w^\circ(\mathbf{x}, \mathbf{z}, \mathbf{F}_j\mathbf{z}) \leq \left(1 + \sup_{k\in\mathbb{N}} |\mathbf{F}_k|^p\right) h_1(\mathbf{z}), \qquad j \in \mathbb{N}, \quad \mathbf{z} \in \mathbb{S}^{n-1},$$

where $h_1 \in L^1(\mathbb{S}^{n-1})$ is the function occurring in Lemma 2. By dominated convergence, $\bar{w}(\mathbf{x}, \mathbf{F}_j) \to \bar{w}(\mathbf{x}, \mathbf{F})$ as $j \to \infty$, which shows the continuity of $\bar{w}(\mathbf{x}, \cdot)$. Inequality $\bar{w} \geq 0$, on the other hand, is obvious.

Again by (22),

$$c_0 \int_{\mathbb{S}^{n-1}} |\mathbf{F}\mathbf{z}|^p \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) - \int_{\mathbb{S}^{n-1}} \tilde{\psi}_0(\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \leq \bar{w}(\mathbf{x}, \mathbf{F}) \leq \int_{\mathbb{S}^{n-1}} h_1(\mathbf{z}) (1 + |\mathbf{F}\mathbf{z}|^p) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).$$

Now,

$$\int_{\mathbb{S}^{n-1}} h_1(\mathbf{z})\,(1+|\mathbf{F}\mathbf{z}|^p)\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \leq \int_{\mathbb{S}^{n-1}} h_1(\mathbf{z})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,(1+|\mathbf{F}|^p)\,,$$

while, on the other hand, using Minkowski's inequality, it is immediate to check that the quantity

$$\mathbf{F} \mapsto \left(\int_{\mathbb{S}^{n-1}} |\mathbf{F}\mathbf{z}|^p\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\right)^{1/p}$$

defines a norm in $\mathbb{R}^{d\times n}$, so it is equivalent to the standard norm. This finishes the proof. □

The following notions are standard (see, e.g., [13]). A Borel and locally bounded function $v:\mathbb{R}^{d\times n}\to\mathbb{R}$ is *quasiconvex* if

$$v(\mathbf{F}) \leq \int_{(0,1)^n} v(\mathbf{F}+D\boldsymbol{\varphi}(\mathbf{x}))\,\mathrm{d}\mathbf{x}$$

for all $\boldsymbol{\varphi}\in C_c^1((0,1)^n,\mathbb{R}^d)$. The *quasiconvexification* (or *quasiconvex envelope*) $Q\bar{w}$ of $\bar{w}$ is defined, for a.e. $\mathbf{x}\in\Omega$, as

$$Q\bar{w}(\mathbf{x},\mathbf{F}) := \sup\left\{v(\mathbf{x},\mathbf{F}) : v(\mathbf{x},\cdot)\leq\bar{w}(\mathbf{x},\cdot) \text{ and } v(\mathbf{x},\cdot) \text{ is quasiconvex}\right\}, \qquad \mathbf{F}\in\mathbb{R}^{d\times n}.$$

The following result follows from Lemma 3 and the standard theory of quasiconvex functions (see, e.g., [13, Sect. 6.3]).

Lemma 4. *The quasiconvexification $Q\bar{w}:\Omega\times\mathbb{R}^{d\times n}\to\mathbb{R}$ is well defined. Moreover, $Q\bar{w}(\mathbf{x},\cdot)$ is continuous for all $\mathbf{x}\in\Omega$, and $Q\bar{w}(\cdot,\mathbf{F})$ is measurable for all $\mathbf{F}\in\mathbb{R}^{d\times n}$. Furthermore, $Q\bar{w}\geq 0$ and*

$$C_0\,(|\mathbf{F}|^p-1) \leq Q\bar{w}(\mathbf{x},\mathbf{F}) \leq \bar{w}(\mathbf{x},\mathbf{F}) \leq c_1\,(1+|\mathbf{F}|^p)\,, \qquad \mathbf{x}\in\Omega, \quad \mathbf{F}\in\mathbb{R}^{d\times n}.$$

5. Preliminary results

In this section, we collect a few results that will be used throughout the paper.

The following simple calculation will be used several times.

Lemma 5. *Let $\delta>0$ and $\mathbf{x}\in\Omega$ be such that $B(\mathbf{x},\delta)\subset\tilde{\Omega}$. Then, for any $\psi\in L^1(\tilde{\Omega})$ or any measurable $\psi:\tilde{\Omega}\to[0,\infty]$,*

$$\int_\Omega\int_{B(\mathbf{x},\delta)} \psi(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = \mathcal{L}^n(\Omega)\int_0^\delta\int_{\mathbb{S}^{n-1}} t^{n-1}\,\psi(t\mathbf{z})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}t.$$

Proof. A change of variables shows that

$$\int_\Omega\int_{B(\mathbf{x},\delta)} \psi(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} = \mathcal{L}^n(\Omega)\int_{B(\mathbf{0},\delta)} \psi(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}}$$

while the coarea formula yields

$$\int_{B(\mathbf{0},\delta)} \psi(\tilde{\mathbf{x}})\,\mathrm{d}\tilde{\mathbf{x}} = \int_0^\delta\int_{\partial B(\mathbf{0},t)} \psi(\tilde{\mathbf{x}})\,\mathrm{d}\mathcal{H}^{n-1}(\tilde{\mathbf{x}})\,\mathrm{d}t = \int_0^\delta\int_{\mathbb{S}^{n-1}} t^{n-1}\,\psi(t\mathbf{z})\,\mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\,\mathrm{d}t. \tag{24}$$

□

Bourgain, Brezis & Mironescu [8, Th. 4] proved a compactness criterion in $W^{1,p}$ involving nonlocal integrals, within a study of a nonlocal characterization of $W^{1,p}$. It was later generalized by Ponce [28, Th. 1.2]. In both approaches, they consider a sequence of (not necessarily smooth) mollifiers $\rho_\delta:\mathbb{R}^n\to\mathbb{R}$ with some generic properties and show that the boundedness of the quantity

$$\int_\Omega |\mathbf{u}_\delta(\mathbf{x})|^p\;\mathrm{d}\mathbf{x} + \int_\Omega\int_\Omega \frac{|\mathbf{u}_\delta(\mathbf{x}')-\mathbf{u}_\delta(\mathbf{x})|^p}{|\mathbf{x}'-\mathbf{x}|^p}\,\rho_\delta(\mathbf{x}'-\mathbf{x})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \tag{25}$$

implies that $\{\mathbf{u}_\delta\}_\delta$ is relatively compact in $L^p(\Omega,\mathbb{R}^d)$, and that any limit point lies in $W^{1,p}(\Omega,\mathbb{R}^d)$. The particular choice of the (non-smooth) mollifier $\rho_\delta:\mathbb{R}^n\to\mathbb{R}$

$$\rho_\delta(\tilde{\mathbf{x}}) := \frac{n+\beta}{\sigma_{n-1}}\,\frac{1}{\delta^{n+\beta}}\,|\tilde{\mathbf{x}}|^\beta\,\chi_{B(\mathbf{0},\delta)}(\tilde{\mathbf{x}}) \tag{26}$$

(where χ indicates characteristic function) gives rise to the following result.

Proposition 6. *Fix $M > 0$. For each $\delta > 0$ let $\mathbf{u}_\delta \in L^p(\Omega, \mathbb{R}^d)$ satisfy*

$$\frac{n+\beta}{\sigma_{n-1}} \frac{1}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} \frac{|\mathbf{u}_\delta(\mathbf{x}') - \mathbf{u}_\delta(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq M.$$

The following assertions hold:

a) If $\int_\Omega \mathbf{u}_\delta \, \mathrm{d}\mathbf{x} = \mathbf{0}$ for all δ, then $\{\mathbf{u}_\delta\}_\delta$ is bounded in $L^p(\Omega, \mathbb{R}^d)$.

b) If $\{\mathbf{u}_\delta\}_\delta$ is bounded in $L^p(\Omega, \mathbb{R}^d)$, then there exists $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$ such that, for a subsequence, $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$. Moreover,

$$\int_\Omega |D\mathbf{u}|^p \, \mathrm{d}\mathbf{x} \leq C\, M,$$

for some $C > 0$ depending only on p and Ω.

Incidentally, the verification that the integral of the function ρ_δ in (26) is 1 requires formula (24).

In both papers [8, 28], it is capital a preliminary study of (25) when $\mathbf{u}_\delta = \mathbf{u}$ for all δ. In particular, when the result of [8, Th. 1] is specialized to the mollifier (26) and to the mollifier

$$\bar{\rho}_\delta(\tilde{\mathbf{x}}) := \frac{n+\beta}{\|h\|_{L^1(\mathbb{S}^{n-1})}} \frac{1}{\delta^{n+\beta}} h\left(\frac{\tilde{\mathbf{x}}}{|\tilde{\mathbf{x}}|}\right) |\tilde{\mathbf{x}}|^\beta \chi_{B(\mathbf{0},\delta)}(\tilde{\mathbf{x}}),$$

(where h is the function of (W4)), one finds the following.

Lemma 7. *There exists $C > 0$, depending only p and Ω, such that for all $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$ and $\delta > 0$,*

$$\frac{n+\beta}{\sigma_{n-1}} \frac{1}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} \frac{|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq C \int_\Omega |D\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x},$$

$$\frac{n+\beta}{\|h\|_{L^1(\mathbb{S}^{n-1})}} \frac{1}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} h\left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}\right) \frac{|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq C \int_\Omega |D\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x}.$$

6. Compactness

In this section we prove the compactness part of the Γ-limit from I_δ to I.

Since both functionals I_δ and I are invariant under translations, it becomes natural the need of a nonlocal version of the Poincaré inequality. The corresponding inequality for Neumann boundary conditions is implicit in Bourgain, Brezis & Mironescu [8] (see *a)* of Proposition 6 above), but it was made explicit by Ponce [28] as a truly inequality, who also provided some generalizations. It was Aksoylu & Mengesha [1] who observed that the inequalities of [8, 28] lent themselves to Poincaré inequalities for Neumann, Dirichlet and mixed boundary conditions suitable for peridynamics. Moreover, [1] also noticed that the corresponding inequality could be quantified in terms of δ, as needed for their numerical analysis, and, as a matter of fact, also for our study. That quantification was not available in other studies where nonlocal Poincaré inequalities were proved (see, e.g., [2]). In our case for Dirichlet conditions, we have not found exactly the inequality needed, so we follow the idea of [1] of adapting [8] in order to prove the quantified Poincaré inequality suitable for our statement with Dirichlet conditions.

Proposition 8. *Let $Q \subset \mathbb{R}^n$ be a bounded domain such that $\Omega \subset Q$ and $\mathcal{L}^n(Q \setminus \Omega) > 0$. Let $\bar{\mathbf{u}}_0 \in W^{1,p}(Q, \mathbb{R}^d)$. Then there exists $C_0 > 0$ such that for all $C > C_0$ there is a $\delta_0 > 0$ for which, if $0 < \delta < \delta_0$ and $\bar{\mathbf{u}} \in L^p(Q, \mathbb{R}^d)$ with $\bar{\mathbf{u}} = \bar{\mathbf{u}}_0$ a.e. in $Q \setminus \Omega$, one has*

$$\frac{1}{C} \int_Q |\bar{\mathbf{u}}|^p \, \mathrm{d}\mathbf{x} \leq \frac{1}{\delta^{n+\beta}} \int_Q \int_{Q \cap B(\mathbf{x},\delta)} \frac{|\bar{\mathbf{u}}(\mathbf{x}') - \bar{\mathbf{u}}(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} + \int_Q \left[|\bar{\mathbf{u}}_0|^p + |D\bar{\mathbf{u}}_0|^p \right] \mathrm{d}\mathbf{x}.$$

Proof. Suppose, additionally, $\bar{\mathbf{u}}_0 = \mathbf{0}$, and assume the result to be false. Then, for each $j \in \mathbb{N}$, there are $\delta_j > 0$ and $\bar{\mathbf{u}}_j \in L^p(\Omega, \mathbb{R}^d)$ with

$$\int_Q |\bar{\mathbf{u}}_j|^p \, \mathrm{d}\mathbf{x} = 1 \tag{27}$$

and $\bar{\mathbf{u}}_j = \mathbf{0}$ a.e. in $Q \setminus \Omega$ such that

$$\lim_{j \to \infty} \delta_j = 0, \qquad \lim_{j \to \infty} \frac{1}{\delta_j^{n+\beta}} \int_Q \int_{Q \cap B(\mathbf{x},\delta_j)} \frac{|\bar{\mathbf{u}}_j(\mathbf{x}') - \bar{\mathbf{u}}_j(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = 0.$$

By Proposition 6, there exists $\bar{\mathbf{u}} \in W^{1,p}(Q,\mathbb{R}^d)$ such that, for a subsequence, $\bar{\mathbf{u}}_j \to \bar{\mathbf{u}}$ in $L^p(Q,\mathbb{R}^d)$ as $j \to \infty$. Moreover, $D\bar{\mathbf{u}} = \mathbf{0}$ a.e., so $\bar{\mathbf{u}}$ is constant. In addition, $\bar{\mathbf{u}} = \mathbf{0}$ a.e. in $Q \setminus \Omega$, so, in fact, $\bar{\mathbf{u}} = \mathbf{0}$ a.e. in Q. But, on the other hand, due to (27) we must have

$$\int_Q |\bar{\mathbf{u}}|^p \, \mathrm{d}\mathbf{x} = 1,$$

which is a contradiction. This proves the result under the additional assumption that $\bar{\mathbf{u}}_0 = \mathbf{0}$.

We have therefore shown that there exists $C_0 > 0$ such that for all $C > C_0$ there is a $\delta_0 > 0$ for which, if $0 < \delta < \delta_0$ and $\bar{\mathbf{u}} \in L^p(Q,\mathbb{R}^d)$ with $\bar{\mathbf{u}} = \bar{\mathbf{u}}_0$ a.e. in $Q \setminus \Omega$, one has

$$\frac{1}{C}\int_Q |\bar{\mathbf{u}} - \bar{\mathbf{u}}_0|^p \, \mathrm{d}\mathbf{x} \le \frac{1}{\delta^{n+\beta}} \int_Q \int_{Q\cap B(\mathbf{x},\delta)} \frac{|(\bar{\mathbf{u}} - \bar{\mathbf{u}}_0)(\mathbf{x}') - (\bar{\mathbf{u}} - \bar{\mathbf{u}}_0)(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \tag{28}$$

We notice the pointwise inequalities

$$|(\bar{\mathbf{u}} - \bar{\mathbf{u}}_0)(\mathbf{x}') - (\bar{\mathbf{u}} - \bar{\mathbf{u}}_0)(\mathbf{x})|^p \le 2^{p-1}\left[|\bar{\mathbf{u}}(\mathbf{x}') - \bar{\mathbf{u}}(\mathbf{x})|^p + |\bar{\mathbf{u}}_0(\mathbf{x}') - \bar{\mathbf{u}}_0(\mathbf{x})|^p\right], \qquad \mathbf{x},\mathbf{x}' \in Q \tag{29}$$

and

$$|\bar{\mathbf{u}}|^p \le 2^{p-1}\left[|\bar{\mathbf{u}} - \bar{\mathbf{u}}_0|^p + |\bar{\mathbf{u}}_0|^p\right]. \tag{30}$$

An application of Lemma 7 yields

$$\frac{1}{\delta^{n+\beta}} \int_Q \int_{Q\cap B(\mathbf{x},\delta)} \frac{|\bar{\mathbf{u}}_0(\mathbf{x}') - \bar{\mathbf{u}}_0(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le C_1 \int_Q |D\bar{\mathbf{u}}_0|^p \, \mathrm{d}\mathbf{x}, \tag{31}$$

with $C_1 > 0$ depending on p, Ω, n and α. Putting together (28), (29), (30) and (31), we obtain the inequality of the statement. □

We are ready to prove the compactness part of the Γ-limit from I_δ to I.

Proof of D1) and N1) of Theorem 1. The first part of the proof is common for both *D1)* and *N1)*.

By (W3), for any $\mathbf{u} \in L^p(\Omega,\mathbb{R}^d)$,

$$I_\delta(\mathbf{u}) \ge c_0 \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega\cap B(\mathbf{x},\delta)} \frac{|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} - \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega\cap B(\mathbf{x},\delta)} \psi_0(\mathbf{x}' - \mathbf{x}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \tag{32}$$

Thanks to Lemma 5,

$$\int_\Omega \int_{\Omega\cap B(\mathbf{x},\delta)} \psi_0(\mathbf{x}' - \mathbf{x}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le \mathcal{L}^n(\Omega) \int_0^\delta \int_{\mathbb{S}^{n-1}} t^{n-1} \, \psi_0(t\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}t. \tag{33}$$

By (W3) again, there exists $t_0 > 0$ such that for all $0 < t \le t_0$ and $\mathcal{H}^{n-1}$-a.e. $\mathbf{z} \in \mathbb{S}^{n-1}$,

$$\frac{\psi_0(t\mathbf{z})}{t^\beta} \le 1 + \tilde{\psi}_0(\mathbf{z}).$$

Thus, if $0 < \delta \le t_0$,

$$\begin{aligned} \frac{n+\beta}{\delta^{n+\beta}} \int_0^\delta \int_{\mathbb{S}^{n-1}} t^{n-1} \, \psi_0(t\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}t &\le \frac{n+\beta}{\delta^{n+\beta}} \int_0^\delta t^{n+\beta-1} \, \mathrm{d}t \int_{\mathbb{S}^{n-1}} \left(1 + \tilde{\psi}_0(\mathbf{z})\right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \\ &= \int_{\mathbb{S}^{n-1}} \left(1 + \tilde{\psi}_0(\mathbf{z})\right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) < \infty. \end{aligned} \tag{34}$$

Putting together (33) and (34), we conclude that

$$\sup_{0<\delta\le t_0} \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega\cap B(\mathbf{x},\delta)} \psi_0(\mathbf{x}' - \mathbf{x}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} < \infty. \tag{35}$$

Using the assumption (12), inequalities (32) and (35) show that

$$\sup_{0<\delta\le t_0} \frac{1}{\delta^{n+\beta}} \int_\Omega \int_{\Omega\cap B(\mathbf{x},\delta)} \frac{|\mathbf{u}_\delta(\mathbf{x}') - \mathbf{u}_\delta(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} < \infty. \tag{36}$$

From this point, the argument is different for *D1)* and *N1)*.

Suppose $\mathbf{u}_\delta \in \mathcal{A}_N$ for all δ. In this case, Proposition 6 shows that, there exists $\mathbf{u} \in W^{1,p}(\Omega,\mathbb{R}^d)$ such that, for a subsequence, $\mathbf{u}_\delta \to \mathbf{u}$ in $L^p(\Omega,\mathbb{R}^d)$. Clearly, $\int_\Omega \mathbf{u} \, \mathrm{d}\mathbf{x} = \mathbf{0}$, so $\mathbf{u} \in \mathcal{B}_N$. This proves part *N1)*.

We assume now that $\mathbf{u}_\delta \in \mathcal{A}_\delta$ for all δ. Let Q be a bounded smooth domain containing $\bar{\Omega}$. Because Ω has a Lipschitz boundary, there exists $\bar{\mathbf{u}}_0 \in W^{1,p}(Q, \mathbb{R}^d)$ such that $\bar{\mathbf{u}}_0|_\Omega = \mathbf{u}_0$. For each $\delta > 0$, define $\bar{\mathbf{u}}_\delta : Q \to \mathbb{R}^d$ by

$$\bar{\mathbf{u}}_\delta := \begin{cases} \mathbf{u}_\delta & \text{in } \Omega, \\ \bar{\mathbf{u}}_0 & \text{in } Q \setminus \Omega. \end{cases}$$

Recalling, (16), put

$$A_\delta := \left\{ (\mathbf{x}, \mathbf{x}') \in \left[\Omega^\delta \times (Q \setminus \Omega) \right] \cup \left[(Q \setminus \Omega) \times \left(\Omega^\delta \cup (Q \setminus \Omega) \right) \right] : |\mathbf{x}' - \mathbf{x}| < \delta \right\}$$

and notice that

$$\int_Q \int_{Q \cap B(\mathbf{x},\delta)} \frac{|\bar{\mathbf{u}}_\delta(\mathbf{x}') - \bar{\mathbf{u}}_\delta(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = \left[\int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} + \iint\limits_{A_\delta} \right] \frac{|\bar{\mathbf{u}}_\delta(\mathbf{x}') - \bar{\mathbf{u}}_\delta(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \tag{37}$$

Moreover,

$$\iint\limits_{A_\delta} \frac{|\bar{\mathbf{u}}_\delta(\mathbf{x}') - \bar{\mathbf{u}}_\delta(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = \iint\limits_{A_\delta} \frac{|\bar{\mathbf{u}}_0(\mathbf{x}') - \bar{\mathbf{u}}_0(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \tag{38}$$

As in (31), Lemma 7 yields

$$\frac{1}{\delta^{n+\beta}} \iint\limits_{A_\delta} \frac{|\bar{\mathbf{u}}_0(\mathbf{x}') - \bar{\mathbf{u}}_0(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le C_1 \int_Q |D\bar{\mathbf{u}}_0|^p \, \mathrm{d}\mathbf{x}, \tag{39}$$

with $C_1 > 0$ depending on p, Ω, n, and α. Equations (36), (37), (38), and (39) imply that

$$\sup_{0<\delta\le t_0} \frac{1}{\delta^{n+\beta}} \int_Q \int_{Q \cap B(\mathbf{x},\delta)} \frac{|\bar{\mathbf{u}}_\delta(\mathbf{x}') - \bar{\mathbf{u}}_\delta(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} < \infty. \tag{40}$$

An application of Proposition 8 shows that there exists $t_1 > 0$ with

$$\sup_{0<\delta\le t_1} \int_Q |\mathbf{u}_\delta|^p \, \mathrm{d}\mathbf{x} < \infty. \tag{41}$$

Thanks to (40) and (41), we can apply Proposition 6, and conclude that there exists $\bar{\mathbf{u}} \in W^{1,p}(Q, \mathbb{R}^d)$ such that, for a subsequence, $\bar{\mathbf{u}}_\delta \to \bar{\mathbf{u}}$ in $L^p(Q, \mathbb{R}^d)$. Since $\bar{\mathbf{u}}_\delta = \bar{\mathbf{u}}_0$ in $Q \setminus \Omega$ for all δ, we have that $\bar{\mathbf{u}} = \bar{\mathbf{u}}_0$ a.e. in $Q \setminus \Omega$. Hence, $\mathbf{u} - \mathbf{u}_0 \in W^{1,p}(Q, \mathbb{R}^d)$ with $\mathbf{u} - \mathbf{u}_0 = \mathbf{0}$ a.e. in $Q \setminus \Omega$. A classic characterization of the space $W_0^{1,p}$ (see, e.g., [10, Prop. IX.18]) shows that, calling $\mathbf{u} := \bar{\mathbf{u}}|_\Omega$, we have that $\mathbf{u} - \mathbf{u}_0 \in W_0^{1,p}(\Omega, \mathbb{R}^d)$, so $\mathbf{u} = \mathbf{u}_0$ in $\partial\Omega$ in the sense of traces. Therefore, $\mathbf{u} \in \mathcal{B}$, which shows part *D1)*. □

7. Pointwise limit

We define the functional $\bar{I} : W^{1,p}(\Omega, \mathbb{R}^d) \to \mathbb{R}$ as

$$\bar{I}(\mathbf{u}) := \int_\Omega \bar{w}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x}.$$

It will be shown to play an important intermediate step for both the lower and upper bound inequalities. In this section, we prove that $I_\delta(\mathbf{u}) \to \bar{I}(\mathbf{u})$ when $\mathbf{u}$ is smooth. In fact, it is convenient to ascertain to what extent that limit is uniform in $\mathbf{u}$. The proof follows the lines of Ponce [29, Prop. 1].

Recall that $C^1(\bar{\Omega}, \mathbb{R}^d)$ denotes the set of $\mathbf{u} \in C(\bar{\Omega}, \mathbb{R}^d)$ such that $\mathbf{u}$ is differentiable in Ω, and $D\mathbf{u}$ is uniformly continuous in Ω; in this case, $D\mathbf{u}$ is extended to $\bar{\Omega}$ by continuity.

Proposition 9. *Let $\mathcal{F} \subset C^1(\bar{\Omega}, \mathbb{R}^d)$ satisfy that $\{D\mathbf{u} : \mathbf{u} \in \mathcal{F}\}$ is bounded in $C(\bar{\Omega}, \mathbb{R}^d)$ and equicontinuous. Then*

$$\lim_{\delta\to 0} \sup_{\mathbf{u}\in\mathcal{F}} \left| I_\delta(\mathbf{u}) - \bar{I}(\mathbf{u}) \right| = 0. \tag{42}$$

Proof. Let M be the supremum of the Lipschitz constants of $\mathbf{u} \in \mathcal{F}$. Since $\{D\mathbf{u} : \mathbf{u} \in \mathcal{F}\}$ is equicontinuous, there exists a decreasing function $\mu : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t \to 0^+} \mu(t) = \mu(0) = 0$$

and, for all $\mathbf{x}, \mathbf{x}' \in \Omega$, and $\mathbf{u} \in \mathcal{F}$,

$$|D\mathbf{u}(\mathbf{x}') - D\mathbf{u}(\mathbf{x})| \leq \mu(|\mathbf{x}' - \mathbf{x}|). \tag{43}$$

Limit (42) will be a consequence of the following three:

$$\lim_{\delta \to 0} \sup_{\mathbf{u} \in \mathcal{F}} \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} |w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) - w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))| \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = 0, \tag{44}$$

$$\lim_{\delta \to 0} \sup_{\mathbf{u} \in \mathcal{F}} \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} |w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) - w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x}))| \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = 0, \tag{45}$$

$$\lim_{\delta \to 0} \sup_{\mathbf{u} \in \mathcal{F}} \left| \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} - \bar{I}(\mathbf{u}) \right| = 0. \tag{46}$$

Fix $\mathbf{u} \in \mathcal{F}$. We start with (44). Thanks to Lemma 2, for a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{x}' \in \Omega \setminus \{\mathbf{x}\}$,

$$\begin{aligned} &w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) - w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \\ =&w(\mathbf{x}, |\mathbf{x}' - \mathbf{x}| \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}, |\mathbf{x}' - \mathbf{x}| \frac{\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})}{|\mathbf{x}' - \mathbf{x}|}) - |\mathbf{x}' - \mathbf{x}|^\beta w^\circ(\mathbf{x}, \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}, \frac{\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})}{|\mathbf{x}' - \mathbf{x}|}). \end{aligned} \tag{47}$$

Let $\varepsilon > 0$. By (W5), there exists $t_0 > 0$ such that if $0 < t \leq t_0$, then for a.e. $\mathbf{x} \in \Omega$, all $\tilde{\mathbf{x}} \in \mathbb{S}^{n-1}$ and all $\tilde{\mathbf{y}} \in \mathbb{R}^d$ with $|\tilde{\mathbf{y}}| \leq M$,

$$\left| w(\mathbf{x}, t\tilde{\mathbf{x}}, t\tilde{\mathbf{y}}) - t^\beta w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \right| \leq t^\beta \varepsilon. \tag{48}$$

Let $0 < \delta \leq t_0$. Using (47) and (48), we find that for a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{x}' \in \Omega \cap B(\mathbf{x}, \delta) \setminus \{\mathbf{x}\}$,

$$|w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) - w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))| \leq |\mathbf{x}' - \mathbf{x}|^\beta \varepsilon,$$

so

$$\begin{aligned} &\frac{n+\beta}{\delta^{n+\beta}} \left| \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} [w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) - w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))] \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \right| \\ \leq &\frac{n+\beta}{\delta^{n+\beta}} \, \varepsilon \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} |\mathbf{x}' - \mathbf{x}|^\beta \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq \mathcal{L}^n(\Omega) \, \sigma_{n-1} \, \varepsilon, \end{aligned}$$

where in the last inequality we have used Lemma 5. This shows (44).

We pass to (45). Let $\varepsilon > 0$. By (W5), w° is uniformly continuous in $\Omega \times \mathbb{S}^{n-1} \times \bar{B}(\mathbf{0}, M)$. It is then easy to prove by contradiction that there exists $C_\varepsilon > 0$ such that for all $\mathbf{x} \in \Omega$, $\tilde{\mathbf{x}} \in \mathbb{S}^{n-1}$ and $\tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2 \in \bar{B}(\mathbf{0}, M)$,

$$|w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}_1) - w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}_2)| \leq C_\varepsilon \, |\tilde{\mathbf{y}}_1 - \tilde{\mathbf{y}}_2| + \frac{\varepsilon}{2}.$$

Then, for all $\mathbf{x}, \mathbf{x}' \in \Omega$ with $\mathbf{x}' \neq \mathbf{x}$,

$$\left| w^\circ(\mathbf{x}, \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}, \frac{\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})}{|\mathbf{x}' - \mathbf{x}|}) - w^\circ(\mathbf{x}, \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}, D\mathbf{u}(\mathbf{x}) \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}) \right| \leq C_\varepsilon \left| \frac{\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) - D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \right| + \frac{\varepsilon}{2}. \tag{49}$$

By the fundamental theorem of Calculus,

$$\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) - D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x}) = \int_0^1 [D\mathbf{u}(\mathbf{x} + t(\mathbf{x}' - \mathbf{x})) - D\mathbf{u}(\mathbf{x})] \, \mathrm{d}t \, (\mathbf{x}' - \mathbf{x}),$$

so, thanks to (43), if $\mathbf{x} \in \Omega$ and $\mathbf{x}' \in \Omega \cap B(\mathbf{x}, \delta) \setminus \{\mathbf{x}\}$,

$$\left| \frac{\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}) - D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|} \right| \leq \int_0^1 |D\mathbf{u}(\mathbf{x} + t(\mathbf{x}' - \mathbf{x})) - D\mathbf{u}(\mathbf{x})| \, \mathrm{d}t \leq \int_0^1 \mu(t|\mathbf{x}' - \mathbf{x}|) \, \mathrm{d}t \leq \mu(\delta). \tag{50}$$

Choose $\delta_0 > 0$ such that

$$\mu(\delta_0) \leq \frac{\varepsilon}{2\, C_\varepsilon} \tag{51}$$

and let $0 < \delta \leq \delta_0$. Putting together (49), (50) and (51) we obtain

$$\left| w^\circ(\mathbf{x}, \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}, \frac{\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})}{|\mathbf{x}' - \mathbf{x}|}) - w^\circ(\mathbf{x}, \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}, D\mathbf{u}(\mathbf{x}) \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}) \right| \leq \varepsilon.$$

Thus, using also Lemmas 2 and 5,

$$\begin{aligned}
&\frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} |w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) - w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x}))| \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\
=&\frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} |\mathbf{x}' - \mathbf{x}|^\beta \left| w^\circ(\mathbf{x}, \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}, \frac{\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})}{|\mathbf{x}' - \mathbf{x}|}) - w^\circ(\mathbf{x}, \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}, D\mathbf{u}(\mathbf{x}) \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}) \right| \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\
\leq&\frac{n+\beta}{\delta^{n+\beta}} \, \varepsilon \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} |\mathbf{x}' - \mathbf{x}|^\beta \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq \mathcal{L}^n(\Omega) \, \sigma_{n-1} \, \varepsilon,
\end{aligned}$$

so (45) holds.

Finally, we show (46). We note that

$$\int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = \int_\Omega \int_{(\Omega - \mathbf{x}) \cap B(\mathbf{0},\delta)} w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, D\mathbf{u}(\mathbf{x})\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} \, \mathrm{d}\mathbf{x} \tag{52}$$

and, for all $\mathbf{F} \in \mathbb{R}^{d \times n}$, arguing as in (24) and using (21) and the definition (23),

$$\begin{aligned}
\int_{B(\mathbf{0},\delta)} w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{F}\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} &= \int_0^\delta \int_{\mathbb{S}^{n-1}} t^{n-1} \, w^\circ(\mathbf{x}, t\mathbf{z}, t\mathbf{F}\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}t \\
&= \int_0^\delta t^{n+\beta-1} \, \mathrm{d}t \int_{\mathbb{S}^{n-1}} w^\circ(\mathbf{x}, \mathbf{z}, \mathbf{F}\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) = \frac{\delta^{n+\beta}}{n+\beta} \, \bar{w}(\mathbf{x}, \mathbf{F}).
\end{aligned} \tag{53}$$

Recall the notation (16). We can split

$$\int_\Omega \int_{(\Omega - \mathbf{x}) \cap B(\mathbf{0},\delta)} = \int_{\Omega_\delta} \int_{B(\mathbf{0},\delta)} + \int_{\Omega \setminus \Omega_\delta} \int_{(\Omega - \mathbf{x}) \cap B(\mathbf{0},\delta)} \quad \text{and} \quad \int_\Omega = \int_{\Omega_\delta} + \int_{\Omega \setminus \Omega_\delta}. \tag{54}$$

Using (52), (53) and (54), we find that

$$\begin{aligned}
&\frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\
= &\int_{\Omega_\delta} \bar{w}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} + \frac{n+\beta}{\delta^{n+\beta}} \int_{\Omega \setminus \Omega_\delta} \int_{(\Omega - \mathbf{x}) \cap B(\mathbf{0},\delta)} w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, D\mathbf{u}(\mathbf{x})\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} \, \mathrm{d}\mathbf{x}
\end{aligned}$$

and

$$\bar{I}(\mathbf{u}) = \int_{\Omega_\delta} \bar{w}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} + \frac{n+\beta}{\delta^{n+\beta}} \int_{\Omega \setminus \Omega_\delta} \int_{B(\mathbf{0},\delta)} w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, D\mathbf{u}(\mathbf{x})\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} \, \mathrm{d}\mathbf{x}$$

so

$$\begin{aligned}
&\left| \bar{I}(\mathbf{u}) - \frac{n+\beta}{\delta^{n+\beta}} \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} w^\circ(\mathbf{x}, \mathbf{x}' - \mathbf{x}, D\mathbf{u}(\mathbf{x})(\mathbf{x}' - \mathbf{x})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \right| \\
=&\frac{n+\beta}{\delta^{n+\beta}} \int_{\Omega \setminus \Omega_\delta} \int_{B(\mathbf{0},\delta) \setminus (\Omega - \mathbf{x})} w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, D\mathbf{u}(\mathbf{x})\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} \, \mathrm{d}\mathbf{x} \\
\leq&\frac{n+\beta}{\delta^{n+\beta}} \int_{\Omega \setminus \Omega_\delta} \int_{B(\mathbf{0},\delta)} w^\circ(\mathbf{x}, \tilde{\mathbf{x}}, D\mathbf{u}(\mathbf{x})\tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}} \, \mathrm{d}\mathbf{x} = \int_{\Omega \setminus \Omega_\delta} \bar{w}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x},
\end{aligned} \tag{55}$$

where in the last equality we have used (53). By Lemma 3,

$$\int_{\Omega \setminus \Omega_\delta} \bar{w}(\mathbf{x}, D\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} \leq c_1 \int_{\Omega \setminus \Omega_\delta} (1 + |D\mathbf{u}(\mathbf{x})|^p) \, \mathrm{d}\mathbf{x}. \tag{56}$$

As $\{D\mathbf{u} : \mathbf{u} \in \mathcal{F}\}$ is bounded in $C(\bar{\Omega}, \mathbb{R}^d)$, it follows that $\{1 + |D\mathbf{u}|^p : \mathbf{u} \in \mathcal{F}\}$ is also bounded in $C(\bar{\Omega}, \mathbb{R}^d)$, so, in particular, it is equiintegrable. Therefore,

$$\lim_{\delta \to 0} \sup_{\mathbf{u} \in \mathcal{F}} \int_{\Omega \setminus \Omega_\delta} (1 + |D\mathbf{u}(\mathbf{x})|^p) \, \mathrm{d}\mathbf{x} = 0,$$

which, thanks to (55) and (56), completes the proof of (46). □

Proposition 9 will be used in the following form.

Corollary 10. *For each $\delta > 0$, let $\mathbf{u}_\delta, \mathbf{u} \in C^1(\bar\Omega, \mathbb{R}^d)$ be such that $\mathbf{u}_\delta \to \mathbf{u}$ in $C^1(\bar\Omega, \mathbb{R}^d)$ as $\delta \to 0$. Then*

$$\lim_{\delta\to 0} I_\delta(\mathbf{u}_\delta) = \bar I(\mathbf{u}).$$

Proof. For each $\delta > 0$,

$$\left|I_\delta(\mathbf{u}_\delta) - \bar I(\mathbf{u}_\delta)\right| \le \sup_{\delta'>0} \left|I_\delta(\mathbf{u}_{\delta'}) - \bar I(\mathbf{u}_{\delta'})\right|. \tag{57}$$

The convergence $\mathbf{u}_\delta \to \mathbf{u}$ in $C^1(\bar\Omega, \mathbb{R}^d)$ implies that for each $\delta > 0$ the set $\{D\mathbf{u}_{\delta'}\}_{0<\delta'\le\delta}$ is bounded in $C(\bar\Omega, \mathbb{R}^d)$ and equicontinuous. From (57) and Proposition 9, we obtain that

$$\lim_{\delta\to 0} \left|I_\delta(\mathbf{u}_\delta) - \bar I(\mathbf{u}_\delta)\right| = 0,$$

while by dominated convergence, we infer that

$$\lim_{\delta\to 0} \bar I(\mathbf{u}_\delta) = \bar I(\mathbf{u}),$$

thanks to the bounds of Lemma 3. This concludes the proof. □

As in Ponce [29, Th. 4], one may show that the convergence $I_\delta(\mathbf{u}) \to \bar I(\mathbf{u})$ holds for $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$, for some choices of w. In the proof of Theorem 1 (see Section 9) only one inequality is needed, and, besides, just for Lipschitz functions; we hence provide a proof for this case.

Lemma 11. *For any $\mathbf{u} \in W^{1,\infty}(\Omega, \mathbb{R}^d)$,*

$$\limsup_{\delta\to 0} I_\delta(\mathbf{u}) \le \bar I(\mathbf{u}).$$

Proof. A standard approximation result (see, e.g., [18, Th. 6.6.3]) shows that there exists a sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ in $C^1(\bar\Omega, \mathbb{R}^d)$ with

$$\mathbf{u}_j \to \mathbf{u} \text{ in } W^{1,p}(\Omega, \mathbb{R}^d) \quad \text{as } j \to \infty, \tag{58}$$

and, calling O_j the set of $\mathbf{x} \in \Omega$ such that $\mathbf{u}_j(\mathbf{x}) \ne \mathbf{u}(\mathbf{x})$, we have that

$$\lim_{j\to\infty} \mathcal{L}^n(O_j) = 0. \tag{59}$$

The bounds of Lemma 3 allow us to apply dominated convergence to (58), and conclude that

$$\lim_{j\to\infty} \bar I(\mathbf{u}_j) = \bar I(\mathbf{u}), \tag{60}$$

while Proposition 9 provides, for each $j \in \mathbb{N}$,

$$\lim_{\delta\to 0} I_\delta(\mathbf{u}_j) = \bar I(\mathbf{u}_j). \tag{61}$$

In addition, for each $\delta > 0$ and $j \in \mathbb{N}$, putting

$$A_{\delta,j} := \{(\mathbf{x}, \mathbf{x}') \in [(\Omega \setminus O_j) \times O_j] \cup [O_j \times \Omega] : |\mathbf{x}' - \mathbf{x}| < \delta\},$$
$$B_{\delta,j} := \{(\mathbf{x}, \mathbf{x}') \in (\Omega \setminus O_j) \times (\Omega \setminus O_j) : |\mathbf{x}' - \mathbf{x}| < \delta\},$$

and using (W4), we have that

$$\begin{aligned} I_\delta(\mathbf{u}) &= \frac{n+\beta}{\delta^{n+\beta}} \left[\iint_{B_{\delta,j}} w(\mathbf{x}, \mathbf{x}'-\mathbf{x}, \mathbf{u}_j(\mathbf{x}') - \mathbf{u}_j(\mathbf{x}))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} + \iint_{A_{\delta,j}} w(\mathbf{x}, \mathbf{x}'-\mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}))\mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \right] \\ &\le I_\delta(\mathbf{u}_j) + \frac{n+\beta}{\delta^{n+\beta}} \iint_{A_{\delta,j}} \left[h\left(\frac{\mathbf{x}'-\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|}\right) \frac{|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^p}{|\mathbf{x}'-\mathbf{x}|^\alpha} + \psi_1(\mathbf{x}'-\mathbf{x}) \right] \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}. \end{aligned} \tag{62}$$

Thanks to Lemma 5,

$$\int_{O_j} \int_{\Omega\cap B(\mathbf{x},\delta)} \psi_1(\mathbf{x}'-\mathbf{x})\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \le \mathcal{L}^n(O_j) \int_0^\delta \int_{\mathbb{S}^{n-1}} t^{n-1}\, \psi_1(t\mathbf{z})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z})\, \mathrm{d}t. \tag{63}$$

Using (W4) again, the same argument that led to (34) also shows there exists $t_0 > 0$ such that for all $0 < \delta \le t_0$,

$$\frac{n+\beta}{\delta^{n+\beta}} \int_0^\delta \int_{\mathbb{S}^{n-1}} t^{n-1} \, \psi_1(t\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \, \mathrm{d}t \le \int_{\mathbb{S}^{n-1}} \left(1 + \tilde{\psi}_1(\mathbf{z})\right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}). \tag{64}$$

Putting (63) and (64) together, we find that

$$\frac{n+\beta}{\delta^{n+\beta}} \int_{O_j} \int_{\Omega \cap B(\mathbf{x},\delta)} \psi_1(\mathbf{x}' - \mathbf{x}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le \mathcal{L}^n(O_j) \int_{\mathbb{S}^{n-1}} \left(1 + \tilde{\psi}_1(\mathbf{z})\right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}). \tag{65}$$

Moreover, since

$$\int_{\Omega \setminus O_j} \int_{O_j \cap B(\mathbf{x},\delta)} \psi_1(\mathbf{x}' - \mathbf{x}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = \int_{O_j} \int_{(\Omega \setminus O_j) \cap B(\mathbf{x},\delta)} \psi_1(-(\mathbf{x}' - \mathbf{x})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}$$

and, for each $t > 0$,

$$\int_{\mathbb{S}^{n-1}} t^{n-1} \, \psi_1(-t\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) = \int_{\mathbb{S}^{n-1}} t^{n-1} \, \psi_1(t\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}),$$

estimate (65) also yields

$$\frac{n+\beta}{\delta^{n+\beta}} \int_{\Omega \setminus O_j} \int_{O_j \cap B(\mathbf{x},\delta)} \psi_1(\mathbf{x}' - \mathbf{x}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le \mathcal{L}^n(O_j) \int_{\mathbb{S}^{n-1}} \left(1 + \tilde{\psi}_1(\mathbf{z})\right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}),$$

so, altogether,

$$\frac{n+\beta}{\delta^{n+\beta}} \iint_{A_{\delta,j}} \psi_1(\mathbf{x}' - \mathbf{x}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le 2 \, \mathcal{L}^n(O_j) \int_{\mathbb{S}^{n-1}} \left(1 + \tilde{\psi}_1(\mathbf{z})\right) \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}). \tag{66}$$

Let L be the Lipschitz constant of $\mathbf{u}$. Using Lemma 5, we find that

$$\begin{aligned}
\int_{O_j} \int_{\Omega \cap B(\mathbf{x},\delta)} h\left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}\right) \frac{|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} &\le L^p \int_{O_j} \int_{B(\mathbf{x},\delta)} h\left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}\right) |\mathbf{x}' - \mathbf{x}|^\beta \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\
&= L^p \, \mathcal{L}^n(O_j) \int_0^\delta t^{n+\beta-1} \, \mathrm{d}t \int_{\mathbb{S}^{n-1}} h(\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}) \\
&= L^p \, \mathcal{L}^n(O_j) \, \frac{\delta^{n+\beta}}{n+\beta} \int_{\mathbb{S}^{n-1}} h(\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}).
\end{aligned}$$

The same argument that yielded (66) from (65) now shows that

$$\iint_{A_{\delta,j}} h\left(\frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|}\right) \frac{|\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})|^p}{|\mathbf{x}' - \mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le 2 \, L^p \, \mathcal{L}^n(O_j) \, \frac{\delta^{n+\beta}}{n+\beta} \int_{\mathbb{S}^{n-1}} h(\mathbf{z}) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{z}). \tag{67}$$

Putting together (62), (66), and (67) let us have that

$$I_\delta(\mathbf{u}) \le I_\delta(\mathbf{u}_j) + C \, \mathcal{L}^n(O_j) \tag{68}$$

for some $C > 0$ independent of j and δ. Equations (59), (60), (61), and (68) conclude the proof by taking limits, first as $\delta \to 0$, and then as $j \to \infty$. □

8. Lower bound

In this section, we show the lower bound part of the Γ-limit of I_δ to I. The proof initially follows the lines of Ponce [29]: given $\mathbf{u}_\delta \to \mathbf{u}$, it exploits the limit of Corollary 10 applied to mollified versions of $\mathbf{u}_\delta$ and $\mathbf{u}$. The bounds of Lemma 4 are also useful to pass to the limit as the mollification parameter goes to zero. The continuity of w in the variable $\mathbf{x}$, as stated in (W4), is essential to control the inhomogeneity, and, hence the error made when $\mathbf{x}$ is replaced by a nearby point.

We fix an $\eta \in C_c^\infty(\mathbb{R}^n)$ with support in $B(\mathbf{0}, 1)$ such that $\int_{\mathbb{R}^n} \eta \, \mathrm{d}\mathbf{x} = 1$ and $\eta \ge 0$. Recall the notation (16). For each $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ and each $r > 0$, we define the mollified function $\mathbf{u}_r : \Omega_r \to \mathbb{R}^d$ by

$$\mathbf{u}_r(\mathbf{x}) := \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \mathbf{u}(\mathbf{x} - \tilde{\mathbf{x}}) \, \mathrm{d}\tilde{\mathbf{x}}. \tag{69}$$

Apart from the general scheme explained above, we also require an inequality between the energy of $\mathbf{u}_r$ and the energy of $\mathbf{u}$, whose proof relies on Jensen's inequality applied to the function (2) (this idea was taken from [29, Lemma 4]), and on a careful error estimate.

Lemma 12. *There exists $c > 0$, depending on w, such that for any $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$, $r > 0$ and $\delta > 0$,*

$$\int_{\Omega_r} \int_{\Omega_r \cap B(\mathbf{x},\delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}_r(\mathbf{x}') - \mathbf{u}_r(\mathbf{x})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq \int_{\Omega} \int_{\Omega \cap B(\mathbf{x},\delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} + c\, \sigma(r)\, \delta^{n+p},$$

where σ is the function appearing in (W4).

Proof. Since for all $\mathbf{x}, \mathbf{x}' \in \Omega_r$,

$$\mathbf{u}_r(\mathbf{x}') - \mathbf{u}_r(\mathbf{x}) = \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) [\mathbf{u}(\mathbf{x}' - \tilde{\mathbf{x}}) - \mathbf{u}(\mathbf{x} - \tilde{\mathbf{x}})] \, \mathrm{d}\tilde{\mathbf{x}},$$

we can apply Jensen's inequality to the convex function (17), and find that for a.e. $\mathbf{x} \in \Omega_r$,

$$\begin{aligned}
&\int_{\Omega_r \cap B(\mathbf{x},\delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}_r(\mathbf{x}') - \mathbf{u}_r(\mathbf{x})) \, \mathrm{d}\mathbf{x}' \\
\leq& \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \int_{\Omega_r \cap B(\mathbf{x},\delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}(\mathbf{x}' - \tilde{\mathbf{x}}) - \mathbf{u}(\mathbf{x} - \tilde{\mathbf{x}})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\tilde{\mathbf{x}} \\
=& \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \int_{(\Omega_r - \tilde{\mathbf{x}}) \cap B(\mathbf{x} - \tilde{\mathbf{x}},\delta)} w(\mathbf{x}, \mathbf{z}' + \tilde{\mathbf{x}} - \mathbf{x}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{x} - \tilde{\mathbf{x}})) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}\tilde{\mathbf{x}},
\end{aligned}$$

where we have performed a change of variables. Integrating the latter inequality, using Fubini's theorem and a further change of variables, we arrive at

$$\begin{aligned}
&\int_{\Omega_r} \int_{\Omega_r \cap B(\mathbf{x},\delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}_r(\mathbf{x}') - \mathbf{u}_r(\mathbf{x})) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \\
\leq& \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \int_{\Omega_r} \int_{(\Omega_r - \tilde{\mathbf{x}}) \cap B(\mathbf{x} - \tilde{\mathbf{x}},\delta)} w(\mathbf{x}, \mathbf{z}' + \tilde{\mathbf{x}} - \mathbf{x}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{x} - \tilde{\mathbf{x}})) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}\mathbf{x} \, \mathrm{d}\tilde{\mathbf{x}} \\
=& \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \int_{\Omega_r - \tilde{\mathbf{x}}} \int_{(\Omega_r - \tilde{\mathbf{x}}) \cap B(\mathbf{z},\delta)} w(\tilde{\mathbf{x}} + \mathbf{z}, \mathbf{z}' - \mathbf{z}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{z})) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}\mathbf{z} \, \mathrm{d}\tilde{\mathbf{x}} \\
=& \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \int_{\Omega_r - \tilde{\mathbf{x}}} \int_{(\Omega_r - \tilde{\mathbf{x}}) \cap B(\mathbf{z},\delta)} w(\mathbf{z}, \mathbf{z}' - \mathbf{z}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{z})) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}\mathbf{z} \, \mathrm{d}\tilde{\mathbf{x}} + E_{r,\delta},
\end{aligned} \tag{70}$$

where we have called

$$E_{r,\delta} := \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \int_{\Omega_r - \tilde{\mathbf{x}}} \int_{(\Omega_r - \tilde{\mathbf{x}}) \cap B(\mathbf{z},\delta)} \big[w(\tilde{\mathbf{x}} + \mathbf{z}, \mathbf{z}' - \mathbf{z}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{z})) - w(\mathbf{z}, \mathbf{z}' - \mathbf{z}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{z}))\big] \mathrm{d}\mathbf{z}' \, \mathrm{d}\mathbf{z} \, \mathrm{d}\tilde{\mathbf{x}}.$$

Now, using the inequality $w \geq 0$ of (W3), we find that

$$\begin{aligned}
&\frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \int_{\Omega_r - \tilde{\mathbf{x}}} \int_{(\Omega_r - \tilde{\mathbf{x}}) \cap B(\mathbf{z},\delta)} w(\mathbf{z}, \mathbf{z}' - \mathbf{z}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{z})) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}\mathbf{z} \, \mathrm{d}\tilde{\mathbf{x}} \\
\leq& \frac{1}{r^n} \int_{B(\mathbf{0},r)} \eta\left(\frac{\tilde{\mathbf{x}}}{r}\right) \int_{\Omega} \int_{\Omega \cap B(\mathbf{z},\delta)} w(\mathbf{z}, \mathbf{z}' - \mathbf{z}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{z})) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}\mathbf{z} \, \mathrm{d}\tilde{\mathbf{x}} \\
=& \int_{\Omega} \int_{\Omega \cap B(\mathbf{z},\delta)} w(\mathbf{z}, \mathbf{z}' - \mathbf{z}, \mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{z})) \, \mathrm{d}\mathbf{z}' \, \mathrm{d}\mathbf{z}.
\end{aligned} \tag{71}$$

We next estimate $E_{r,\delta}$. For each $M > 0$ and $\tau > 0$, define the sets

$$O_M := \{(\mathbf{z}, \mathbf{z}') \in \Omega \times \Omega : |\mathbf{u}(\mathbf{z}') - \mathbf{u}(\mathbf{z})| > M|\mathbf{z}' - \mathbf{z}|\}, \qquad D_\tau := \{(\mathbf{z}, \mathbf{z}') \in \Omega \times \Omega : |\mathbf{z}' - \mathbf{z}| > \tau\}.$$

They satisfy that

$$M\,\tau < M|\mathbf{z}' - \mathbf{z}| < |\mathbf{u}(\mathbf{z}')| + |\mathbf{u}(\mathbf{z})|, \qquad (\mathbf{z}, \mathbf{z}') \in O_M \cap D_\tau,$$

so

$$M\,\tau\,\mathcal{L}^{2n}(O_M\cap D_\tau)\le\iint\limits_{O_M\cap D_\tau}[|\mathbf{u}(\mathbf{z}')|+|\mathbf{u}(\mathbf{z})|]\,\mathrm{d}\mathbf{z}'\,\mathrm{d}\mathbf{z}\le 2\,\mathcal{L}^n(\Omega)\int_\Omega|\mathbf{u}|\,\mathrm{d}\mathbf{z}.$$

Therefore, $\lim_{M\to\infty}\mathcal{L}^{2n}(O_M\cap D_\tau)=0$ for each $\tau>0$, and, hence,

$$\limsup_{M\to\infty}\mathcal{L}^{2n}(O_M)\le\mathcal{L}^{2n}\left((\Omega\times\Omega)\setminus D_\tau\right)$$

As the quantity $\mathcal{L}^{2n}\left((\Omega\times\Omega)\setminus D_\tau\right)$ can be done as small as we wish, by choosing τ small, we conclude that

$$\lim_{M\to\infty}\mathcal{L}^{2n}(O_M)=0. \tag{72}$$

Coming back to $E_{r,\delta}$ we find, by (W4), that

$$|E_{r,\delta}|\le\frac{1}{r^n}\int_{B(\mathbf{0},r)}\eta\left(\frac{\tilde{\mathbf{x}}}{r}\right)\int_\Omega\int_{\Omega\cap B(\mathbf{z},\delta)}\sigma(|\tilde{\mathbf{x}}|)\left[h\left(\frac{\mathbf{z}'-\mathbf{z}}{|\mathbf{z}'-\mathbf{z}|}\right)\frac{|\mathbf{u}(\mathbf{z}')-\mathbf{u}(\mathbf{z})|^p}{|\mathbf{z}'-\mathbf{z}|^\alpha}+\psi_1(\mathbf{z}'-\mathbf{z})\right]\mathrm{d}\mathbf{z}'\,\mathrm{d}\mathbf{z}\,\mathrm{d}\tilde{\mathbf{x}}. \tag{73}$$

Now,

$$\frac{1}{r^n}\int_{B(\mathbf{0},r)}\eta\left(\frac{\tilde{\mathbf{x}}}{r}\right)\sigma(|\tilde{\mathbf{x}}|)\,\mathrm{d}\tilde{\mathbf{x}}\le\sigma(r) \tag{74}$$

and, arguing, as in (63)–(66), we find that

$$\frac{n+\beta}{\delta^{n+\beta}}\int_\Omega\int_{\Omega\cap B(\mathbf{x},\delta)}\psi_1(\mathbf{z}'-\mathbf{z})\,\mathrm{d}\mathbf{z}'\,\mathrm{d}\mathbf{z}\le\mathcal{L}^n(\Omega)\int_{\mathbb{S}^{n-1}}\left(1+\tilde{\psi}_1\right)\mathrm{d}\mathcal{H}^{n-1}, \tag{75}$$

while, arguing, as in (67), we obtain that

$$\iint\limits_{O_{M,\delta}}h\left(\frac{\mathbf{z}'-\mathbf{z}}{|\mathbf{z}'-\mathbf{z}|}\right)\frac{|\mathbf{u}(\mathbf{z}')-\mathbf{u}(\mathbf{z})|^p}{|\mathbf{z}'-\mathbf{z}|^\alpha}\,\mathrm{d}\mathbf{z}'\,\mathrm{d}\mathbf{z}\le 2\,M^p\,\mathcal{L}^n(\Omega)\,\frac{\delta^{n+\beta}}{n+\beta}\int_{\mathbb{S}^{n-1}}h\,\mathrm{d}\mathcal{H}^{n-1}, \tag{76}$$

where $O_{M,\delta}:=\{(\mathbf{z},\mathbf{z}')\in O_M:|\mathbf{z}'-\mathbf{z}|<\delta\}$. In fact, thanks to Lemma 7,

$$\int_\Omega\int_{\Omega\cap B(\mathbf{z},\delta)}h\left(\frac{\mathbf{z}'-\mathbf{z}}{|\mathbf{z}'-\mathbf{z}|}\right)\frac{|\mathbf{u}(\mathbf{z}')-\mathbf{u}(\mathbf{z})|^p}{|\mathbf{z}'-\mathbf{z}|^\alpha}\,\mathrm{d}\mathbf{z}'\,\mathrm{d}\mathbf{z}<\infty,$$

so by (72) we can find $M>0$ such that

$$\left[\int_\Omega\int_{\Omega\cap B(\mathbf{z},\delta)}-\iint\limits_{O_{M,\delta}}\right]h\left(\frac{\mathbf{z}'-\mathbf{z}}{|\mathbf{z}'-\mathbf{z}|}\right)\frac{|\mathbf{u}(\mathbf{z}')-\mathbf{u}(\mathbf{z})|^p}{|\mathbf{z}'-\mathbf{z}|^\alpha}\,\mathrm{d}\mathbf{z}'\,\mathrm{d}\mathbf{z}\le\delta^{n+\beta}. \tag{77}$$

Putting together estimates (70), (71), (73), (74), (75), (76) and (77), we arrive at the conclusion of the lemma. □

We are in a position to prove *D2)* and *N2)* of Theorem 1. In this case, the boundary conditions do not play any role, so we state and prove the result without any mention to them.

Proposition 13. *Let $\{\mathbf{u}_\delta\}_\delta$ be a sequence in $L^p(\Omega,\mathbb{R}^d)$ such that $\mathbf{u}_\delta\to\mathbf{u}$ in $L^p(\Omega,\mathbb{R}^d)$ for some $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^d)$. Then*

$$I(\mathbf{u})\le\liminf_{\delta\to 0}I_\delta(\mathbf{u}_\delta).$$

Proof. For each $\delta>0$ and $r>0$, let $\mathbf{u}_{\delta,r}$ and $\mathbf{u}_r$ be as in (69), the mollified functions of $\mathbf{u}_\delta$ and $\mathbf{u}$, respectively. By Lemma 12,

$$\frac{n+\beta}{\delta^{n+\beta}}\int_{\Omega_r}\int_{\Omega_r\cap B(\mathbf{x},\delta)}w(\mathbf{x},\mathbf{x}'-\mathbf{x},\mathbf{u}_{\delta,r}(\mathbf{x}')-\mathbf{u}_{\delta,r}(\mathbf{x}))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\le I_\delta(\mathbf{u}_\delta)+c\,(n+\beta)\,\sigma(r). \tag{78}$$

Fix $r_0 > 0$, and $0 < r \le r_0$. Using standard properties of mollifiers, it is easy to check that $\mathbf{u}_{\delta,r} \to \mathbf{u}_r$ in $C^1(\bar{\Omega}_{r_0}, \mathbb{R}^d)$ as $\delta \to 0$. By the inequalities $0 \le Q\bar{w} \le \bar{w}$ (see Lemma 4), Corollary 10 and (78),

$$\begin{aligned}\int_{\Omega_{r_0}} Q\bar{w}(\mathbf{x}, D\mathbf{u}_r)\,\mathrm{d}\mathbf{x} \le \int_{\Omega_r} Q\bar{w}(\mathbf{x}, D\mathbf{u}_r)\,\mathrm{d}\mathbf{x} \le \int_{\Omega_r} \bar{w}(\mathbf{x}, D\mathbf{u}_r)\,\mathrm{d}\mathbf{x} \\ = \lim_{\delta\to 0} \frac{n+\beta}{\delta^{n+\beta}} \int_{\Omega_r} \int_{\Omega_r \cap B(\mathbf{x},\delta)} w(\mathbf{x}, \mathbf{x}' - \mathbf{x}, \mathbf{u}_{\delta,r}(\mathbf{x}') - \mathbf{u}_{\delta,r}(\mathbf{x}))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \\ \le \liminf_{\delta\to 0} I_\delta(\mathbf{u}_\delta) + c\,(n+\beta)\,\sigma(r).\end{aligned} \tag{79}$$

Again by well-known properties of mollifiers, for each $r_0 > 0$, the convergence $\mathbf{u}_r \to \mathbf{u}$ holds in $W^{1,p}(\Omega_{r_0}, \mathbb{R}^d)$ as $r \to 0$. Using a standard argument based on dominated convergence and Lemma 4, we conclude that

$$\lim_{r\to 0} \int_{\Omega_{r_0}} Q\bar{w}(\mathbf{x}, D\mathbf{u}_r)\,\mathrm{d}\mathbf{x} = \int_{\Omega_{r_0}} Q\bar{w}(\mathbf{x}, D\mathbf{u})\,\mathrm{d}\mathbf{x}, \tag{80}$$

while monotone convergence shows that

$$\lim_{r_0\to 0} \int_{\Omega_{r_0}} Q\bar{w}(\mathbf{x}, D\mathbf{u})\,\mathrm{d}\mathbf{x} = \int_{\Omega} Q\bar{w}(\mathbf{x}, D\mathbf{u})\,\mathrm{d}\mathbf{x} = I(\mathbf{u}). \tag{81}$$

The conclusion follows from (79), (80), and (81). □

9. Upper bound

In this section, we construct the recovery sequence that shows the upper bound inequality, thus finishing the proof of Theorem 1.

Proof of N3) of Theorem 1. Let $\mathbf{u} \in \mathcal{B}_N$. The bounds of Lemma 3 allow us to apply the relaxation theorem of Dacorogna [12] (see [13, Th. 9.8] for the inhomogeneous version, which is the one used here) to have that there exists a sequence $\{\mathbf{v}_i\}_{i\in\mathbb{N}}$ in $W^{1,p}(\Omega, \mathbb{R}^d)$ such that

$$\mathbf{v}_i \to \mathbf{u} \text{ in } L^p(\Omega, \mathbb{R}^d), \quad \text{and} \quad \bar{I}(\mathbf{v}_i) \to I(\mathbf{u}) \qquad \text{as } i \to \infty. \tag{82}$$

Because Ω has a Lipschitz boundary, for each $i \in \mathbb{N}$ there exists a sequence $\{\mathbf{v}_i^j\}_{j\in\mathbb{N}}$ in $C^1(\bar{\Omega}, \mathbb{R}^d)$ such that

$$\mathbf{v}_i^j \to \mathbf{v}_i \text{ in } W^{1,p}(\Omega, \mathbb{R}^d) \qquad \text{as } j \to \infty. \tag{83}$$

Thanks again to the bounds of Lemma 3, we can apply dominated convergence to (83) so as to obtain that

$$\lim_{j\to\infty} \bar{I}(\mathbf{v}_i^j) = \bar{I}(\mathbf{v}_i). \tag{84}$$

Now, for each $i, j \in \mathbb{N}$, Proposition 9 yields

$$\lim_{\delta\to 0} I_\delta(\mathbf{v}_i^j) = \bar{I}(\mathbf{v}_i^j). \tag{85}$$

Bearing in mind limits (82) and (83) on the one hand, and (82), (84) and (85) on the other, a standard diagonalization argument shows that there exist sequences $\{j_i\}_{i\in\mathbb{N}}$ in $\mathbb{N}$, and $\{\delta_i\}_{i\in\mathbb{N}}$ in $(0,\infty)$ such that $j_i \nearrow \infty$, $\delta_i \searrow 0$ and

$$\mathbf{v}_i^{j_i} \to \mathbf{u} \text{ in } L^p(\Omega, \mathbb{R}^d), \quad \text{and} \quad I_{\delta_i}(\mathbf{v}_i^{j_i}) \to I(\mathbf{u}) \qquad \text{as } i \to \infty. \tag{86}$$

The first limit of (86) and the inclusion $\mathbf{u} \in \mathcal{B}_N$ imply that

$$\lim_{i\to\infty} \int_\Omega \mathbf{v}_i^{j_i}\,\mathrm{d}\mathbf{x} = \mathbf{0}. \tag{87}$$

In addition, since the functional I_{δ_i} is invariant under translations, when, for each $i \in \mathbb{N}$, we define

$$\mathbf{u}_i^{j_i} := \mathbf{v}_i^{j_i} - \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega \mathbf{v}_i^{j_i}\,\mathrm{d}\mathbf{x},$$

we have that $\mathbf{u}_i^{j_i} \in \mathcal{A}_N$ and, thanks to (86) and (87),

$$\mathbf{u}_i^{j_i} \to \mathbf{u} \text{ in } L^p(\Omega, \mathbb{R}^d) \quad \text{and} \quad I_{\delta_i}(\mathbf{u}_i^{j_i}) \to I(\mathbf{u}) \qquad \text{as } i \to \infty.$$

This proves (14), having in mind the notational convention about sequences explained in Section 3. □

The construction of the recovery sequence is more involved for Dirichlet conditions. We start with a result that compares the functional $\bar{I}$ for two closeby sequences, one of each has p-equiintegrable gradients. This method of comparing energies is standard in the theory of Γ-convergence, as is the method of matching the boundary values based on the decomposition lemma of Fonseca, Müller & Pedregal [20] that will be used in the proof of *D3)* of Theorem 1. We took, in particular, some ideas from the exposition of Braides [9, Sect. 4.2].

Lemma 14. *For each $i \in \mathbb{N}$, let $\mathbf{u}_i, \mathbf{v}_i \in W^{1,p}(\Omega, \mathbb{R}^d)$ be such that $\{|D\mathbf{v}_i|^p\}_{i\in\mathbb{N}}$ is equiintegrable and*

$$\lim_{i\to\infty} \mathcal{L}^n\left(\{\mathbf{x} \in \Omega : D\mathbf{v}_i(\mathbf{x}) \neq D\mathbf{u}_i(\mathbf{x})\}\right) = 0.$$

Then

$$\limsup_{i\to\infty} \bar{I}(\mathbf{v}_i) \leq \limsup_{i\to\infty} \bar{I}(\mathbf{u}_i). \tag{88}$$

Proof. For each $i \in \mathbb{N}$, let O_i be the set of $\mathbf{x} \in \Omega$ such that $D\mathbf{v}_i(\mathbf{x}) \neq D\mathbf{u}_i(\mathbf{x})$. Then, by Lemma 3,

$$\bar{I}(\mathbf{v}_i) = \int_{\Omega\setminus O_i} \bar{w}(\mathbf{x}, D\mathbf{u}_i)\,\mathrm{d}\mathbf{x} + \int_{O_i} \bar{w}(\mathbf{x}, D\mathbf{v}_i)\,\mathrm{d}\mathbf{x} \leq \bar{I}(\mathbf{u}_i) + c_1 \int_{O_i} (1 + |D\mathbf{v}_i|^p)\,\mathrm{d}\mathbf{x},$$

Taking limits and using equiintegrability, we obtain the conclusion of the lemma. □

We are in a position to build the recovery sequence under Dirichlet conditions.

Proof of D3) of Theorem 1. We first prove the result for $\mathbf{u} \in \mathcal{B} \cap W^{1,\infty}(\Omega, \mathbb{R}^d)$.

The bounds of Lemma 3 allow us to apply the relaxation theorem of Dacorogna [13, Th. 9.8], and obtain that there exists a bounded sequence $\{\mathbf{u}_i\}_{i\in\mathbb{N}}$ in $W^{1,p}(\Omega, \mathbb{R}^d)$ such that

$$\mathbf{u}_i \to \mathbf{u} \text{ in } L^p(\Omega, \mathbb{R}^d) \quad \text{and} \quad \bar{I}(\mathbf{u}_i) \to I(\mathbf{u}) \qquad \text{as } i \to \infty. \tag{89}$$

An application of [20, Lemma 1.2] (see, if necessary, [9, Th. 4.11] for an alternative formulation) shows that, for a subsequence, there exists a sequence $\{\mathbf{v}_i\}_{i\in\mathbb{N}}$ in $W^{1,\infty}(\Omega, \mathbb{R}^d)$ such that $\{|D\mathbf{v}_i|^p\}_{i\in\mathbb{N}}$ is equiintegrable,

$$\lim_{i\to\infty} \mathcal{L}^n\left(\{\mathbf{x} \in \Omega : \mathbf{v}_i(\mathbf{x}) \neq \mathbf{u}_i(\mathbf{x}) \text{ or } D\mathbf{v}_i(\mathbf{x}) \neq D\mathbf{u}_i(\mathbf{x})\}\right) = 0. \tag{90}$$

and

$$\mathbf{v}_i \to \mathbf{u} \text{ in } L^p(\Omega, \mathbb{R}^d) \quad \text{as } i \to \infty. \tag{91}$$

By Lemma 14, inequality (88) holds.

For each $j \in \mathbb{N}$, let $\eta^j \in C^1(\bar{\Omega})$ satisfy

$$0 \leq \eta^j \leq 1, \quad \eta^j = 1 \text{ in } \Omega_{1/j}, \quad \eta^j = 0 \text{ in } \Omega^{\frac{1}{2j}}, \quad |D\eta^j| \leq C\,j$$

for some $C > 0$ independent of j; recall that we are using notation (16). Notice that

$$\lim_{j\to\infty} \mathcal{L}^n\left(\Omega \setminus \Omega_{1/j}\right) = 0. \tag{92}$$

For each $i, j \in \mathbb{N}$ define

$$\mathbf{v}_i^j := \eta^j \mathbf{v}_i + (1 - \eta^j)\mathbf{u}_0$$

and note that $\mathbf{v}_i^j \in \mathcal{A}_{\frac{1}{2j}} \cap W^{1,\infty}(\Omega, \mathbb{R}^d)$ and

$$D\mathbf{v}_i^j = (\mathbf{v}_i - \mathbf{u}_0) \otimes D\eta^j + (1 - \eta^j) D\mathbf{u}_0 + \eta^j D\mathbf{v}_i,$$

where $\otimes$ indicates tensor product. As a consequence,

$$\left|D\mathbf{v}_i^j\right| \leq C\,j\,|\mathbf{v}_i - \mathbf{u}_0|\,\chi_{\Omega\setminus\Omega_{1/j}} + |D\mathbf{u}_0| + |D\mathbf{v}_i|\,. \tag{93}$$

Moreover, thanks to (91), we have that, for each $j \in \mathbb{N}$,

$$\mathbf{v}_i^j \to \eta^j \mathbf{u} + (1 - \eta^j)\mathbf{u}_0 \text{ in } L^p(\Omega, \mathbb{R}^d) \qquad \text{as } i \to \infty, \tag{94}$$

while by (92),

$$\eta^j \mathbf{u} + (1 - \eta^j)\mathbf{u}_0 \to \mathbf{u} \text{ in } L^p(\Omega, \mathbb{R}^d) \qquad \text{as } j \to \infty. \tag{95}$$

From (94) and (95) on the one hand, and from (91) on the other, we can construct an increasing sequence $\{i_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ tending to ∞ such that

$$\mathbf{v}^j_{i_j} \to \mathbf{u} \quad \text{and} \quad j\left(\mathbf{v}_{i_j} - \mathbf{u}\right) \to \mathbf{0} \qquad \text{in } L^p(\Omega, \mathbb{R}^d) \quad \text{as } j \to \infty. \tag{96}$$

Now, for each $\mathbf{x} \in \Omega \setminus \Omega_{1/j}$, let $\mathbf{p}(\mathbf{x}) \in \partial\Omega$ be such that $|\mathbf{x} - \mathbf{p}(\mathbf{x})| \leq 1/j$. Then, using $\mathbf{u}|_{\partial\Omega} = \mathbf{u}_0|_{\partial\Omega}$ and calling L, L_0 the Lipschitz constants of $\mathbf{u}, \mathbf{u}_0$, respectively, we find that

$$\begin{aligned} |\mathbf{v}_i(\mathbf{x}) - \mathbf{u}_0(\mathbf{x})| &\leq |\mathbf{v}_i(\mathbf{x}) - \mathbf{u}(\mathbf{x})| + |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{p}(\mathbf{x}))| + |\mathbf{u}_0(\mathbf{p}(\mathbf{x})) - \mathbf{u}_0(\mathbf{x})| \\ &\leq |\mathbf{v}_i(\mathbf{x}) - \mathbf{u}(\mathbf{x})| + \frac{L + L_0}{j}. \end{aligned} \tag{97}$$

Inequalities (93) and (97) yield

$$\left|D\mathbf{v}^j_i\right| \leq C\, j\, |\mathbf{v}_i - \mathbf{u}| + C\, L + (C+1)\, L_0 + |D\mathbf{v}_i|\,. \tag{98}$$

Equations (96) and (98), and the equiintegrability of $\{|D\mathbf{v}_i|^p\}_{i\in\mathbb{N}}$ imply that $\{|D\mathbf{v}^j_{i_j}|^p\}_{j\in\mathbb{N}}$ is equiintegrable. Using (92) and Lemma 14, we infer that

$$\limsup_{j\to\infty} \bar{I}(\mathbf{v}^j_{i_j}) \leq \limsup_{j\to\infty} \bar{I}(\mathbf{v}_{i_j}) \leq \limsup_{i\to\infty} \bar{I}(\mathbf{v}_i). \tag{99}$$

Putting together (89), (88), and (99), we have that

$$\limsup_{j\to\infty} \bar{I}(\mathbf{v}^j_{i_j}) \leq I(\mathbf{u}). \tag{100}$$

On the other hand, Lemma 11 shows that for each $j \in \mathbb{N}$,

$$\limsup_{\delta\to 0} I_\delta(\mathbf{v}^j_{i_j}) \leq \bar{I}(\mathbf{v}^j_{i_j}). \tag{101}$$

From equations (100) and (101), we can see that there exists a decreasing sequence $\{\delta_j\}_{j\in\mathbb{N}}$ in $(0,\infty)$ such that $\delta_j \leq \frac{1}{2j}$ for all $j \in \mathbb{N}$ and

$$\limsup_{j\to\infty} I_{\delta_j}(\mathbf{v}^j_{i_j}) \leq I(\mathbf{u}).$$

Thus, $\mathbf{v}^j_{i_j} \in \mathcal{A}_{\delta_j}$ for all $j \in \mathbb{N}$. This shows *D3)* under the additional assumption $\mathbf{u} \in W^{1,\infty}(\Omega, \mathbb{R}^d)$.

Now, take a general $\mathbf{u} \in \mathcal{B}$. By definition, $C^1_c(\Omega, \mathbb{R}^d)$ is dense in $W^{1,p}_0(\Omega, \mathbb{R}^d)$. Therefore, $\mathbf{u}_0 + C^1_c(\Omega, \mathbb{R}^d)$, and, hence, $\mathcal{B} \cap W^{1,\infty}(\Omega, \mathbb{R}^d)$ are dense in $\mathcal{B}$ with respect to the topology of $W^{1,p}(\Omega, \mathbb{R}^d)$. Accordingly, consider a sequence $\{\mathbf{w}_k\}_{k\in\mathbb{N}}$ in $\mathcal{B} \cap W^{1,\infty}(\Omega, \mathbb{R}^d)$ such that

$$\mathbf{w}_k \to \mathbf{u} \text{ in } W^{1,p}(\Omega, \mathbb{R}^d) \quad \text{as } k \to \infty. \tag{102}$$

By Lemma 4 and dominated convergence,

$$\lim_{k\to\infty} I(\mathbf{w}_k) = I(\mathbf{u}). \tag{103}$$

Thanks to the result just proved for $\mathcal{B} \cap W^{1,\infty}(\Omega, \mathbb{R}^d)$, for each $k \in \mathbb{N}$ we can find $\delta_k > 0$ and $\mathbf{u}^k \in \mathcal{A}_{\delta_k}$ such that

$$\int_\Omega \left|\mathbf{u}^k - \mathbf{w}_k\right|^p \,\mathrm{d}\mathbf{x} \leq \frac{1}{k}, \qquad I_{\delta_k}(\mathbf{u}^k) \leq I(\mathbf{w}_k) + \frac{1}{k} \tag{104}$$

and the sequence $\{\delta_k\}_{k\in\mathbb{N}}$ decreases to zero. Equations (102), (103) and (104) show that $\mathbf{u}^k \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ as $k \to \infty$ and

$$\limsup_{k\to\infty} I_{\delta_k}(\mathbf{u}^k) \leq I(\mathbf{u}).$$

This concludes the proof of Theorem 1. □

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