

# LOWER SEMICONTINUITY AND RELAXATION VIA YOUNG MEASURES FOR NONLOCAL VARIATIONAL PROBLEMS AND APPLICATIONS TO PERIDYNAMICS

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ABSTRACT. We study nonlocal variational problems in  $L^p$ , like those that appear in peridynamics. The functional object of our study is given by a double integral. We establish characterizations of weak lower semicontinuity of the functional in terms of nonlocal versions of either a convexity notion of the integrand, or a Jensen inequality for Young measures. Existence results, obtained through the direct method of the Calculus of variations, are also established. We cover different boundary conditions, for which the coercivity is obtained from nonlocal Poincaré inequalities. Finally, we analyze the relaxation (that is, the computation of the lower semicontinuous envelope) for this problem when the lower semicontinuity fails. We state a general relaxation result in terms of Young measures and show, by means of two examples, the difficulty of having a relaxation in  $L^p$  in an integral form. At the root of this difficulty lies the fact that, contrary to what happens for local functionals, non-positive integrands may give rise to positive nonlocal functionals.

## 1. INTRODUCTION

This paper studies functionals  $I$  of the form

$$I(u) = \int_{\Omega} \int_{\Omega} w(x, x', u(x), u(x')) \, dx \, dx',$$

where  $\Omega \subset \mathbb{R}^n$  is an open subset,  $u : \Omega \rightarrow \mathbb{R}^d$  is in some Lebesgue space  $L^p$ , and the integrand  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  has some measurability and continuity properties. This kind of functionals appears in many contexts in the mathematical modelling of some processes, whose common feature is their *nonlocal* nature; we mention here micromagnetics [35], phase transitions [4], peridynamics [36], pattern formation [23], image processing [25], population dispersal [20], diffusion [8] and optimal design [5]. It also has applications in the characterization of Sobolev spaces [16]. Finally, although not strictly relevant to the current work, functionals of the style of  $I$  share many features with the (linear and nonlinear) fractional Laplacian [18]. Our main motivation, though, comes from peridynamics: in this context,  $\Omega$  represents the body in its reference configuration,  $u$  is the deformation of the body and  $I$  is the energy of the deformation.

Apart from the nonlocality, another significant attribute of  $I$  is the absence of derivatives of  $u$ , which makes the Lebesgue space  $L^p$  a natural set of admissible functions. In fact, in most of the examples cited before, the nonlocal derivative-free modelling of  $I$  substitutes a more common local model involving derivatives.

In this work we do not deal with the evolution problem associated to  $I$  but rather its equilibrium solutions, in particular, minimizers  $u$  of  $I$ , which is usually given the interpretation of an energy (the total *macroelastic* potential energy, in the context of peridynamics). We will carry out the direct method of the Calculus of variations in order to establish the existence of minimizers. The two main ingredients of this method are *coercivity* and *lower semicontinuity*; the topology chosen in  $L^p$  is the weak topology.

The issue of coercivity has been addressed in several papers [16, 17, 34, 6, 7, 1, 2, 27, 28, 11, 12], and, besides nonlocality, the main difficulty was that typically  $w$  vanishes in a great part of the domain, namely, in points  $(x, x') \in \Omega \times \Omega$  for which  $|x - x'| > \delta$  for some fixed  $\delta > 0$  called the *horizon* (or *interaction*) distance. In those papers several nonlocal versions of Poincaré's inequality are given for different cases: Dirichlet, Neumann or mixed boundary conditions. Clarified formulations of Poincaré's inequality useful for our proposes were presented in [11], which will be recalled here for proving the existence results.

Different characterizations of the lower semicontinuity property were obtained in [22, 15, 11, 33] and, within a different context for functionals involving derivatives, in [31, 30]. In this paper we further explore those characterizations of lower semicontinuity, unify the previous approaches, establish their equivalence, and point out and fix misleading statements appearing in some of those references. More explicit formulations of this nonlocal convexity

were obtained for the one-dimensional case ( $n = 1$ ) in [29, 13, 19], although, even in this situation, lower semicontinuity is characterized through a nonlocal convexity notion, as shown in a counterexample in [11]. In this paper, weak lower semicontinuity of the functional  $I$  in  $L^p(\Omega, \mathbb{R}^d)$ , for  $p \geq 1$ , is characterized through two equivalent notions: one involving the convexity of certain integrals (already introduced in [22, 11]), and another one in terms of Young measures (first introduced for functionals depending on derivatives in [31]). We also cover the case  $p = 1$ , which is treated somewhat separately.

In the absence of lower semicontinuity, the existence of minimizers is not guaranteed and a usual approach is the *relaxation*, which consists in finding the lower semicontinuous envelope of  $I$  in the relevant topology. In the classical context of nonlinear elasticity, understanding the relaxation is capital to study the microstructure of the material [10]. Relaxation for nonlocal functionals similar to  $I$  but depending on  $\nabla u$  was first studied in [31, 29, 13, 19]. In this paper, we first analyze the relaxation of the functional  $I$  in terms of Young measures. We proceed by extending  $L^p$  to the space of Young measures equipped with the narrow topology. We conclude with a relaxed formulation of the functional  $I$  in terms of Young measures, so providing a full characterization of the relaxation. In fact, Young measures appear throughout the paper as a useful tool to analyze both lower semicontinuity and relaxation. Good accounts on  $L^p$  Young measures can be found in [32, 9, 24].

The relaxation in  $L^p$  turns out to be a considerably difficult issue; in fact, the existence of a relaxed formulation in an integral form defined on  $L^p$  is not clear at all. In this respect, we construct an explicit example ruling out the natural candidate for the relaxed formulation in the homogeneous case (i.e., when the integrand  $w$  does not depend on the independent variables  $x, x'$ ), namely, the functional in which the integrand  $w$  is replaced by its separately convex envelope. In addition, we give another example in which, assuming that there exists a relaxation in  $L^p$  of an integral form, we prove that the integrand must be a separately convex function which lies sometimes above and sometimes below the original integrand  $w$ . This unexpected fact makes it complicated the possible definition of an integrand of a relaxed formulation of integral form in  $L^p$ . This is the first incursion in this issue, but more work in the future will be needed to understand this interesting question.

One of the reasons for the difficulty of the  $L^p$  relaxation are the surprising facts appearing in nonlocal functionals. Possibly, the primary unexpected fact is that different integrands  $w$  may have the same functional  $I$ , which cannot happen in the local case. In this paper, we characterize those integrands giving rise to the null functional; this suffices to characterize the integrands with the same functional  $I$ . This question was first addressed in [22], and here we weaken the hypotheses and simplify the proof provided therein. We also give a striking example of a positive functional whose integrand takes negative values in an open set; this integrand is the same one used in the example for the relaxation in  $L^p$  mentioned in the previous paragraph. The problem of characterizing those integrands giving rise to positive functionals is left open.

This paper is organized as follows. In Section 2 we set the general notation. Section 3 explains the results on Young measures that will be used throughout the paper. Section 4 is one of the central parts of this paper: it shows several necessary and sufficient conditions for the lower semicontinuity of  $I$  in  $L^p$ , so proving their equivalence. Section 5 collects the results of other works about coercivity and, using the analysis of the previous section, establishes the existence of minimizers. We also explain how the boundary conditions are defined in this nonlocal context. Section 6 is an aside where we study when two integrands  $w$  give rise to the same functional  $I$ . Section 7 computes the relaxation of  $I$  in the space of Young measures. Finally, in Section 8 we make, by means of two examples, some remarks about the difficulty of computing the relaxation of  $I$  in  $L^p(\Omega, \mathbb{R}^d)$ .

## 2. NOTATION

In this section we set the general notation of the paper, most of which is standard.

Given  $E \subset \mathbb{R}^n$ ,  $C(E)$  is the set of continuous functions from  $E$  to  $\mathbb{R}$ , while  $C_0(E)$  is its subset of functions in  $C(E)$  that vanish at infinity; in other words, a function  $u \in C(E)$  belongs to  $C_0(E)$  whenever for every  $\varepsilon > 0$  there exists a compact  $K \subset E$  such that  $|u(x)| < \varepsilon$  for all  $x \in E \setminus K$ . The subset of bounded functions in  $C(E)$  is denoted by  $C_b(E)$ . The supremum norm in  $C_b(E)$  is denoted by  $\|\cdot\|_\infty$ .

For  $1 \leq p < \infty$ , the Lebesgue  $L^p$  space is defined in the usual way. We anticipate that this  $p \geq 1$  will always be finite. In function spaces, we will indicate the domain and target sets, as in, for example,  $L^p(E, \mathbb{R}^d)$ , except if the

target space is  $\mathbb{R}$ , in which case we will simply write  $L^p(E)$ ; here  $E$  is a measurable set of  $\mathbb{R}^n$ . The norm in  $L^p(E, \mathbb{R}^d)$  is denoted by  $\|\cdot\|_{L^p(E, \mathbb{R}^d)}$ .

We denote by  $\mathcal{M}(E)$  the set of (positive) measures in  $E$ . A probability measure in  $E$  is a  $\mu \in \mathcal{M}(E)$  such that  $\mu(E) = 1$ . Given  $a \in E$ , the Dirac delta at  $a$  is denoted by  $\delta_a$ ; it is, of course, a probability measure in  $E$ .

Given  $\mu_1 \in \mathcal{M}(E_1)$  and  $\mu_2 \in \mathcal{M}(E_2)$ , its product  $\mu_1 \otimes \mu_2 \in \mathcal{M}(E_1 \times E_2)$  is defined as the only measure satisfying  $(\mu_1 \otimes \mu_2)(B_1 \times B_2) = \mu_1(B_1)\mu_2(B_2)$  for all Borel  $B_1 \subset E_1$  and  $B_2 \subset E_2$ . Analogously, given two functions  $u_1 : \Omega_1 \rightarrow E_1$  and  $u_2 : \Omega_2 \rightarrow E_2$ , its product  $u_1 \otimes u_2 : \Omega_1 \times \Omega_2 \rightarrow E_1 \times E_2$  is defined as  $(u_1 \otimes u_2)(x_1, x_2) := (u_1(x_1), u_2(x_2))$ .

We will deal with two types of measurability: Lebesgue and Borel. Lebesgue measurability will be in a Lebesgue measurable subset  $\Omega$  of  $\mathbb{R}^n$ , while Borel measurability will be in  $\mathbb{R}^d$ . The Lebesgue measure in any Lebesgue measurable subset  $\Omega$  of  $\mathbb{R}^n$  will be denoted by  $\mathcal{L}^n$ . When we just write *measurable* it means Lebesgue measurable, while when we say  $\mathcal{B}^d$ -measurable it means Borel measurable in  $\mathbb{R}^d$ . Likewise,  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable means measurable in  $\Omega \times \mathbb{R}^d$  with respect to the product measure: the  $\mathcal{L}^n \otimes \mathcal{B}^d$ -sigma algebra will always be understood when working in  $\Omega \times \mathbb{R}^d$ . In fact, most of the paper deals with functions defined in  $\Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d$  that are  $\mathcal{L}^n \otimes \mathcal{L}^n \otimes \mathcal{B}^d \otimes \mathcal{B}^d$ -measurable. For this kind of functions  $w = w(x, x', y, y')$  we will often use expressions like “for a.e.  $x, x' \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ , property (P) holds”. We specify its meaning, since it may cause ambiguity: there exists a measurable set  $M \subset \Omega \times \Omega$  with  $\mathcal{L}^{2n}(\Omega \times \Omega) = \mathcal{L}^{2n}(M)$  for which for all  $(x, x', y, y') \in M \times \mathbb{R}^d \times \mathbb{R}^d$  property (P) holds.

The characteristic function of a  $B \subset \mathbb{R}^n$  is denoted by  $\chi_B$ . The average integral  $f_B$  denotes the integral in  $B$  divided by  $\mathcal{L}^n(B)$ . The negative part of a function  $f$  is denoted by  $f^-$ . Given  $A \subset \Omega$ , we denote  $A^c = \Omega \setminus A$ .

In the proofs of convergence, we will continuously use subsequences, which not be relabelled.

Weak convergence in  $L^p$  is denoted by  $\rightharpoonup$ . We will also use *biting* convergence, defined as follows (see, e.g., [32, Sect. 6.4] or [24, Def. 2.65]). We say that  $u_j \xrightarrow{b} u$  in  $L^1(\Omega, \mathbb{R}^d)$  (convergence in the biting sense) when  $\{u_j\}_{j \in \mathbb{N}}$  is bounded in  $L^1(\Omega, \mathbb{R}^d)$  and there exists a decreasing sequence  $\{E_k\}_{k \in \mathbb{N}}$  of measurable subsets of  $\Omega$  such that  $\mathcal{L}^n(E_k) \rightarrow 0$  as  $k \rightarrow \infty$ , and, for any  $k \in \mathbb{N}$ ,

$$u_j \rightharpoonup u \quad \text{in } L^1(\Omega \setminus E_k, \mathbb{R}^d) \quad \text{as } j \rightarrow \infty.$$

Of course, weak convergence in  $L^1$  implies biting convergence. Consequently, if a functional is lower semicontinuous with respect to the biting convergence then it also lower semicontinuous with respect to the weak convergence in  $L^1$ .

A function  $g : \mathbb{R}^d \times \mathbb{R}^d$  is *separately convex* if  $g(\cdot, y)$  and  $g(y, \cdot)$  are convex for each  $y \in \mathbb{R}^d$ .

### 3. YOUNG MEASURES IN $L^p$

In this section we briefly recall the definitions and results concerning Young measures that are needed in the paper; for the proofs and general expositions, we refer the reader to [37, 38, 32, 9, 24] as well as the references therein. We only provide a proof for those results for which we have not found a precise reference. In this section, we follow the exposition in [9, Sect. 4.3], which is based on Prokhorov’s theorem, instead of other usual approaches based on the duality between  $L^1(\Omega, C_0(\mathbb{R}^d))$  and  $L^\infty(\Omega, \mathcal{M}(\mathbb{R}^d))$ .

Let  $\Omega \subset \mathbb{R}^n$  be an open bounded subset. A *Young measure* in  $\Omega \times \mathbb{R}^d$ , equipped with the  $\mathcal{L}^n \otimes \mathcal{B}^d$ -sigma algebra, is a measure  $\nu$  in  $\Omega \times \mathbb{R}^d$  such that for any measurable  $E \subset \Omega$ ,

$$\nu(E \times \mathbb{R}^d) = \mathcal{L}^n(E).$$

We denote by  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  the set of Young measures in  $\Omega \times \mathbb{R}^d$ .

The procedure of *disintegration* (or *slicing*; see, e.g., [9, Th. 4.2.4]) allows us for an alternative description of Young measures. Accordingly, we can identify  $\nu$  with a family  $(\nu_x)_{x \in \Omega}$  of probability measures on  $\mathbb{R}^d$  such that for all  $f \in C_0(\Omega \times \mathbb{R}^d)$ , the map

$$\Omega \ni x \mapsto \int_{\mathbb{R}^d} f(x, y) d\nu_x(y)$$

is measurable and

$$\int_{\Omega \times \mathbb{R}^d} f(x, y) d\nu(x, y) = \int_{\Omega} \left( \int_{\mathbb{R}^d} f(x, y) d\nu_x(y) \right) dx.$$

We write  $\nu = (\nu_x)_{x \in \Omega}$ , although a more proper notation would be  $\nu = \mathcal{L}^n \otimes (\nu_x)_{x \in \Omega}$ . That is why Young measures are also called *parametrized measures*. In the sequel, we will use both approaches.

The sets  $\mathcal{M}(\Omega \times \mathbb{R}^d)$ , and, hence,  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  can be given a variety of topologies (see, e.g., [14]). The most relevant to the current work is the narrow topology in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$ : it is weakest topology that makes the maps

$$\nu \mapsto \int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\nu(x, y)$$

continuous, for all  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable  $\varphi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\varphi(x, \cdot) \in C_b(\mathbb{R}^d) \text{ for a.e. } x \in \Omega \quad \text{and} \quad \int_{\Omega} \|\varphi(x, \cdot)\|_{\infty} dx < \infty. \quad (1)$$

In particular, it induces the following convergence: a sequence  $\{\mu^j\}_{j \in \mathbb{N}} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$  narrowly converges to a  $\mu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ , and write  $\mu^j \xrightarrow{\text{nar}} \mu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$ , when for all  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable  $\varphi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  with property (1), one has

$$\lim_{j \rightarrow \infty} \int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\mu^j(x, y) = \int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\mu(x, y).$$

Moreover, with the identification  $\mu^j = (\mu_x^j)_{x \in \Omega}$  and  $\mu = (\mu_x)_{x \in \Omega}$ , we have that  $\mu^j \xrightarrow{\text{nar}} \mu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$  if and only if for all  $g \in L^1(\Omega)$  and  $h \in C_0(\mathbb{R}^d)$ ,

$$\lim_{j \rightarrow \infty} \int_{\Omega} g(x) \left( \int_{\mathbb{R}^d} h(y) d\mu_x^j(y) \right) dx = \int_{\Omega} g(x) \left( \int_{\mathbb{R}^d} h(y) d\mu_x(y) \right) dx$$

(see, e.g., [9, Th. 4.3.1], which states that narrow convergence of Young measures and weak convergence of their corresponding probability measures are equivalent). The narrow topology is not metrizable, but the relevance of working with sequences (instead of nets) will become clear in Theorem 4; we anticipate that convergence of sequences is enough for the purposes of this work.

The following concept is of central importance.

**Definition 1.** A set  $\mathcal{H} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$  is tight when for all  $\varepsilon > 0$  there exists a compact  $K \subset \mathbb{R}^d$  such that

$$\sup_{\nu \in \mathcal{H}} \nu(\Omega \times (\mathbb{R}^d \setminus K)) < \varepsilon.$$

Prokhorov's theorem states the relatively compactness of bounded tight sets of measures (see, e.g., [14, Sect. 5]). When applied to sequences of Young measures, it can be stated as follows.

**Theorem 2.** Let  $\{\nu^j\}_{j \in \mathbb{N}} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$ .

- a) If  $\{\nu^j\}_{j \in \mathbb{N}}$  is tight, then there exist a subsequence (not relabelled) and a  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$  such that  $\nu^j \xrightarrow{\text{nar}} \nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$ .
- b) If there exists  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$  such that  $\nu^j \xrightarrow{\text{nar}} \nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$  then  $\{\nu^j\}_{j \in \mathbb{N}}$  is tight.

Tightness can also be characterized by the following criterion, similar in spirit to de la Vallée-Poussin criterion for equiintegrability; see [37, Prop. 8] or [38, Comment 2, p. 369].

**Proposition 3.** A set  $\mathcal{H} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$  is tight if and only if there exists a function  $h : [0, \infty) \rightarrow [0, \infty]$  such that

$$\lim_{t \rightarrow \infty} h(t) = \infty \quad (2)$$

and

$$\sup_{\nu \in \mathcal{H}} \int_{\Omega \times \mathbb{R}^d} h(|y|) d\nu(x, y) < \infty.$$

Property (2) is sometimes called *coercivity*. In our finite-dimensional context such  $h$  are also characterized by being *inf-compact*; see [9, Sect. 3.2.5].

Despite the narrow topology is not metrizable, the following result holds; it is a consequence of [37, Thms. 1, 2 and 11].

**Theorem 4.** Let  $\mathcal{H}$  be a tight subset of  $\mathcal{Y}(\Omega, \mathbb{R}^d)$ . Then the narrow topology on  $\mathcal{H}$  is metrizable.

Any measurable function  $u : \Omega \rightarrow \mathbb{R}^d$  can be identified with the Young measure  $\nu = (\nu_x)_{x \in \Omega}$  given by  $\nu_x = \delta_{u(x)}$ , i.e.,

$$\int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\nu(x, y) = \int_{\Omega} \varphi(x, u(x)) dx$$

for all  $\varphi \in C_0(\Omega \times \mathbb{R}^d)$ . With a small abuse of notation, we can write  $u \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ ; analogously, we can talk about narrow convergence of a sequence of measurable functions, meaning narrow convergence of their associated Young measures. Thus, a sequence  $\{u_j\}_{j \in \mathbb{N}}$  of measurable functions from  $\Omega$  to  $\mathbb{R}^d$  is tight if and only if for every  $\varepsilon > 0$  there exists  $M > 0$  such that

$$\sup_{j \in \mathbb{N}} \mathcal{L}^n(\{x \in \Omega : |u_j(x)| > M\}) < \varepsilon;$$

equivalently, in view of Proposition 3, there exists a function  $h : [0, \infty) \rightarrow [0, \infty]$  with property (2) such that

$$\sup_{j \in \mathbb{N}} \int_{\Omega} h(|u_j(x)|) dx < \infty.$$

In this case, Theorem 2 provides a version of the existence result for Young measures (see, e.g, [24, Th. 8.6]).

**Proposition 5.** *Let  $\{u_j\}_{j \in \mathbb{N}}$  be a tight sequence of measurable functions from  $\Omega$  to  $\mathbb{R}^d$ . Then there exist a subsequence (not relabelled) and a  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$  such that  $u_j \xrightarrow{\text{nar}} \nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$ .*

When  $u_j \xrightarrow{\text{nar}} \nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$ , we say that the sequence of functions  $\{u_j\}_{j \in \mathbb{N}}$  generates the Young measure  $\nu$ . Recall that it means that for all  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable  $\varphi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  with property (1), one has

$$\lim_{j \rightarrow \infty} \int_{\Omega} \varphi(x, u_j(x)) dx = \int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\nu(x, y). \quad (3)$$

In fact, the following continuity result shows that the above limit holds for a larger family of test functions (see, e.g., [9, Th. 4.3.3] or [24, Th. 8.6]).

**Proposition 6.** *Let  $\{u_j\}_{j \in \mathbb{N}}$  be a sequence of measurable functions from  $\Omega$  to  $\mathbb{R}^d$  narrowly converging to some  $\nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$ . Let  $\varphi : \Omega \times \mathbb{R}^d \rightarrow [-\infty, \infty]$  be  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable and satisfy*

- a) for a.e.  $x \in \Omega$ , the function  $\varphi(x, \cdot)$  is continuous;
- b) the sequence of functions  $\Omega \ni x \mapsto \varphi(x, u_j(x))$  is equiintegrable.

Then limit (3) holds.

Functions  $\varphi : \Omega \times \mathbb{R}^d \rightarrow [-\infty, \infty]$  that are  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable and satisfy a) of Proposition 6 are called *Carathéodory integrands* (see, e.g., [24, Def. 6.33]).

The following semicontinuity property also holds (see [9, Prop. 4.3.4] or [24, Th. 8.6]).

**Proposition 7.** *Let  $\{u_j\}_{j \in \mathbb{N}}$  be a sequence of measurable functions from  $\Omega$  to  $\mathbb{R}^d$  narrowly converging to some  $\nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$ . Let  $\varphi : \Omega \times \mathbb{R}^d \rightarrow [-\infty, \infty]$  be  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable and satisfy*

- a) for a.e.  $x \in \Omega$ , the function  $\varphi(x, \cdot)$  is lower semicontinuous;
- b) the sequence of functions  $\Omega \ni x \mapsto \varphi^-(x, u_j(x))$  is equiintegrable.

Then

$$\int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\nu(x, y) \leq \liminf_{j \rightarrow \infty} \int_{\Omega} \varphi(x, u_j(x)) dx.$$

Functions  $\varphi : \Omega \times \mathbb{R}^d \rightarrow [-\infty, \infty]$  that are  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable and satisfy a) of Proposition 7 are called *normal integrands* (see, e.g., [24, Def. 6.27 and Prop. 6.31]).

Both Propositions 6 and 7 are consequences of the following lower semicontinuity result for the narrow convergence of Young measures (see [9, Prop. 4.3.3]).

**Proposition 8.** *Let  $\{\nu_j\}_{j \in \mathbb{N}} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$  narrowly converge to some  $\nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$ . Let  $\varphi : \Omega \times \mathbb{R}^d \rightarrow [0, \infty]$  be  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable such that property a) of Proposition 7 holds. Then*

$$\int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\nu(x, y) \leq \lim_{j \rightarrow \infty} \int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\nu^j(x, y). \quad (4)$$

As a consequence of the definition of narrow convergence, the following slight generalization of Proposition 8 follows.

**Corollary 9.** *Let  $\{\nu_j\}_{j \in \mathbb{N}} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$  narrowly converge to some  $\nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$ . Let  $\varphi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  be  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable such that property a) of Proposition 7 holds, and, in addition,  $\varphi \geq \psi$  for some  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable  $\psi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  such that*

$$\psi(x, \cdot) \in C_b(\mathbb{R}^d) \text{ for a.e. } x \in \Omega \quad \text{and} \quad \int_{\Omega} \|\psi(x, \cdot)\|_{\infty} dx < \infty.$$

Then inequality (4) holds.

Given  $p \geq 1$ , we call  $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$  the set of  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$  such that

$$\int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y) < \infty.$$

As a consequence of Hölder's inequality,  $\mathcal{Y}^p(\Omega, \mathbb{R}^d) \subset \mathcal{Y}^q(\Omega, \mathbb{R}^d)$  if  $1 \leq q \leq p$ .

Note that  $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$  is not closed in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  under the narrow topology. Nevertheless, given a bounded sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$ , thanks to Proposition 5, for a subsequence, there exists  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$  such that  $\{u_j\}_{j \in \mathbb{N}}$  generates  $\nu$ ; moreover, due to Proposition 7,  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . The converse result also holds (see, e.g., [32, Th. 7.7]); we present both in the following proposition.

**Proposition 10.** *Let  $p \geq 1$ . If  $\{u_j\}_{j \in \mathbb{N}}$  is a bounded sequence in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ , then  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . Conversely, for any  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  there exists a bounded sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable.*

Proposition 10 can be restated in the following somewhat abstract way.

**Proposition 11.** *Let  $p \geq 1$  and  $M > 0$ . Then the closure of  $\{u \in L^p(\Omega, \mathbb{R}^d) : \|u\|_{L^p(\Omega, \mathbb{R}^d)}^p \leq M\}$  in the narrow topology of  $\mathcal{Y}(\Omega, \mathbb{R}^d)$  is  $\{\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d) : \int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y) \leq M\}$ .*

*Proof.* Let  $\{u_j\}_{j \in \mathbb{N}}$  be a sequence in  $L^p(\Omega, \mathbb{R}^d)$  such that  $\|u_j\|_{L^p(\Omega, \mathbb{R}^d)}^p \leq M$  for all  $j \in \mathbb{N}$  converging narrowly to a  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ . By Proposition 7,  $\int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y) \leq M$ .

Conversely, given a  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$  with  $\int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y) \leq M$ , by Proposition 10 there exists a sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  converging narrowly to  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. By Proposition 6,

$$\lim_{j \rightarrow \infty} \|u_j\|_{L^p(\Omega, \mathbb{R}^d)}^p = \int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y) \leq M.$$

Then the sequence  $\{v_j\}_{j \in \mathbb{N}}$  defined by

$$v_j := \frac{\int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y)}{\|u_j\|_{L^p(\Omega, \mathbb{R}^d)}^p} u_j, \quad j \in \mathbb{N}$$

satisfies that  $v_j - u_j$  converges to zero in  $L^p(\Omega, \mathbb{R}^d)$ , so in measure, hence (see, e.g., [9, Prop. 4.3.8])  $v_j \xrightarrow{\text{nar}} \nu$  as  $j \rightarrow \infty$ , and, in addition,  $\|v_j\|_{L^p(\Omega, \mathbb{R}^d)}^p \leq M$  for all  $j \in \mathbb{N}$ .  $\square$

The following property is immediate (see, if necessary, the proof of [31, Prop. 2.3] or [22, Th. 11]).

**Lemma 12.** *If  $\{u_j\}_{j \in \mathbb{N}}$  is a bounded sequence in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ , then  $\{u_j \otimes u_j\}_{j \in \mathbb{N}}$  generates  $\nu \otimes \nu$ . Moreover, if  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable then  $\{|u_j \otimes u_j|^p\}_{j \in \mathbb{N}}$  is also equiintegrable.*

Given  $\nu \in \mathcal{Y}^1(\Omega, \mathbb{R}^d)$ , the first moment of  $\nu$  is defined as the measurable function  $u : \Omega \rightarrow \mathbb{R}^d$

$$u(x) := \int_{\mathbb{R}^d} y d\nu_x(y), \quad \text{a.e. } x \in \Omega.$$

Jensen's inequality shows at once that  $u \in L^p(\Omega, \mathbb{R}^d)$  whenever  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  for a given  $p \geq 1$ . The following classical result shows the relationship between narrow convergence and weak convergence in  $L^p(\Omega, \mathbb{R}^d)$  (see, e.g., [24, Th. 8.11] or [32, Th. 6.8]).

**Lemma 13.** *Let  $p \geq 1$ . Let  $\{u_j\}_{j \in \mathbb{N}}$  be a bounded sequence in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu$ , and let  $u$  be the first moment of  $\nu$ . Then  $u \in L^p(\Omega, \mathbb{R}^d)$  and  $u_j \xrightarrow{b} u$  as  $j \rightarrow \infty$ . If, in addition,  $p > 1$  then  $u_j \rightharpoonup u$  in  $L^p(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$ .*

#### 4. NECESSARY AND SUFFICIENT CONDITIONS FOR WEAK LOWER SEMICONTIINUITY

In this section we study the lower semicontinuity of the functional

$$I : L^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}, \quad I(u) := \int_{\Omega} \int_{\Omega} w(x, x', u(x), u(x')) \, dx \, dx' \quad (5)$$

under the weak topology of  $L^p(\Omega, \mathbb{R}^d)$  (and the biting convergence when  $p = 1$ ).

In fact, Elbau [22, Th. 11] (see also [15, Prop. 8.8]) found the following necessary and sufficient condition, which we call *nonlocal convexity*, for the lower semicontinuity of  $I$  in terms of  $w$ :

(NC) For a.e.  $x \in \Omega$  and every  $v \in L^p(\Omega, \mathbb{R}^d)$ , the function

$$\Phi_{x,v} : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}, \quad \Phi_{x,v}(y) := \int_{\Omega} w(x, x', y, v(x')) \, dx'$$

is convex.

This section aims to understand this condition and provide more characterizations of the lower semicontinuity, as well as an alternative proof to that of [22]. We will make use of the following extended functional  $\bar{I}$  of  $I$ :

$$\bar{I} : \mathcal{Y}^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}, \quad \bar{I}(\nu) := \int_{\Omega \times \mathbb{R}^d} \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') \, d\nu(x, y) \, d\nu(x', y'). \quad (6)$$

A generalization of condition (NC), already appearing in [22], is the following:

(NY) For a.e.  $x \in \Omega$  and every  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ , the function

$$\Phi_{x,\nu} : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}, \quad \Phi_{x,\nu}(y) := \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') \, d\nu(x', y')$$

is convex.

Condition (NY) is called *nonlocal convexity for Young measures*. Note that, via the identification of a function  $u \in L^p(\Omega, \mathbb{R}^d)$  with its associated Young measure, we have that  $\Phi_{x,u}$  according to both definitions in (NC) and (NY) coincide, so condition (NY) is, in principle, stronger than condition (NC).

On the other hand, and in a slightly different context (namely, functionals involving derivatives), Pedregal [31, Prop. 3.1 and eq. (4.3)] showed a necessary and sufficient condition for the lower semicontinuity of the analogue of our functional  $I$ . It reads as follows:

(NJ) For any  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ , and letting  $u$  be its first moment, we have

$$I(u) \leq \bar{I}(\nu).$$

Condition (NJ) is called *nonlocal Jensen's inequality*.

The functional  $\bar{I}$ , as well as being a useful theoretical tool in order to study the lower semicontinuity of  $I$ , turns out to be the relaxation of  $I$  in terms of Young measures, as will be shown in Section 7.

Before stating the equivalence between conditions (NC), (NY), (NJ) and the lower semicontinuity of  $I$ , we first establish several sets of assumptions on the integrand  $w$  that will be continuously used in the sequel. We assume  $p \geq 1$  is given.

The first set of conditions deals with the finiteness of the integral in  $\bar{I}$ :

(F0) For every  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  there exist  $\alpha \in L^1(\Omega)$  and  $C > 0$  such that

$$\int_{\Omega \times \mathbb{R}^d} w^-(x, x', y, y') \, d\nu(x', y') \leq \alpha(x) + C|y|^p \quad \text{for a.e. } x \in \Omega \text{ and all } y \in \mathbb{R}^d.$$

(F1) For every  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  there exist  $\alpha \in L^1(\Omega)$  and  $C > 0$  such that

$$\int_{\Omega \times \mathbb{R}^d} |w(x, x', y, y')| \, d\nu(x', y') \leq \alpha(x) + C|y|^p \quad \text{for a.e. } x \in \Omega \text{ and all } y \in \mathbb{R}^d.$$

It is easy to see that condition (F0) implies that  $\bar{I}(\nu)$  is well defined as a member of  $\mathbb{R} \cup \{\infty\}$ , for each  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ , and that condition (F1) implies that  $\bar{I}(\nu)$  is well defined and finite. One could also define the analogous conditions for  $I$ , but they are not used in this paper. This is, in fact, what is done in [22, Prop. 3], where it is also proved that the analogue of (F0) and (F1) for  $I$  are necessary and sufficient for  $I$  to be well-defined and finite, respectively.

Now we present the set of assumptions (A), which is taylor-made so as to apply Proposition 6:

- (A0)  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [-\infty, \infty]$  is  $\mathcal{L}^n \otimes \mathcal{L}^n \otimes \mathcal{B}^d \otimes \mathcal{B}^d$ -measurable and  $w(x, x', y, y') = w(x', x, y', y)$  for a.e.  $x, x' \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ .
- (A1) For a.e.  $x, x' \in \Omega$ , the function  $w(x, x', \cdot, \cdot)$  is continuous.
- (A2) There exist  $a \in L^1(\Omega \times \Omega)$  and  $c > 0$  such that

$$|w(x, x', y, y')| \leq a(x, x') + c|y|^p$$

for a.e.  $x, x' \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ .

Note that the symmetry of  $w$  stated in (A0) can be imposed without loss of generality, since, by Fubini's theorem, given any  $\mathcal{L}^n \otimes \mathcal{L}^n \otimes \mathcal{B}^d \otimes \mathcal{B}^d$ -measurable  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow [-\infty, \infty]$  with suitable integrability properties, the integrands  $w$  and

$$\frac{w(x, x', y, y') + w(x', x, y', y)}{2}$$

give rise to the same functional.

The set of assumptions (B) is taylor-made so as to apply Proposition 7, and consists of conditions (A0), (B1) and (B2), where (B1) and (B2) are as follows:

- (B1) For a.e.  $x, x' \in \Omega$ , the function  $w(x, x', \cdot, \cdot)$  is lower semicontinuous.
- (B2) There exist  $a \in L^1(\Omega \times \Omega)$  and a continuous strictly increasing  $g : [0, \infty) \rightarrow [0, \infty)$  with

$$\lim_{t \rightarrow \infty} \frac{g(t)}{t^p} = 0 \tag{7}$$

such that

$$w^-(x, x', y, y') \leq a(x, x') + g(|y|)$$

for a.e.  $x, x' \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ .

The relevance of assumption (B2) comes from the following fact.

**Lemma 14.** *Let  $p \geq 1$ . Assume that  $\{u_j\}_{j \in \mathbb{N}}$  is a bounded sequence in  $L^p(\Omega, \mathbb{R}^d)$  and let  $g : [0, \infty) \rightarrow [0, \infty)$  be continuous, strictly increasing such that (7) holds. Then the sequence of functions  $\{g(|u_j|)\}_{j \in \mathbb{N}}$  is equiintegrable.*

*Proof.* As  $g$  is continuous and strictly increasing, it has an inverse  $g^{-1}$  defined on  $[g(0), g(\infty))$ , where  $g(\infty)$  stands for  $\lim_{t \rightarrow \infty} g(t)$ . If  $g(\infty) < \infty$  then  $\{g(|u_j|)\}_{j \in \mathbb{N}}$  is bounded in  $L^\infty(\Omega)$ , so equiintegrable. If  $g(\infty) = \infty$  then we define  $h : [g(0), \infty) \rightarrow [0, \infty)$  as  $h(s) := g^{-1}(s)^p$ , which satisfies

$$\lim_{s \rightarrow \infty} \frac{h(s)}{s} = \lim_{t \rightarrow \infty} \frac{h(g(t))}{g(t)} = \lim_{t \rightarrow \infty} \frac{t^p}{g(t)} = \infty$$

and

$$\sup_{j \in \mathbb{N}} \int_{\Omega} h(g(|u_j|)) \, dx = \sup_{j \in \mathbb{N}} \int_{\Omega} |u_j|^p \, dx < \infty.$$

The conclusion follows from de la Vallée-Poussin criterion. □

Note that assumption (A) implies (F1), whereas assumption (B) implies (F0).

In full rigour, conditions (NC), (NY) and (NJ) should be called  $(NC)_p$  or with a similar symbol indicating its dependence on  $p$ . Similarly, the set of assumptions (A) and (B) also depend on  $p$ , but, for simplicity of notation, we do not indicate that dependence and assume that the exponent  $p$  is fixed throughout the work. This dependence is especially significant when distinguishing the cases  $p > 1$  and  $p = 1$ , as will be seen in the next proposition, where we establish the equivalence between condition (NJ) and the lower semicontinuity of  $I$ . Its proof is an adaptation of that of [31, Prop. 3.1] for functionals without derivatives and with growth conditions of the style (A) or (B).

**Proposition 15.** *Let  $p \geq 1$ . The following statements hold:*

- a) If  $w$  satisfies assumptions (A) and
- i)  $p > 1$  and  $I$  is lower semicontinuous in  $L^p(\Omega, \mathbb{R}^d)$  with respect to the weak convergence; or
  - ii)  $p = 1$  and  $I$  is lower semicontinuous in  $L^p(\Omega, \mathbb{R}^d)$  with respect to the biting convergence,
- then condition (NJ) holds.
- b) If  $w$  satisfies assumptions (B) and condition (NJ) holds then the functional  $I$  is lower semicontinuous in  $L^p(\Omega, \mathbb{R}^d)$
- i) with respect to the weak convergence if  $p > 1$ ; and
  - ii) with respect to the biting convergence if  $p = 1$ .

*Proof.* We first prove a). Consider  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  and let  $u \in L^p(\Omega, \mathbb{R}^d)$  be its first moment. By Proposition 10, there exists a sequence  $\{u_j\}_{j \in \mathbb{N}}$  generating  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. By (A2), the sequence of functions

$$\Omega \times \Omega \ni (x, x') \mapsto w(x, x', u_j(x), u_j(x'))$$

is equiintegrable. Hence, by Proposition 6,

$$\lim_{j \rightarrow \infty} I(u_j) = \bar{I}(\nu). \quad (8)$$

On the other hand, we have by Lemma 13 that  $u_j \rightharpoonup u$  in  $L^p(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$  if  $p > 1$ , and  $u_j \xrightarrow{b} u$  if  $p = 1$ . Applying the lower semicontinuity of  $I$ , we find that

$$I(u) \leq \liminf_{j \rightarrow \infty} I(u_j). \quad (9)$$

Comparing (8) and (9), we obtain condition (NJ).

We now prove b). Let  $\{u_j\}_{j \in \mathbb{N}}$  be a bounded sequence in  $L^p(\Omega, \mathbb{R}^d)$  weakly converging to  $u$  if  $p > 1$ , and  $u_j \xrightarrow{b} u$  as  $j \rightarrow \infty$  if  $p = 1$ . Passing to a subsequence, we can assume by Propositions 5 and 10 that  $\{u_j\}_{j \in \mathbb{N}}$  generates a  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . Moreover, by Lemma 13,  $u$  is the first moment of  $\nu$ . By (B2) and Lemma 14, the sequence of functions

$$\Omega \times \Omega \ni (x, x') \mapsto w^-(x, x', u_j(x), u_j(x'))$$

is equiintegrable. By (NJ) and Proposition 7,

$$I(u) \leq \bar{I}(\nu) \leq \liminf_{j \rightarrow \infty} I(u_j),$$

so  $I$  is lower semicontinuous. □

We will use several times the following auxiliary result, which is a version of Lebesgue's differentiation theorem for double integrals.

**Lemma 16.** *Let  $h \in L^1_{\text{loc}}(\Omega \times \Omega)$ . Then, for a.e.  $x_0 \in \Omega$ ,*

$$\lim_{r \searrow 0} \int_{B(x_0, r)} \int_{B(x_0, r)} h(x, x') \, dx' \, dx = 0 \quad (10)$$

and

$$\lim_{r \searrow 0} \int_{B(x_0, r)} \int_{\Omega \setminus B(x_0, r)} h(x, x') \, dx' \, dx = \lim_{r \searrow 0} \int_{B(x_0, r)} \int_{\Omega} h(x, x') \, dx' \, dx = \int_{\Omega} h(x_0, x') \, dx'. \quad (11)$$

*Proof.* By Fubini's theorem,  $\int_{\Omega} h(\cdot, x') \, dx' \in L^1_{\text{loc}}(\Omega)$ , hence by Lebesgue's differentiation theorem, for a.e.  $x_0 \in \Omega$ ,

$$\lim_{r \searrow 0} \int_{B(x_0, r)} \int_{\Omega} h(x, x') \, dx' \, dx = \int_{\Omega} h(x_0, x') \, dx'.$$

This shows the second equality of (11); the first one follows from the second and (10). It, therefore, remains to prove (10).

By Lebesgue's differentiation theorem, for a.e.  $(x_0, x'_0) \in \Omega \times \Omega$ ,

$$\lim_{r \searrow 0} \int_{B(x_0, r)} \int_{B(x'_0, r)} h(x, x') \, dx' \, dx = h(x_0, x'_0);$$

in particular,

$$\lim_{r \searrow 0} \int_{B(x_0, r)} \int_{B(x'_0, r)} h(x, x') \, dx' \, dx = 0. \quad (12)$$

Fix  $x_0 \in \Omega$  such that limit (12) holds for a.e.  $x'_0 \in \Omega$ ; note that a.e.  $x_0 \in \Omega$  satisfies that. Observe now (e.g., by Lebesgue's dominated theorem) that the function

$$(x_1, r) \mapsto \int_{B(x_0, r)} \int_{B(x_1, r)} h(x, x') dx' dx$$

is continuous in the set  $\{(x_1, r) \in \Omega \times [0, \infty) : B(x_1, r) \subset \Omega\}$ , hence uniformly continuous in compact subsets. Thus, given  $\varepsilon > 0$  there exists  $\delta > 0$  such that for any  $x'_0 \in \bar{B}(x_0, \delta)$  and any  $0 < r \leq \delta$  one has

$$\left| \int_{B(x_0, r)} \int_{B(x_0, r)} h(x, x') dx' dx - \int_{B(x_0, r)} \int_{B(x'_0, r)} h(x, x') dx' dx \right| < \varepsilon.$$

Taking, additionally, an  $x'_0 \in \Omega$  such that (12) holds, we obtain that there exists  $r_0 \in (0, \delta]$  such that for any  $0 < r < r_0$ ,

$$\left| \int_{B(x_0, r)} \int_{B(x'_0, r)} h(x, x') dx' dx \right| < \varepsilon,$$

so

$$\left| \int_{B(x_0, r)} \int_{B(x_0, r)} h(x, x') dx' dx \right| < 2\varepsilon$$

and the lemma is proved.  $\square$

The main result in this section is the following theorem, which establishes the equivalence between (NC), (NY) and (NJ).

**Theorem 17.** *Let  $p \geq 1$ . The following implications hold:*

- a) *Under assumption (A), condition (NC) implies (NY).*
- b) *Under assumptions (A0) and (F0), condition (NY) implies (NJ).*
- c) *Under assumptions (A0) and (F1), condition (NJ) implies (NC).*

*Proof.* We first prove a). Fix  $x \in \Omega$  such that for every  $v \in L^p(\Omega, \mathbb{R}^d)$  the function  $\Phi_{x, v}$  is convex. Let  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ ; by Proposition 10, there exists a sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu$  such that  $\{|u_j|^p\}$  is equiintegrable. Fix  $y \in \mathbb{R}^d$ . Thanks to (A2), the sequence of functions  $\{f_{j, x, y}\}_{j \in \mathbb{N}}$  in  $\Omega$  defined by

$$f_{j, x, y}(x') := w(x, x', y, u_j(x')), \quad x' \in \Omega$$

is equiintegrable. Therefore, by Proposition 6,

$$\lim_{j \rightarrow \infty} \Phi_{x, u_j}(y) = \lim_{j \rightarrow \infty} \int_{\Omega} f_{j, x, y}(x') dx' = \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') d\nu(x', y') = \Phi_{x, \nu}(y).$$

Consequently,  $\Phi_{x, \nu}$  is convex as a pointwise limit of convex functions.

Now we prove b). Take  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ , and let  $u \in L^p(\Omega, \mathbb{R}^d)$  be its first moment. Note that

$$\bar{I}(\nu) = \int_{\Omega \times \mathbb{R}^d} \Phi_{x, \nu}(y) d\nu(x, y), \quad I(u) = \int_{\Omega} \Phi_{x, u}(u(x)) dx, \quad \int_{\Omega} \Phi_{x, \nu}(u(x)) dx = \int_{\Omega \times \mathbb{R}^d} \Phi_{x, u}(y) d\nu(x, y). \quad (13)$$

Thanks to (NY), we can use Jensen's inequality to the convex function  $\Phi_{x, \nu}$  and the probability measure  $\nu_x$ , and obtain that

$$\Phi_{x, \nu}(u(x)) \leq \int_{\mathbb{R}^d} \Phi_{x, \nu}(y) d\nu_x(y), \quad \text{a.e. } x \in \Omega,$$

so

$$\int_{\Omega} \Phi_{x, \nu}(u(x)) dx \leq \int_{\Omega \times \mathbb{R}^d} \Phi_{x, \nu}(y) d\nu(x, y). \quad (14)$$

Analogously,

$$\int_{\Omega} \Phi_{x, u}(u(x)) dx \leq \int_{\Omega \times \mathbb{R}^d} \Phi_{x, u}(y) d\nu(x, y). \quad (15)$$

Putting together the relations (13), (14) and (15) we obtain

$$I(u) = \int_{\Omega} \Phi_{x, u}(u(x)) dx \leq \int_{\Omega \times \mathbb{R}^d} \Phi_{x, u}(y) d\nu(x, y) = \int_{\Omega} \Phi_{x, \nu}(u(x)) dx \leq \int_{\Omega \times \mathbb{R}^d} \Phi_{x, \nu}(y) d\nu(x, y) = \bar{I}(\nu),$$

as desired.

We now show *c*). For this we follow the idea of the proof of [33, Th. 2.6]: we shall construct a family in  $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$  such that, when (NJ) is imposed, we will arrive at (NC). Let  $u \in L^p(\Omega, \mathbb{R}^d)$ . Fix  $y_0, y_1, y_2 \in \mathbb{R}^d$  and  $\alpha \in [0, 1]$  such that  $y_0 = \alpha y_1 + (1 - \alpha)y_2$ . Consider a measurable subdomain  $A \subset \Omega$ , and define the parametrized measures  $\nu$  and  $\mu$  by

$$\nu_x = \begin{cases} \delta_{y_0}, & \text{for } x \in A, \\ \delta_{u(x)}, & \text{for a.e. } x \in A^c, \end{cases} \quad \mu_x = \begin{cases} \alpha \delta_{y_1} + (1 - \alpha) \delta_{y_2}, & \text{for } x \in A, \\ \delta_{u(x)}, & \text{for a.e. } x \in A^c. \end{cases} \quad (16)$$

Clearly,  $\nu, \mu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . Furthermore, the parametrized measure  $\mu^t := (t\mu_x + (1-t)\nu_x)_{x \in \Omega}$  also belongs to  $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$  for each  $t \in [0, 1]$ . Define the function  $f : [0, 1] \rightarrow \mathbb{R}$  as  $f(t) := \bar{I}(\mu^t)$  and note that

$$f(t) = t^2 \bar{I}(\mu) + 2t(1-t) \int_{\Omega \times \mathbb{R}^d} \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') d\nu(x, y) d\mu(x', y') + (1-t)^2 \bar{I}(\nu).$$

Observe that all coefficients of the second-order polynomial  $f$  are finite, thanks to (F1).

The first moment  $\tilde{u}$  of  $\mu^t$  turns out to be independent of  $t \in [0, 1]$ , and is

$$\tilde{u}(x) = \begin{cases} y_0, & \text{for } x \in A, \\ u(x), & \text{for a.e. } x \in A^c. \end{cases}$$

Imposing condition (NJ) to  $\mu^t$  yields

$$f(t) \geq \int_{\Omega} \int_{\Omega} w(x, x', \tilde{u}(x), \tilde{u}(x')) dx dx', \quad t \in [0, 1], \quad (17)$$

We now observe that in (17) we have equality for  $t = 0$ , as is immediate to check. Therefore,  $f$  attains its minimum at  $t = 0$ , and, hence,  $f'(0) \geq 0$ , obtaining the inequality

$$\int_{\Omega \times \mathbb{R}^d} \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') d\nu(x, y) d\mu(x', y') \geq \bar{I}(\nu).$$

Having in mind expressions (16), and simplifying common terms in both sides, the last inequality becomes

$$\begin{aligned} & \int_A \int_A [\alpha w(x, x', y_1, y_0) + (1 - \alpha) w(x, x', y_2, y_0)] dx' dx \\ & + \int_A \int_{A^c} [\alpha w(x, x', y_1, u(x')) + (1 - \alpha) w(x, x', y_2, u(x'))] dx' dx \\ & \geq \int_A \int_A w(x, x', y_0, y_0) dx' dx + \int_A \int_{A^c} w(x, x', y_0, u(x')) dx' dx. \end{aligned}$$

We now take  $A$  to be a ball, divide by  $|A|$  and take limits when the radius of  $A$  goes to zero; by Lemma 16 we find that for a.e.  $x \in \Omega$ ,

$$\int_{\Omega} [\alpha w(x, x', y_1, u(x')) + (1 - \alpha) w(x, x', y_2, u(x'))] dx' \geq \int_{\Omega} w(x, x', y_0, u(x')) dx',$$

that is to say,

$$\alpha \Phi_{x,u}(y_1) + (1 - \alpha) \Phi_{x,u}(y_2) \geq \Phi_{x,u}(y_0),$$

so  $\Phi_{x,u}$  is convex. □

Of course, Proposition 15 and Theorem 17 show at once that under assumption (A), conditions (NC), (NY), (NJ) and the lower semicontinuity of  $I$  in  $L^p(\Omega, \mathbb{R}^d)$  are equivalent.

Contrarily to what is claimed in [30, Th. 5.1], it is not true that if  $\Phi_{x,a}$  is convex for a.e.  $x \in \Omega$  and all  $a \in \mathbb{R}^d$  then condition (NC) holds, as the following example shows, which is a small adaptation to that in [11, Sect. 3] showing that (NC) is weaker than separate convexity. Both examples meet the requirements of peridynamics.

**Example 18.** Let  $n = d = 1$  and  $\Omega = (0, 1)$ . Let  $h : (-1, 1) \rightarrow \mathbb{R}$  be any bounded smooth function such that

- $h(t) = h(-t)$  for all  $t \in (-1, 1)$ .
- $h > 0$  in  $(-1 + \delta, 1 - \delta)$ , and  $h < 0$  in  $(-1, -1 + \delta) \cup (1 - \delta, 1)$ , for some  $0 < \delta < \frac{1}{2}$ .
- $\int_{-1+t}^t h \geq 0$  for all  $t \in (0, 1)$ .

Define  $w : \Omega \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  as  $w(x, x', y, y') := \frac{1}{12}h(x - x')(y - y')^4$ . Then, for any  $x \in \Omega$  and  $a \in \mathbb{R}$ , the function  $\Phi_{x,a}$  is convex, but for all  $x \in (1 - \delta, 1)$  there exists  $u \in L^p(\Omega)$  such that  $\Phi_{x,u}$  is not convex.

*Proof.* For any  $x \in \Omega$  and  $u \in L^p(\Omega)$ ,

$$\Phi''_{x,v}(y) = \int_0^1 h(x - x') (y - u(x'))^2 dx'$$

so, for all  $a \in \mathbb{R}$ ,

$$\Phi''_{x,a}(y) = (y - a)^2 \int_0^1 h(x - x') dx' \geq 0,$$

hence  $\Phi_{x,a}$  is convex. Now fix  $x \in (1 - \delta, 1)$  and choose  $u = b\chi_{(0, x-1+\delta)}$  for  $b > 0$  big enough. Then

$$\Phi''_{x,u}(y) = (y - b)^2 \int_0^{x-1+\delta} h(x - x') dx' + y^2 \int_{x-1+\delta}^1 h(x - x') dx'.$$

Therefore, there exist  $b > 0$  and  $y \in \mathbb{R}$  such that  $\Phi''_{x,u}(y) < 0$ , so  $\Phi_{x,u}$  is not convex.  $\square$

What is true, nevertheless, is that, given a dense subset  $D$  of  $L^p(\Omega, \mathbb{R}^d)$ , and assuming that  $w$  satisfies assumption (A), if  $\Phi_{x,u}$  is convex for a.e.  $x \in \Omega$  and all  $u \in D$ , then condition (NC) holds. The proof of this fact is similar (in fact, easier, since it does not involve Young measures) to the proof of *a*) of Theorem 17.

Condition (NC) (or, equivalently, (NY) and (NJ)) depends on the domain  $\Omega$ ; this is awkward in view of the applications in peridynamics, when one would expect that the lower semicontinuity property of the energy density of a material does not depend on the reference configuration of the body. The relevance of having the lower semicontinuity property for a collection of domains became more apparent in the work [12], where that assumption was needed in order to pass to the limit as the horizon tends to zero in the peridynamic model to obtain the classical nonlinearly elastic model. Recently, Pedregal [33, Th. 2.6] showed that condition (NC) for all domains  $\Omega$  reduces to separate convexity. We provide a simpler proof of this fact.

**Proposition 19.** *Assume condition (A0) and that  $w(x, \cdot, y, y') \in L^1_{\text{loc}}(\Omega)$  for a.e.  $x \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ . Suppose that for a.e.  $x \in \Omega$ , every  $y' \in \mathbb{R}^d$  and every measurable  $D \subset \Omega$ , the function*

$$\Phi_{x,y',D} : \mathbb{R}^d \rightarrow \mathbb{R}, \quad \Phi_{x,y',D}(y) := \int_D w(x, x', y, y') dx'$$

*is convex. Then for a.e.  $x, x' \in \Omega$ , the function  $w(x, x', \cdot, \cdot)$  is separately convex.*

*Proof.* Fix  $x \in \Omega$  such that for every  $y' \in \mathbb{R}^d$  and every measurable  $D \subset \Omega$ , the function  $\Phi_{x,y',D}$  is convex. Take  $y \in \mathbb{R}^d$  and let  $x'$  be a Lebesgue point of  $w(x, \cdot, y, y')$ . Then

$$\lim_{r \searrow 0} \frac{1}{\mathcal{L}^n(B(x', r))} \Phi_{x,y',B(x',r)}(y) = w(x, x', y, y').$$

Hence  $w(x, x', \cdot, y')$  is convex as a pointwise limit of convex functions. Thanks to (A0),  $w(x, x', \cdot, \cdot)$  is separately convex.  $\square$

In contrast with the functional  $I$ , and as a consequence of Corollary 9, the functional  $\bar{I}$  is lower semicontinuous for integrands with a suitable lower bound, without the need of any convexity assumption.

**Proposition 20.** *Let  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  satisfy conditions (A0), (B1) and, additionally,  $w \geq h$  for some  $\mathcal{L}^n \otimes \mathcal{L}^n \otimes \mathcal{B}^d \otimes \mathcal{B}^d$ -measurable  $h : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  such that*

$$h(x, x', \cdot, \cdot) \in C_b(\mathbb{R}^d \times \mathbb{R}^d) \text{ for a.e. } x, x' \in \Omega, \quad \text{and} \quad \int_{\Omega} \int_{\Omega} \|h(x, x', \cdot, \cdot)\|_{\infty} dx' dx < \infty.$$

*Let  $\{\nu_j\}_{j \in \mathbb{N}} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$  narrowly converge to some  $\nu$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$ . Then*

$$\bar{I}(\nu) \leq \liminf_{j \rightarrow \infty} \bar{I}(\nu_j).$$

## 5. BOUNDARY CONDITIONS, COERCIVITY AND EXISTENCE OF MINIMIZERS

In this section we give conditions for the existence of minimizers of  $I$  and  $\bar{I}$ . Once the lower semicontinuity has been analyzed in Section 4, we ought to study the issue of coercivity in order to carry out the direct method of the Calculus of variations.

Typically, a lower bound for the integrand  $w$  together with some adequate boundary conditions yield the coercivity of the functional  $I$ , so we start explaining the type of boundary conditions normally used in nonlocal problems (see, e.g., [8, 26, 21, 11]), with the caveat that they are slightly different than in local problems, one of the reason being that  $L^p$  functions do not have traces of the boundary  $\partial\Omega$ .

First we establish the precise meaning of *translation invariance* of the functionals  $I$  and  $\bar{I}$ .

**Definition 21.** *The functional  $I$  is invariant under translations if  $I(u) = I(u + a)$  for all  $u \in L^p(\Omega, \mathbb{R}^d)$  and  $a \in \mathbb{R}^d$ .*

*The functional  $\bar{I}$  is invariant under translation if  $\bar{I}(\nu) = \bar{I}(\nu^a)$  for all  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  and  $a \in \mathbb{R}^d$ , where  $\nu^a$  is defined as the only  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measure in  $\Omega \times \mathbb{R}^d$  that satisfies, for any measurable  $E \subset \Omega$  and any Borel  $F \subset \mathbb{R}^d$ ,*

$$\nu^a(E \times F) = \nu(E \times (F + a)),$$

where  $F + a$  is the translated set of  $F$  by  $a$ .

It is immediate to check that  $\nu^a \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  whenever  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . Moreover,

$$\int_{\Omega \times \mathbb{R}^d} \varphi(x, y) d\nu^a(x, y) = \int_{\Omega \times \mathbb{R}^d} \varphi(x, y - a) d\nu(x, y) \quad (18)$$

for all  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable  $\varphi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfying condition (1). In fact, by monotone convergence, equality (18) also holds for any  $\mathcal{L}^n \otimes \mathcal{B}^d$ -measurable  $\varphi : \Omega \times \mathbb{R}^d \rightarrow [0, \infty]$ .

Note that a sufficient condition for  $I$  and  $\bar{I}$  to be invariant under translations is that the integrand  $w$  depends on  $(x, x', y, y')$  through  $(x, x', y - y')$ , but the analysis of Section 6 shows that there are more possibilities.

We now explain the nonlocal analogue of *Dirichlet* and *mixed* boundary conditions. We require the choice of a non-empty open set  $\Omega_0 \subset \Omega$  (which plays the role of *nonlocal interior*) and a  $\delta > 0$  such that  $\Omega_0 + B(0, \delta) \subset \Omega$ . In the context of peridynamics, this  $\delta$  is also the *horizon distance*: particles  $x, x' \in \Omega$  with  $|x - x'| \geq \delta$  do not interact, although this condition is not required in the paper. Of course,  $\Omega_0 + B(0, \delta)$  denotes the set of points in  $\mathbb{R}^n$  that can be expressed as a sum of an element of  $\Omega_0$  plus an element of  $B(0, \delta)$  (the open ball of centre 0 and radius  $\delta$ ). Pure Dirichlet conditions, in this context, prescribe the value of  $u$  in  $\Omega \setminus \Omega_0$ , while mixed Dirichlet–Neumann conditions prescribe the value of  $u$  in a measurable subset  $\Omega_D \subset \Omega \setminus \Omega_0$  with  $0 < \mathcal{L}^n(\Omega_D) < \mathcal{L}^n(\Omega \setminus \Omega_0)$ ; minimizers automatically satisfy a *nonlocal natural boundary condition* in  $\Omega \setminus (\Omega_0 \cup \Omega_D)$ . Pure *Neumann* conditions, which, again, are not imposed explicitly, require that the functional  $I$  is invariant under translations; in this case, the restriction  $\int_{\Omega} u dx = 0$  is made, so as to avoid that invariance. As before, minimizers of this problem satisfy a nonlocal natural boundary condition, which can be consulted in [11, Sect. 8]. Moreover, this kind of nonlocal boundary conditions can be given an interpretation of a nonlocal flux through the boundary, thus mimicking what happens for the local equations. This nonlocal calculus is developed in [26, 21, 3], to which we refer for further explanation.

The coercivity for the functional  $I$  was studied in [11] by collecting several *nonlocal Poincaré inequalities* that had appeared in the literature: they were suitable for integrands depending on  $(x, x', y, y')$  through  $(x, x', y - y')$ , as in the case of translation-invariant functionals  $I$ , and, in particular, in peridynamics. They also took into account the fact that the integrand  $w(x, x', y, y')$  can vanish when  $|x - x'| \geq \delta$ .

The first nonlocal Poincaré inequality that we present is suitable for pure Dirichlet and mixed Dirichlet–Neumann conditions. It has been proved, within several contexts and with slightly different versions, in [7, Prop. 2.5], [1, Lemma 2.4], [2, Prop. 4.1], [27, Lemma 3.5] and [12, Prop. 8]. The current formulation is taken from [11, Cor. 4.4].

**Proposition 22.** *Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^n$ , fix  $\delta > 0$  and let  $p \geq 1$ . Let  $\Omega_0$  be a non-empty open subset of  $\Omega$  satisfying  $\Omega_0 + B(0, \delta) \subset \Omega$ . Let  $\Omega_D$  be a measurable subset of  $\Omega \setminus \Omega_0$  with positive measure. Then there exists  $\lambda > 0$  such that for all  $u \in L^p(\Omega, \mathbb{R}^d)$ ,*

$$\int_{\Omega} |u(x)|^p dx \leq \lambda \int_{\Omega} \int_{\Omega \cap B(x, \delta)} |u(x) - u(x')|^p dx' dx + \lambda \int_{\Omega_D} |u(x)|^p dx.$$

The second nonlocal Poincaré inequality that we show is adequate for Neumann conditions. Again, it has been proved, with different versions, in [16], [17, Th. 1], [34, Th. 1.1], [6, Prop. 4.1], [1, Cor. 3.4] and [28, Cor. 4.6]. The following formulation is taken from [11, Prop. 4.2].

**Proposition 23.** *Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^n$ , fix  $\delta > 0$  and let  $p \geq 1$ . Then there exists  $\lambda > 0$  such that for all  $u \in L^p(\Omega, \mathbb{R}^d)$ ,*

$$\int_{\Omega} \left| u(x) - \int_{\Omega} u \right|^p dx \leq \lambda \int_{\Omega} \int_{\Omega \cap B(x, \delta)} |u(x) - u(x')|^p dx' dx.$$

In order to prove existence for  $\bar{I}$ , we will need the following versions of Propositions 22 and 23 for Young measures.

**Proposition 24.** *Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^n$ , fix  $\delta > 0$  and let  $p \geq 1$ . Let  $\Omega_0$  be a non-empty open subset of  $\Omega$  satisfying  $\Omega_0 + B(0, \delta) \subset \Omega$ . Let  $\Omega_D$  be a measurable subset of  $\Omega \setminus \Omega_0$  with positive measure. Then there exists  $\lambda > 0$  such that for all  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ ,*

$$\int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y) \leq \lambda \int_{\Omega \times \mathbb{R}^d} \int_{(\Omega \cap B(x, \delta)) \times \mathbb{R}^d} |y - y'|^p d\nu(x', y') d\nu(x, y) + \lambda \int_{\Omega_D \times \mathbb{R}^d} |y|^p d\nu(x, y).$$

*Proof.* Let  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . By Proposition 10, there exists a sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. According to Proposition 22, there exists  $\lambda > 0$  such that for all  $j \in \mathbb{N}$ ,

$$\int_{\Omega} |u_j(x)|^p dx \leq \lambda \int_{\Omega} \int_{\Omega \cap B(x, \delta)} |u_j(x) - u_j(x')|^p dx' dx + \lambda \int_{\Omega_D} |u_j(x)|^p dx.$$

By Proposition 6,

$$\lim_{j \rightarrow \infty} \int_{\Omega} |u_j(x)|^p dx = \int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y) \quad \text{and} \quad \lim_{j \rightarrow \infty} \int_{\Omega_D} |u_j(x)|^p dx = \int_{\Omega_D \times \mathbb{R}^d} |y|^p d\nu(x, y).$$

Similarly, having in mind Lemma 12, we also obtain

$$\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega \cap B(x, \delta)} |u_j(x) - u_j(x')|^p dx' dx = \int_{\Omega \times \mathbb{R}^d} \int_{(\Omega \cap B(x, \delta)) \times \mathbb{R}^d} |y - y'|^p d\nu(x', y') d\nu(x, y).$$

This concludes the proof.  $\square$

**Proposition 25.** *Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^n$ , fix  $\delta > 0$  and let  $p \geq 1$ . Then there exists  $\lambda > 0$  such that for all  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ ,*

$$\int_{\Omega \times \mathbb{R}^d} \left| y - \int_{\Omega} u \right|^p d\nu(x, y) \leq \lambda \int_{\Omega \times \mathbb{R}^d} \int_{(\Omega \cap B(x, \delta)) \times \mathbb{R}^d} |y - y'|^p d\nu(x', y') d\nu(x, y),$$

where  $u$  is the first moment of  $\nu$ .

*Proof.* Let  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  be such that its first moment  $u$  satisfies  $\int_{\Omega} u = 0$ . As in Proposition 24, there exists a sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. By Lemma 13,  $u_j \rightarrow u$  in  $L^p(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$ ; this also holds for  $p = 1$  because  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. In particular,  $\int_{\Omega} u_j \rightarrow 0$  as  $j \rightarrow \infty$ . Define  $v_j := u_j - \int_{\Omega} u_j$  for each  $j \in \mathbb{N}$ , which satisfies  $\int_{\Omega} v_j = 0$ . Then  $v_j - u_j \rightarrow 0$  in measure as  $j \rightarrow \infty$ , and, hence (see, e.g., [9, Prop. 4.3.8]),  $\{v_j\}_{j \in \mathbb{N}}$  generates  $\nu$ . Moreover,  $\{v_j\}_{j \in \mathbb{N}}$  is equiintegrable as the sum of two equiintegrable sequences. Thanks to Proposition 23, there exists  $\lambda > 0$  such that for all  $j \in \mathbb{N}$ ,

$$\int_{\Omega} |v_j(x)|^p dx \leq \lambda \int_{\Omega} \int_{\Omega \cap B(x, \delta)} |v_j(x) - v_j(x')|^p dx' dx.$$

Arguing as in Proposition 24, we obtain that

$$\int_{\Omega \times \mathbb{R}^d} |y|^p d\nu(x, y) \leq \lambda \int_{\Omega \times \mathbb{R}^d} \int_{(\Omega \cap B(x, \delta)) \times \mathbb{R}^d} |y - y'|^p d\nu(x', y') d\nu(x, y),$$

which concludes the proof in this case.

Now let be given a general  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  with first moment  $u$ , call  $a = \int_{\Omega} u$  and consider the Young measure  $\nu^a$  of Definition 21. Clearly, its first moment  $u^a$  satisfies  $\int_{\Omega} u^a = 0$ . By the first part of the proof,

$$\int_{\Omega \times \mathbb{R}^d} |y|^p d\nu^a(x, y) \leq \lambda \int_{\Omega \times \mathbb{R}^d} \int_{(\Omega \cap B(x, \delta)) \times \mathbb{R}^d} |y - y'|^p d\nu^a(x', y') d\nu^a(x, y).$$

The result is concluded by noting that, thanks to (18),

$$\int_{\Omega \times \mathbb{R}^d} |y|^p d\nu^a(x, y) = \int_{\Omega \times \mathbb{R}^d} |y - a|^p d\nu(x, y)$$

and

$$\int_{\Omega \times \mathbb{R}^d} \int_{(\Omega \cap B(x, \delta)) \times \mathbb{R}^d} |y - y'|^p d\nu^a(x', y') d\nu^a(x, y) = \int_{\Omega \times \mathbb{R}^d} \int_{(\Omega \cap B(x, \delta)) \times \mathbb{R}^d} |y - y'|^p d\nu(x', y') d\nu(x, y).$$

□

Finally, when the functional  $I$  is not invariant under translations, one can just impose a lower bound in  $w(x, x', y, y')$  in terms of  $|y|^p$  so as to obtain coercivity trivially, but this assumption is unrealistic in peridynamics.

With the lower semicontinuity and coercivity results at hand, we present the existence theorems: they generalize those of [11, Thms. 5.1 and 5.2]; in particular, the case  $p = 1$  is covered. First, we state the result for the functional  $I$ ; we will show three variants: for no boundary conditions, for Dirichlet (or mixed Dirichlet–Neumann) conditions and for Neumann conditions.

**Theorem 26.** *Let  $p \geq 1$  and let  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  satisfy conditions (B) as well as (NJ). The following hold:*

a) *Assume there exist  $c > 0$  and  $a \in L^1(\Omega \times \Omega)$  such that*

$$w(x, x', y, y') \geq c|y|^p + a(x, x').$$

*Then there exists a minimizer of  $I$  in  $L^p(\Omega, \mathbb{R}^d)$ .*

b) *Assume  $\Omega$  is a Lipschitz domain, fix  $\delta > 0$  and let  $\Omega_0$  be a non-empty open subset of  $\Omega$  satisfying  $\Omega_0 + B(0, \delta) \subset \Omega$ . Let  $\Omega_D$  be a measurable subset of  $\Omega \setminus \Omega_0$  with positive measure. Let  $u_0 \in L^p(\Omega, \mathbb{R}^d)$ . Assume that there exist  $c > 0$  and  $a \in L^1(\Omega \times \Omega)$  such that*

$$w(x, x', y, y') \geq c\chi_{B(0, \delta)}(x - x')|y - y'|^p + a(x, x'). \quad (19)$$

*Then there exists a minimizer of  $I$  in the set of  $u \in L^p(\Omega, \mathbb{R}^d)$  such that  $u = u_0$  in  $\Omega_D$ .*

c) *Assume  $\Omega$  is a Lipschitz domain, fix  $\delta > 0$  and let  $\Omega_0$  be a non-empty open subset of  $\Omega$  satisfying  $\Omega_0 + B(0, \delta) \subset \Omega$ . Assume that there exist  $c > 0$  and  $a \in L^1(\Omega \times \Omega)$  such that inequality (19) holds. If  $p = 1$ , assume, in addition, that  $I$  is invariant under translations. Then there exists a minimizer of  $I$  in the set of  $u \in L^p(\Omega, \mathbb{R}^d)$  such that  $\int_{\Omega} u = 0$ .*

*Proof.* We can assume that  $I$  is not identically infinity. Let  $\{u_j\}_{j \in \mathbb{N}}$  be a minimizing sequence of  $I$  in the corresponding set of admissible functions. Then  $\{u_j\}_{j \in \mathbb{N}}$  is bounded in  $L^p(\Omega, \mathbb{R}^d)$ : this is immediate under assumption a), it is a consequence of Proposition 22 under assumption b), and it is a consequence of Proposition 23 under assumption c). For a subsequence (not relabelled),  $\{u_j\}_{j \in \mathbb{N}}$  converges to some  $u \in L^p(\Omega, \mathbb{R}^d)$  weakly if  $p > 1$  and in the biting sense if  $p = 1$ . By Proposition 15,

$$I(u) \leq \liminf_{j \rightarrow \infty} I(u_j).$$

Under assumption b), it is easy to check that  $u = u_0$  in  $\Omega_D$ , in both cases  $p > 1$  and  $p = 1$ . Under assumption c), it is immediate that  $\int_{\Omega} u = 0$  when  $p > 1$ . Therefore,  $u$  is a minimizer of  $I$  in set of admissible functions in all cases, except perhaps in c) when  $p = 1$ , where the minimizer is  $u - \int_{\Omega} u$ , thanks to the translation-invariance of  $I$ . This concludes the proof. □

Now we study the existence of minimizers for  $\bar{I}$ . Analogously as before, we will use Propositions 24 and 25 to obtain coercivity for translation-invariant functionals, and Proposition 3 otherwise.

**Theorem 27.** *Let  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  satisfy conditions (B). The following hold:*

a) Assume there exist  $h : [0, \infty) \rightarrow [0, \infty)$  and  $a \in L^1(\Omega \times \Omega)$  such that  $\lim_{t \rightarrow \infty} h(t) = \infty$  and

$$w(x, x', y, y') \geq h(|y|) + a(x, x').$$

Then there exists a minimizer of  $\bar{I}$  in  $\mathcal{Y}(\Omega, \mathbb{R}^d)$ .

b) Let  $p \geq 1$ . Assume  $\Omega$  is a Lipschitz domain, fix  $\delta > 0$  and let  $\Omega_0$  be a non-empty open subset of  $\Omega$  satisfying  $\Omega_0 + B(0, \delta) \subset \Omega$ . Let  $\Omega_D$  be a measurable subset of  $\Omega \setminus \Omega_0$  with positive measure. Let  $u_0 \in L^p(\Omega, \mathbb{R}^d)$ . Assume that there exist  $c > 0$  and  $a \in L^1(\Omega \times \Omega)$  such that inequality (19) holds. Then there exists a minimizer of  $\bar{I}$  in the set of  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  such that  $\nu_x = \delta_{u_0(x)}$  for a.e.  $x \in \Omega_D$ .

c) Let  $p \geq 1$ . Assume  $\Omega$  is a Lipschitz domain, fix  $\delta > 0$  and let  $\Omega_0$  be a non-empty open subset of  $\Omega$  satisfying  $\Omega_0 + B(0, \delta) \subset \Omega$ . Assume that there exist  $c > 0$  and  $a \in L^1(\Omega \times \Omega)$  such that inequality (19) holds. If  $p = 1$ , assume, in addition, that  $\bar{I}$  is invariant under translations. Then there exists a minimizer of  $\bar{I}$  in the set of  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  whose first moment  $u$  satisfies  $\int_{\Omega} u = 0$ .

*Proof.* We can assume that  $\bar{I}$  is not identically infinity. Let  $\{\nu^j\}_{j \in \mathbb{N}}$  be a minimizing sequence. Thanks to Theorem 2 and Proposition 3, for a subsequence (not relabelled)  $\{\nu^j\}_{j \in \mathbb{N}}$  converges narrowly to some  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ : this is immediate under assumption a), it is a consequence of Proposition 24 under assumption b), and it is a consequence of Proposition 25 under assumption c). By Proposition 20,

$$\bar{I}(\nu) \leq \liminf_{j \rightarrow \infty} \bar{I}(\nu^j).$$

Moreover,  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  under assumptions b) or c), thanks to Proposition 8. Under assumption b), it is easy to check that  $\nu_x = \delta_{u_0(x)}$  for a.e.  $x \in \Omega_D$ . Under assumption c), let  $u$  be the first moment of  $\nu$ . If  $p > 1$  we have that  $\int_{\Omega} u = 0$  thanks to Lemma 13. Therefore,  $\nu$  is a minimizer of  $\bar{I}$  in the admissible set in all cases, except perhaps in c) with  $p = 1$ , where a minimizer is the Young measure  $\nu^a$  of Definition 21, where  $a := \int_{\Omega} u$ , thanks to the translation-invariance of  $\bar{I}$ . This concludes the proof.  $\square$

## 6. EQUIVALENT INTEGRANDS

In this section we study which integrands  $w$  give rise to a null  $I$ . Two integrands  $w_1$  and  $w_2$  are called *equivalent* if their corresponding functionals (5) coincide; in this case, the functional corresponding to  $w_1 - w_2$  is identically zero. Understanding the equivalent integrands is a prior step in order to derive properties of  $w$  from the properties of  $I$ . We point out that in the local case there are essentially no equivalent integrands to a given one apart from itself (see, e.g., [24, Props. 6.24 and 6.26]).

A study of equivalent integrands was done in [22, Prop. 17]. We present an alternative proof, with weaker assumptions, of that result.

**Theorem 28.** *Let  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfy (A0) and the following condition:*

(G) *for all  $M > 0$  there exists  $a_M \in L^1(\Omega \times \Omega)$  such that*

$$|w(x, x', y, y')| \leq a_M(x, x') \text{ for a.e. } x, x' \in \Omega \text{ and all } y, y' \in \mathbb{R}^d \text{ with } |y|, |y'| \leq M.$$

*Assume that  $I(u) = 0$  for all  $u \in L^\infty(\Omega, \mathbb{R}^d)$ . Then there exist an  $\mathcal{L}^n \otimes \mathcal{L}^n \otimes \mathcal{B}^d$ -measurable  $g : \Omega \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  and a symmetric  $h \in L^1(\Omega \times \Omega)$  such that*

$$w(x, x', y, y') = g(x, x', y) + g(x', x, y') + h(x, x') \text{ for a.e. } x, x' \in \Omega \text{ and all } y, y' \in \mathbb{R}^d,$$

$$\int_{\Omega} g(x, x', y) dx' = 0 \text{ for a.e. } x \in \Omega \text{ and all } y \in \mathbb{R}^d$$

and

$$\int_{\Omega} \int_{\Omega} h(x, x') dx' dx = 0.$$

Moreover, for all  $M > 0$  there exists  $b_M \in L^1(\Omega \times \Omega)$  such that

$$|g(x, x', y)| \leq b_M(x, x') \text{ for a.e. } x, x' \in \Omega \text{ and all } y \in \mathbb{R}^d \text{ with } |y| \leq M. \quad (20)$$

Note that condition (G) guarantees that the integral defining  $I(u)$  converges for all  $u \in L^\infty(\Omega, \mathbb{R}^d)$ , which is a prior hypothesis before assuming  $I(u) = 0$ .

*Proof of Theorem 28.* Fix  $y \in \mathbb{R}^d$ ,  $u \in L^\infty(\Omega, \mathbb{R}^d)$  and  $A \subset \Omega$  measurable. Take  $v = y\chi_A + u\chi_{A^c}$ . Using  $I(v) = 0$  and  $I(u) = 0$  we obtain

$$\begin{aligned} 0 &= \int_{\Omega} \int_{\Omega} w(x, x', v(x), v(x')) \, dx' \, dx \\ &= \int_A \int_A w(x, x', y, y) \, dx' \, dx + 2 \int_A \int_{A^c} w(x, x', y, u(x')) \, dx' \, dx + \int_{A^c} \int_{A^c} w(x, x', u(x), u(x')) \, dx' \, dx \\ &= \int_A \int_A w(x, x', y, y) \, dx' \, dx + 2 \int_A \int_{A^c} w(x, x', y, u(x')) \, dx' \, dx + \left[ -2 \int_A \int_{\Omega} + \int_A \int_A \right] w(x, x', u(x), u(x')) \, dx' \, dx. \end{aligned}$$

Now we take  $A$  to be a ball centred at  $x \in \Omega$ , divide by  $\mathcal{L}^n(A)$  and use Lemma 16 to conclude that

$$\int_{\Omega} w(x, x', y, u(x')) \, dx' = \int_{\Omega} w(x, x', u(x), u(x')) \, dx' \quad (21)$$

for a.e.  $x \in \Omega$ , all  $y \in \mathbb{R}^d$  and all  $u \in L^\infty(\Omega, \mathbb{R}^d)$ . Let  $x \in \Omega$  be such that (21) is satisfied, fix  $y_1, y_2 \in \mathbb{R}^d$  and  $B \subset \Omega$  measurable such that  $x \in B$ . Taking, successively,  $u = y_1$  and  $u = y_1\chi_B + y_2\chi_{B^c}$  in (21), we find that

$$\int_{\Omega} w(x, x', y, y_1) \, dx' = \int_{\Omega} w(x, x', y_1, y_1) \, dx' \quad (22)$$

and

$$\int_B w(x, x', y, y_1) \, dx' + \int_{B^c} w(x, x', y, y_2) \, dx' = \int_B w(x, x', y_1, y_1) \, dx' + \int_{B^c} w(x, x', y_1, y_2) \, dx'. \quad (23)$$

We subtract (22) from (23) and obtain

$$\int_{B^c} w(x, x', y, y_2) \, dx' - \int_{B^c} w(x, x', y, y_1) \, dx' = \int_{B^c} w(x, x', y_1, y_2) \, dx' - \int_{B^c} w(x, x', y_1, y_1) \, dx'.$$

As this is true for all measurable  $B \subset \Omega$  such that  $x \in B$ , we conclude that

$$w(x, x', y, y_2) - w(x, x', y, y_1) = w(x, x', y_1, y_2) - w(x, x', y_1, y_1) \quad (24)$$

for a.e.  $x, x' \in \Omega$  and all  $y, y_1, y_2 \in \mathbb{R}^d$ . We define  $g : \Omega \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$g(x, x', y) := w(x, x', y, 0) - w(x, x', 0, 0),$$

which satisfies (20) due to condition (G), just by taking  $b_M = 2a_M$ . Thanks to (24),

$$g(x, x', y) = w(x, x', y, y') - w(x, x', 0, y')$$

for a.e.  $x, x' \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ , while, by (22),

$$\int_{\Omega} g(x, x', y) \, dx' = \int_{\Omega} [w(x, x', y, 0) - w(x, x', 0, 0)] \, dx' = 0$$

for a.e.  $x \in \Omega$  and all  $y \in \mathbb{R}^d$ . Define, in addition,  $h : \Omega \times \Omega \rightarrow \mathbb{R}$

$$h(x, x') = w(x, x', 0, 0),$$

which is in  $L^1(\Omega \times \Omega)$  thanks to condition (G). Applying  $I(0) = 0$  we find that

$$\int_{\Omega} \int_{\Omega} h(x, x') \, dx' \, dx = \int_{\Omega} \int_{\Omega} w(x, x', 0, 0) \, dx' \, dx = 0.$$

Finally, for a.e.  $x, x' \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ ,

$$\begin{aligned} g(x, x', y) + g(x', x, y') + h(x, x') &= w(x, x', y, y') - w(x, x', 0, y') + w(x', x, y', 0) - w(x', x, 0, 0) + w(x', x, 0, 0) \\ &= w(x, x', y, y'), \end{aligned}$$

thanks to (A0), which concludes the proof.  $\square$

In many real problems (in particular, in Mechanics [36] and diffusion models [8]) the functional  $I$  is invariant under translations, and the dependence of  $w$  on  $(x, x', y, y')$  is through  $(x, x', y - y')$ . For this kind of integrands, the result on null functionals is as follows.

**Corollary 29.** *Let  $\tilde{w} : \Omega \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be  $\mathcal{L}^n \otimes \mathcal{L}^n \otimes \mathcal{B}^d$ -measurable and satisfy*

$$\tilde{w}(x, x', \tilde{y}) = \tilde{w}(x', x, -\tilde{y}) \quad \text{for a.e. } x, x' \in \Omega \text{ and all } \tilde{y} \in \mathbb{R}^d \quad (25)$$

as well as the following condition:

(G') For all  $M > 0$  there exists  $a_M \in L^1(\Omega \times \Omega)$  such that

$$|w(x, x', y)| \leq a_M(x, x') \text{ for a.e. } x, x' \in \Omega \text{ and all } y \in \mathbb{R}^d \text{ with } |y| \leq M.$$

Define  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  as

$$w(x, x', y, y') = \tilde{w}(x, x', y - y')$$

and assume that  $I(u) = 0$  for all  $u \in L^\infty(\Omega, \mathbb{R}^d)$ . Then there exist  $a \in L^1(\Omega \times \Omega, \mathbb{R}^d)$  and  $b \in L^1(\Omega \times \Omega)$  such that

$$a(x, x') = -a(x', x) \quad \text{and} \quad b(x, x') = b(x', x), \quad \text{a.e. } x, x' \in \Omega, \quad (26)$$

$$\int_{\Omega} a(x, x') \, dx' = 0, \quad \text{a.e. } x \in \Omega, \quad (27)$$

$$\int_{\Omega} \int_{\Omega} b(x, x') \, dx' \, dx = 0 \quad (28)$$

and

$$\tilde{w}(x, x', \tilde{y}) = a(x, x') \cdot \tilde{y} + b(x, x') \quad \text{for a.e. } x, x' \in \Omega \text{ and all } \tilde{y} \in \mathbb{R}^d. \quad (29)$$

*Proof.* It is easy to check that  $w$  satisfies the assumptions of Theorem 28. We repeat its proof until equality (24), which in this case reads as

$$\tilde{w}(x, x', y - y_2) - \tilde{w}(x, x', y - y_1) = \tilde{w}(x, x', y_1 - y_2) - \tilde{w}(x, x', 0),$$

for a.e.  $x, x' \in \Omega$  and all  $y, y_1, y_2 \in \mathbb{R}^d$ . Equivalently,

$$\tilde{w}(x, x', \tilde{y}_1) - \tilde{w}(x, x', \tilde{y}_2) = \tilde{w}(x, x', \tilde{y}_2 - \tilde{y}_1) - \tilde{w}(x, x', 0), \quad (30)$$

for a.e.  $x, x' \in \Omega$  and all  $\tilde{y}_1, \tilde{y}_2 \in \mathbb{R}^d$ . This tells us that for a.e.  $x, x' \in \Omega$ , the function  $\tilde{w}(x, x', \cdot)$  is affine. Hence there exist  $a : \Omega \times \Omega \rightarrow \mathbb{R}^d$  and  $b : \Omega \times \Omega \rightarrow \mathbb{R}^d$  such that equality (29) holds. Moreover,

$$a(x, x') = (\tilde{w}(x, x', e_1) - \tilde{w}(x, x', 0), \dots, \tilde{w}(x, x', e_d) - \tilde{w}(x, x', 0)) \quad \text{and} \quad b(x, x') = \tilde{w}(x, x', 0), \quad x, x' \in \Omega,$$

where  $e_i$  is the  $i$ -th vector of the canonical basis of  $\mathbb{R}^d$ ,  $i \in \{1, \dots, d\}$ . Thanks to (G'),  $a \in L^1(\Omega \times \Omega, \mathbb{R}^d)$  and  $b \in L^1(\Omega \times \Omega)$ . Imposing (25) to (29) yields (26), equality (22) leads to (27), while equality  $I(0) = 0$  implies (28).  $\square$

A further particular case of Theorem 28 is the following, in which  $w$  does not depend on  $(x, x')$ . Of course, such  $w$  cannot represent a realistic density function in peridynamics, but it may be useful in other problems.

**Corollary 30.** *Let  $w : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a Borel symmetric function that sends bounded sets into bounded sets. Assume that  $I(u) = 0$  for all  $u \in L^\infty(\Omega, \mathbb{R}^d)$ . Then  $w = 0$ .*

*Proof.* It is clear that  $w$  satisfies the assumptions of Theorem 28. We repeat its proof until equality (22), which, in this case, reads as  $w(y, y_1) = w(y_1, y_1)$  for all  $y, y_1 \in \mathbb{R}^d$ . Consequently,  $w$  does not depend on the first variable. As it is symmetric, it does not depend on the second variable either. Hence it is constant. As  $I(0) = 0$ , it must be  $w = 0$ .  $\square$

Once characterized those densities  $w$  that give rise to a null  $I$ , we show, by means of an example, that it is vain to try to characterize the densities that give rise to a positive  $I$ .

**Example 31.** *Let  $\Omega$  be a bounded open set of  $\mathbb{R}^n$ . Let  $p = 4$  and fix  $\alpha > 0$ . Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the polynomial*

$$f(t) := -2\alpha t^2 + t^4.$$

Let  $w : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $w(y, y') := f(y - y')$  and let  $I$  be its corresponding functional. Then  $I(u) \geq -|\Omega|^2 \alpha^2 / 2$  for all  $u \in L^p(\Omega)$ , with equality if and only if there exist a measurable subset  $A \subset \Omega$  with  $|A| = \frac{1}{2}|\Omega|$  and  $a \in \mathbb{R}$  such that

$$u = \frac{1}{2}\sqrt{\alpha}(\chi_A - \chi_{A^c}) + a.$$

*Proof.* We calculate a lower bound for the values of  $I$ . First we notice that  $I$  is invariant under translations, hence we can assume that  $\int_{\Omega} u = 0$  in the calculation of  $I(u)$ . We have

$$\int_{\Omega} \int_{\Omega} (u(x) - u(x'))^2 \, dx \, dx' = \int_{\Omega} \int_{\Omega} (u(x)^2 - 2u(x)u(x') + u(x')^2) \, dx \, dx' = 2 \|u\|_{L^2(\Omega)}^2 |\Omega|,$$

since  $\int_{\Omega} u = 0$ . Analogously,

$$\int_{\Omega} \int_{\Omega} (u(x) - u(x'))^4 \, dx \, dx' = \sum_{i=0}^4 (-1)^i \binom{4}{i} \int_{\Omega} \int_{\Omega} u(x)^i u(x')^{4-i} \, dx \, dx' = 2 \|u\|_{L^4(\Omega)}^4 |\Omega| + 6 \|u\|_{L^2(\Omega)}^4.$$

Hölder's inequality ensures that  $\|u\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{4}} \|u\|_{L^4(\Omega)}$ . Using this and the fact that the polynomial  $t \rightarrow -4\alpha t|\Omega| + 8t^2$  attains its minimum at  $t = \alpha|\Omega|/4$  and equals  $-|\Omega|^2\alpha^2/2$ , we obtain

$$I(u) = -4\alpha \|u\|_{L^2(\Omega)}^2 |\Omega| + 2 \|u\|_{L^4(\Omega)}^4 |\Omega| + 6 \|u\|_{L^2(\Omega)}^4 \geq -4\alpha \|u\|_{L^2(\Omega)}^2 |\Omega| + 8 \|u\|_{L^2(\Omega)}^4 \geq -\frac{\alpha^2}{2} |\Omega|^2.$$

Moreover, the equality  $I(u) = -|\Omega|^2\frac{\alpha^2}{2}$  holds if and only if

$$\|u\|_{L^2(\Omega)} = |\Omega|^{\frac{1}{4}} \|u\|_{L^4(\Omega)} \quad \text{and} \quad \|u\|_{L^2(\Omega)}^2 = \frac{\alpha|\Omega|}{4}.$$

Now, the first equality holds if and only if the functions  $|u|$  and 1 are parallel, i.e., there exists  $c \geq 0$  such that  $|u| = c$  a.e. Hence,  $u$  must be of the form  $u = c(\chi_A - \chi_{A^c})$  for some measurable set  $A \subset \Omega$ . The restriction  $\int_{\Omega} u = 0$  yields  $|A| = \frac{1}{2}|\Omega|$ . Finally, imposing  $\|u\|_{L^2(\Omega)}^2 = \frac{\alpha|\Omega|}{4}$  leads to  $c = \frac{1}{2}\sqrt{\alpha}$ .  $\square$

Consider now the function  $w_1 := \alpha^2/2 + w$ . Thanks to the previous example, its corresponding functional  $I_1$  satisfies  $I_1 \geq 0$ , but, on the other hand,  $w_1$  takes all values in  $[-\alpha^2/2, \infty)$ . This fact is in contrast with the local case (see, e.g., [24, Prop. 6.24]), where the positivity of the local functional implies the positivity of the integrand. Moreover, as a consequence of Corollary 30, there does not exist an equivalent integrand  $\tilde{w}_1$  of  $w_1$  such that  $\tilde{w}_1 \geq 0$ .

In view of Theorem 17 and the fact that  $f$  is not convex, the functional  $I$  is not lower semicontinuous in the weak topology of  $L^p(\Omega)$ . Nevertheless, it has minimizers, as computed explicitly in Example 31.

Example 31 will be recovered in Section 8 to show that the relaxation of  $I$  in the weak topology of  $L^p(\Omega)$  can be difficult to compute.

## 7. RELAXATION VIA YOUNG MEASURES

In this section, we calculate the relaxation of the functional  $I$  in the space of Young measures with the narrow topology. In general, the *relaxation* is the lower semicontinuous envelope in the chosen topology. We specify in the next paragraphs the precise definition in our case, and, in particular, the domain where the relaxation takes places.

On the one hand, condition (A) together with any of the coercivity inequalities of Theorem 26 imply at once that the quantity  $I(u)$  is well defined as a member of  $\mathbb{R} \cup \{\infty\}$  for any measurable  $u : \Omega \rightarrow \mathbb{R}^d$ , and  $I(u) < \infty$  if and only if  $u \in L^p(\Omega, \mathbb{R}^d)$ . In fact, given a subset  $\mathcal{A} \subset L^p(\Omega, \mathbb{R}^d)$ , we have that  $\mathcal{A}$  is bounded in  $L^p(\Omega, \mathbb{R}^d)$  if and only if  $\{I(u) : u \in \mathcal{A}\}$  is bounded. Similarly, if  $\bar{I}$  is as in (6), we have that the quantity  $\bar{I}(\nu)$  is well defined as a member of  $\mathbb{R} \cup \{\infty\}$  for any  $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ , and given a subset  $\mathcal{A} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$ , we have that

$$\sup_{\nu \in \mathcal{A}} \int_{\Omega \times \mathbb{R}^d} |y|^p \, d\nu(x, y) < \infty \quad \text{if and only if} \quad \sup_{\nu \in \mathcal{A}} \bar{I}(\nu) < \infty. \quad (31)$$

Having in mind now Proposition 11 and Lemma 12, we conclude that the natural domain for extending  $I$  in terms of Young measures is  $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$ .

Thus, considering the inclusion  $L^p(\Omega, \mathbb{R}^d) \subset \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  as explained in Section 3, we first extend  $I$  to a functional  $I_1$  defined in  $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$  by setting  $I_1$  to be  $\infty$  in  $\mathcal{Y}^p(\Omega, \mathbb{R}^d) \setminus L^p(\Omega, \mathbb{R}^d)$ . What we relax is this functional  $I_1$ . In this way, the lower semicontinuous envelope  $\bar{I}$  of  $I_1$  in the narrow topology of  $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$  is the greatest lower semicontinuous function in  $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$  that is below  $I$ , i.e., for each  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ ,

$$\bar{I}(\nu) := \sup \{J(\nu) : J : \mathcal{Y}^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\} \text{ is lower semicontinuous in the narrow topology and } J \leq I_1\}.$$

Because of the comments in the paragraph above, it suffices to consider bounded sets in  $L^p(\Omega, \mathbb{R}^d)$  and, in general, sets  $\mathcal{A} \subset \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  for which any of the two equivalent conditions of (31) hold. In virtue of Proposition 3 and Theorem 4, the topology in those sets  $\mathcal{A}$  is metrizable. In particular, for any  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ ,

$$\bar{I}(\nu) = \inf \left\{ \liminf_{j \rightarrow \infty} I_1(\nu^j) : \{\nu^j\}_{j \in \mathbb{N}} \subset \mathcal{Y}^p(\Omega, \mathbb{R}^d) \text{ and } \nu^j \xrightarrow{\text{nar}} \nu \text{ as } j \rightarrow \infty \right\}$$

(see, e.g., [9, Th. 11.1.1] or [24, Prop. 3.12]). Moreover, having in mind that  $I_1$  is the extension by infinity of  $I$ , we have that a functional  $\bar{I}$  is the relaxation of  $I_1$  if and only if:

(i) For any bounded sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  such that  $u_j \xrightarrow{\text{nar}} \nu$  as  $j \rightarrow \infty$  for some  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ , we have

$$\bar{I}(\nu) \leq \liminf_{j \rightarrow \infty} I(u_j).$$

(ii) For any  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  there exists a sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  such that  $u_j \xrightarrow{\text{nar}} \nu$  as  $j \rightarrow \infty$  and

$$\bar{I}(\nu) = \lim_{j \rightarrow \infty} I(u_j).$$

See, e.g., [9, Prop. 11.1.1] for this equivalence. Note that, even though the two conditions above are sometimes taken as a definition of relaxation, we needed all the preliminaries about metrizability on tight sets to conclude that assertion.

We first present the relaxation result without boundary conditions; a similar result was proved in [31, Th. 5.1].

**Theorem 32.** *Let  $p \geq 1$ . Let  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfy assumptions (A) and, in addition,*

a) *There exist a function  $\alpha \in L^1(\Omega \times \Omega)$  and a constant  $c > 0$  such that*

$$w(x, x', y, y') \geq c|y|^p + \alpha(x, x')$$

*for a.e.  $x, x' \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ .*

*Define  $I_1 : \mathcal{Y}^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}$  and  $\bar{I} : \mathcal{Y}^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R}$  as*

$$I_1(\nu) = \begin{cases} I(u) & \text{if } \nu = (\delta_{u(x)})_{x \in \Omega} \text{ for some } u \in L^p(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise,} \end{cases}$$

$$\bar{I}(\nu) := \int_{\Omega \times \mathbb{R}^d} \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') \, d\nu(x, y) \, d\nu(x', y').$$

*Then, the lower semicontinuous envelope of  $I_1$  with respect to the narrow topology is  $\bar{I}$ .*

*Proof.* By the discussion above, it suffices to prove (i) and (ii). Property (i) is a consequence of Proposition 20.

Let now  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . By Proposition 10, there exists a bounded sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. Due to (A), the sequence of functions

$$\Omega \times \Omega \ni (x, x') \mapsto w(x, x', u_j(x), u_j(x'))$$

is equiintegrable, hence, thanks to Proposition 6,

$$\bar{I}(\nu) = \lim_{j \rightarrow \infty} I_1(u_j)$$

with the identification of a function with its corresponding Young measure. □

Now we present the relaxation result for Dirichlet and mixed Dirichlet–Neumann conditions, as explained in Section 5. The conclusion is that the boundary conditions pass to the relaxation.

**Theorem 33.** *Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^n$ , fix  $\delta > 0$  and let  $p \geq 1$ . Let  $\Omega_0$  be a non-empty open subset of  $\Omega$  satisfying  $\Omega_0 + B(0, \delta) \subset \Omega$ . Let  $\Omega_D$  be a measurable subset of  $\Omega \setminus \Omega_0$  with positive measure. Let  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfy assumptions (A), as well as*

a') *There exist a function  $\alpha \in L^1(\Omega \times \Omega)$  and a constant  $c > 0$  such that*

$$w(x, x', y, y') \geq c\chi_{B(0, \delta)}(x - x')|y - y'|^p + \alpha(x, x'),$$

*for a.e.  $x, x' \in \Omega$  and all  $y, y' \in \mathbb{R}^d$ .*

Let  $u_0 \in L^p(\Omega_D, \mathbb{R}^d)$ . Let  $\mathcal{Y}_{u_0}^p(\Omega, \mathbb{R}^d)$  be the set of  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  such that  $\nu_x = \delta_{u_0(x)}$  for a.e.  $x \in \Omega_D$ . Define  $I_1, \bar{I} : \mathcal{Y}^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}$  as

$$I_1(\nu) = \begin{cases} I(u) & \text{if } \nu = (\delta_{u(x)})_{x \in \Omega} \text{ for some } u \in L^p(\Omega, \mathbb{R}^d) \text{ with } u|_{\Omega_D} = u_0 \text{ a.e.}, \\ \infty & \text{otherwise,} \end{cases}$$

$$\bar{I}(\nu) := \begin{cases} \int_{\Omega \times \mathbb{R}^d} \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') d\nu(x, y) d\nu(x', y') & \text{if } \nu \in \mathcal{Y}_{u_0}^p(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

Then, the lower semicontinuous envelope of  $I_1$  with respect to the narrow topology is  $\bar{I}$ .

*Proof.* Only two additional steps to those of the proof of Theorem 32 are needed.

Let  $\{u_j\}_{j \in \mathbb{N}}$  be a sequence in  $L^p(\Omega, \mathbb{R}^d)$  such that  $u_j|_{\Omega_D} = u_0$  and  $\sup_{j \in \mathbb{N}} I(u_j) < \infty$ . Thanks to *a'*) and Proposition 22,  $\{u_j\}_{j \in \mathbb{N}}$  is bounded in  $L^p(\Omega, \mathbb{R}^d)$ . Then,  $\{u_j\}_{j \in \mathbb{N}}$  generates a  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . It is immediate to see from the definition of narrow convergence (see, e.g., [9, Rk. 4.3.1]) that  $u_j|_{\Omega_D} \xrightarrow{\text{nar}} \nu|_{\Omega_D}$  in  $\mathcal{Y}^p(\Omega_D, \mathbb{R}^d)$ . Since  $u_j|_{\Omega_D}$  is identified with  $\{\delta_{u_0(x)}\}_{x \in \Omega_D}$ , we have that  $\nu_x = \delta_{u_0(x)}$  a.e.  $x \in \Omega_D$ .

Conversely, assume that  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  satisfies  $\nu_x = \delta_{u_0(x)}$  for a.e.  $x \in \Omega_D$ , and let  $\{u_j\}_{j \in \mathbb{N}}$  be any bounded sequence in  $L^p(\Omega; \mathbb{R}^d)$  generating  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. As before,  $\{u_j|_{\Omega_D}\}_{j \in \mathbb{N}}$  generates  $\nu|_{\Omega_D}$  so (see, e.g., [9, Prop. 4.3.8])  $u_j|_{\Omega_D} \rightarrow u_0$  in measure. Define  $v_j := u_j \chi_{\Omega \setminus \Omega_D} + u_0 \chi_{\Omega_D}$ . Then  $v_j - u_j \rightarrow 0$  in measure and, hence (see again [9, Prop. 4.3.8]),  $\{v_j\}_{j \in \mathbb{N}}$  generates  $\nu$ . Moreover,  $\{|v_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable as the sum of two equiintegrable sequences. This is enough to conclude that the proof of Theorem 32 can be adapted to this case.  $\square$

Finally, the relaxation result for Neumann conditions is as follows; in this case, we need  $p > 1$  for the restriction  $\int_{\Omega} u = 0$  to pass to the limit. We will deal in Theorem 35 with the case  $p = 1$ .

**Theorem 34.** *Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^n$ , fix  $\delta > 0$  and let  $p > 1$ . Let  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfy assumptions (A) and assumption *a'*) of Theorem 33. Let  $\mathcal{Y}_0^p(\Omega, \mathbb{R}^d)$  be the set of  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  whose first moment  $u$  satisfies  $\int_{\Omega} u = 0$ . Define  $I_1, \bar{I} : \mathcal{Y}^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}$  as*

$$I_1(\nu) = \begin{cases} I(u) & \text{if } \nu = (\delta_{u(x)})_{x \in \Omega} \text{ for some } u \in L^p(\Omega, \mathbb{R}^d) \text{ with } \int_{\Omega} u = 0, \\ \infty & \text{otherwise,} \end{cases}$$

$$\bar{I}(\nu) := \begin{cases} \int_{\Omega \times \mathbb{R}^d} \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') d\nu(x, y) d\nu(x', y') & \text{if } \nu \in \mathcal{Y}_0^p(\Omega, \mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

Then, the lower semicontinuous envelope of  $I_1$  with respect to the narrow topology is  $\bar{I}$ .

*Proof.* Only two additional steps to those of the proof of Theorem 32 are needed.

Let  $\{u_j\}_{j \in \mathbb{N}}$  be a sequence in  $L^p(\Omega, \mathbb{R}^d)$  such that  $\int_{\Omega} u_j = 0$  for all  $j \in \mathbb{N}$  and  $\sup_{j \in \mathbb{N}} I(u_j) < \infty$ . Thanks to assumption *a'*) of Theorem 33 and Proposition 23,  $\{u_j\}_{j \in \mathbb{N}}$  is bounded in  $L^p(\Omega, \mathbb{R}^d)$ . Then,  $\{u_j\}_{j \in \mathbb{N}}$  generates a  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . Let  $u$  be the first moment of  $\nu$ . By Lemma 13,  $\{u_j\}_{j \in \mathbb{N}}$  converges to  $u$  weakly in  $L^p(\Omega, \mathbb{R}^d)$ , so  $\int_{\Omega} u = 0$ .

Conversely, assume that  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  has a first moment  $u$  satisfying  $\int_{\Omega} u = 0$ , and let  $\{u_j\}_{j \in \mathbb{N}}$  be any bounded sequence in  $L^p(\Omega; \mathbb{R}^d)$  generating  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. Then  $u_j \rightharpoonup u$  in  $L^p(\Omega; \mathbb{R}^d)$ , so  $\int_{\Omega} u_j \rightarrow 0$  as  $j \rightarrow \infty$ . As in the proof of Proposition 25, define  $v_j := u_j - \int_{\Omega} u_j$  for each  $j \in \mathbb{N}$ , which satisfies  $\int_{\Omega} v_j = 0$ . Moreover,  $\{v_j\}_{j \in \mathbb{N}}$  generates  $\nu$  and  $\{|v_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable. This concludes the proof.  $\square$

As a consequence of a general abstract fact (see, e.g., [9, Th. 11.1.2]) and of the existence theorems of Section 5, we have that, in the context of any of Theorems 32, 33 or 34, we have the following facts:

- The functional  $\bar{I}$  has a minimizer, and its minimum coincides with the infimum of  $I$ .
- Minimizing sequences of  $I$  converge narrowly, up to a subsequence, to a minimizer of  $\bar{I}$ .
- Every minimizer of  $\bar{I}$  is a narrow limit of a minimizing sequence of  $I$ .

It only remains to deal with the case  $p = 1$  with Neumann boundary conditions. In this case, the restriction  $\int_{\Omega} u = 0$  does not pass to the relaxation, so we discard it. It was that restriction that allowed us to focus the attention on tight sets of Young measures. Without that restriction, we are still able to show that properties (i) and (ii) of the beginning of this section hold, and that minimizers of  $\bar{I}$  are precisely the limit of suitable *translated* minimizing sequences of  $I$ , so mimicking properties a)–c) above. This is, in fact, a kind of relaxation result, but this itself does not show that  $\bar{I}$  is the lower semicontinuous envelope of  $I$  because the narrow topology is not metrizable outside tight sets.

**Theorem 35.** *Let  $\Omega$  be a Lipschitz domain of  $\mathbb{R}^n$ , fix  $\delta > 0$  and let  $p = 1$ . Let  $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  satisfy assumptions (A) and assumption a') of Theorem 33. Define  $I_1 : \mathcal{Y}^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R} \cup \{\infty\}$  and  $\bar{I} : \mathcal{Y}^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R}$  as in Theorem 32. Then, properties (i) and (ii) hold. Moreover, under the assumption that  $I$  and  $\bar{I}$  are invariant under translations, the following hold:*

a) *Calling  $m := \inf_{u \in L^p(\Omega, \mathbb{R}^d)} I(u)$ , we have that  $m = \min_{\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)} \bar{I}(\nu)$ .*

b) *If  $\{u_j\}_{j \in \mathbb{N}}$  is a sequence in  $L^p(\Omega, \mathbb{R}^d)$  with  $I(u_j) \rightarrow m$  as  $j \rightarrow \infty$  then, for a subsequence (not relabelled), the sequence  $\{v_j\}_{j \in \mathbb{N}}$  defined as*

$$v_j := u_j - \int_{\Omega} u_j, \quad j \in \mathbb{N}$$

*converges narrowly to some  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ , and any such  $\nu$  satisfies  $\bar{I}(\nu) = m$ .*

c) *For every  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$  satisfying  $\bar{I}(\nu) = m$  there exists an equiintegrable sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^p(\Omega, \mathbb{R}^d)$  generating  $\nu$  such that  $I(u_j) \rightarrow m$  as  $j \rightarrow \infty$ .*

*Proof.* The proof of properties (i) and (ii) is identical to that of Theorem 32. In fact, the proof of (ii) shows property c) at once.

Let now  $\{u_j\}_{j \in \mathbb{N}}$  and  $\{v_j\}_{j \in \mathbb{N}}$  be as in b). By assumption a') of Theorem 33 and Proposition 23,  $\{v_j\}_{j \in \mathbb{N}}$  is bounded in  $L^p(\Omega, \mathbb{R}^d)$ , so by Proposition 10, after a subsequence, it converges narrowly to some  $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ . Let  $\nu$  be any such limit. By Proposition 20,  $\bar{I}(\nu) \leq m$ , while part c) allows us to conclude that  $\bar{I}(\nu) = m$  as well as part a), since, by Theorem 26,  $\bar{I}$  has minimizers.  $\square$

## 8. REMARKS ON THE RELAXATION IN $L^p$

In this section we make some comments about the difficulty of computing the relaxation of  $I$  in the weak topology of  $L^p(\Omega, \mathbb{R}^d)$ . An analogous reasoning to that of the beginning of Section 7 allows us to write three equivalent definitions of  $I^* : L^p(\Omega, \mathbb{R}^d) \rightarrow \mathbb{R}$ , the lower semicontinuous envelope of  $I$  the weak topology of  $L^p(\Omega, \mathbb{R}^d)$ , provided that condition (A) and any of the coercivity inequalities of Theorem 26 hold. To be precise, we choose  $\mathcal{A}$  as the closed subspace of  $L^p(\Omega, \mathbb{R}^d)$  codifying the boundary conditions, according to the cases a)–c) of Theorem 26; in other words,  $\mathcal{A} = L^p(\Omega, \mathbb{R}^d)$  in case a),  $\mathcal{A} = L^p_{u_0}(\Omega, \mathbb{R}^d)$  in case b) and  $\mathcal{A} = L^p_0(\Omega, \mathbb{R}^d)$  in case c), where

$$L^p_{u_0}(\Omega, \mathbb{R}^d) := \{u \in L^p(\Omega, \mathbb{R}^d) : u = u_0 \text{ a.e. in } \Omega_D\}, \quad L^p_0(\Omega, \mathbb{R}^d) := \left\{u \in L^p(\Omega, \mathbb{R}^d) : \int_{\Omega} u = 0\right\}.$$

Then, for any  $u \in \mathcal{A}$ ,

$$\begin{aligned} I^*(u) &:= \sup \{J(u) : J : \mathcal{A} \rightarrow \mathbb{R} \text{ is lower semicontinuous in the weak topology and } J \leq I\} \\ &= \inf \left\{ \liminf_{j \rightarrow \infty} I(u_j) : \{u_j\}_{j \in \mathbb{N}} \subset \mathcal{A} \text{ and } u_j \rightharpoonup u \text{ as } j \rightarrow \infty \right\}. \end{aligned}$$

Moreover,  $I^*$  is characterized by the following two facts:

(i) For any sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $\mathcal{A}$  such that  $u_j \rightharpoonup u$  in  $L^p(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$ , we have

$$I^*(u) \leq \liminf_{j \rightarrow \infty} I(u_j).$$

(ii) There exists a sequence  $\{u_j\}_{j \in \mathbb{N}}$  in  $\mathcal{A}$  such that  $u_j \rightharpoonup u$  in  $L^p(\Omega, \mathbb{R}^d)$  as  $j \rightarrow \infty$  and

$$I^*(u) = \lim_{j \rightarrow \infty} I(u_j).$$

Moreover, the abstract properties a)–c) of Section 7 also hold for  $I^*$  replacing  $\bar{I}$ , under any of the coercivity assumptions of Theorems 32, 33 or 34. In particular,

$$\min_{u \in \mathcal{A}} I^*(u) = \min_{\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)} \bar{I}(\nu) = \inf_{u \in L^p(\Omega, \mathbb{R}^d)} I(u). \quad (32)$$

We focus our attention on the simplest case of a  $w : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$  with no spatial dependence. Then, having in mind Corollary 30, conditions (NC) and (NY) trivially reduce to the separate convexity of  $w$ . Thanks to Proposition 15 and Theorem 17, so do condition (NJ) and the lower semicontinuity of  $I$  in  $L^p(\Omega, \mathbb{R}^d)$ , provided that the hypotheses therein hold, which we assume. Therefore,  $I^* = I$  if and only if  $w$  is separately convex. Let now  $I^{sc}$  be the functional associated to  $w^{sc}$ , the *separately convex hull* of  $w$  (i.e.,  $w^{sc}$  is the greatest separately convex function that is below  $w$ ). Then  $I^{sc} \leq I^*$  since  $I^{sc}$  is lower semicontinuous and  $I^{sc} \leq I$ .

We first mention that Pedregal [33, Sect. 3] showed an example of a function  $w : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  for which  $I^{sc} \neq I^*$ . We present here a simpler variant of his construction.

**Example 36.** Let  $p \geq 1$ . Take  $a_1 < a_2 < a_3 < a_4$  and consider the points

$$z_1 := (a_2, a_1), \quad z_2 := (a_4, a_1), \quad z_3 := (a_4, a_3), \quad z_4 := (a_3, a_4), \quad z_5 := (a_1, a_4), \quad z_6 := (a_1, a_2).$$

Let  $w : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$  be a continuous symmetric function such that  $w(z) = 0$  if and only if  $z \in \{z_1, \dots, z_6\}$ , and there exists  $c > 0$  for which

$$w(y, y') \geq c|y|^p - \frac{1}{c}, \quad \text{for all } y, y' \in \mathbb{R}.$$

Then  $I^{sc} \neq I^*$ .

*Proof.* We shall show that  $\min_{\nu \in \mathcal{Y}^p(\Omega)} \bar{I}(\nu) > 0$  and  $\min_{u \in L^p(\Omega)} I^{sc}(u) = 0$ : property (32) will conclude.

Suppose, for a contradiction, that  $\bar{I}(\nu) = 0$  for some  $\nu \in \mathcal{Y}^p(\Omega)$ . Then

$$\int_{\Omega \times \mathbb{R}} \int_{\Omega \times \mathbb{R}} w(y, y') \, d\nu(x, y) \, d\nu(x', y') = 0,$$

hence  $w(y, y') = 0$ , equivalently,  $(y, y') \in \{z_i\}_{i=1}^6$  for  $\nu$ -a.e.  $(x, y), (x', y') \in \Omega \times \mathbb{R}$ . In particular,  $\text{supp } \nu_x \subset \{a_1, a_2, a_3, a_4\}$  for a.e.  $x \in \Omega$ , so  $\nu_x = \sum_{i=1}^4 \lambda_i(x) \delta_{a_i}$  for some measurable  $\lambda_i : \Omega \rightarrow [0, 1]$  ( $i \in \{1, \dots, 4\}$ ) with  $\sum_{i=1}^4 \lambda_i(x) = 1$ , whence

$$\bar{I}(\nu) = \int_{\Omega} \int_{\Omega} \sum_{i,j=1}^4 \lambda_i(x) \lambda_j(x') w(a_i, a_j) \, dx \, dx' \geq \int_{\Omega} \int_{\Omega} \sum_{i=1}^4 \lambda_i(x) \lambda_i(x') w(a_i, a_i) \, dx \, dx' > 0,$$

a contradiction.

Now,  $w^{sc}$  satisfies  $w^{sc} \geq 0$  and  $w^{sc}$  vanishes in  $\{z_i\}_{i=1}^6$ . As  $w^{sc}$  is separately convex, it vanishes in the separately convex hull of  $\{z_i\}_{i=1}^6$ , which is easily seen to include  $[a_2, a_3] \times [a_2, a_3]$  (see Figure 1). Hence  $I^{sc}(u) = 0$  for any  $u \in L^\infty(\Omega)$  such that  $a_2 \leq u(x) \leq a_3$  for a.e.  $x \in \Omega$ . In particular,  $\min_{u \in L^p(\Omega)} I^{sc}(u) = 0$

□

A question of capital importance is whether  $I^*$  admits an integral representation, i.e., whether there exists an  $\mathcal{L}^n \otimes \mathcal{L}^n \otimes \mathcal{B}^d \otimes \mathcal{B}^d$ -measurable function  $w^* : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}$  such that

$$I^*(u) = \int_{\Omega} \int_{\Omega} w^*(x, x', u(x), u(x')) \, dx \, dx' \quad \text{for all } u \in L^p(\Omega, \mathbb{R}^d). \quad (33)$$

The usual theorems on integral representations (see, e.g., [24, Thms. 5.29 and 6.65]) cannot be applied since the functional  $I$  is not additive with respect to the set of integration  $\Omega$ . In [33] it is claimed that if  $I^*$  admits an integral representation then the corresponding integrand must be  $w^{sc}$ , but its proof is dubious, since it seems to use that if a continuous symmetric integrand  $\tilde{w} : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$  is such that its corresponding functional  $\tilde{I}$  satisfies  $\tilde{I} \geq 0$  then  $\tilde{w} \geq 0$ , which is not true in general, as we saw in Example 31. In fact, we are unable to answer the question of whether  $I^*$  admits an integral representation in Example 36.

We now present an example of a different nature, namely, the functional  $I$  of Example 31. Since  $I$  is given through an integrand  $f : \mathbb{R} \rightarrow \mathbb{R}$  depending on  $y - y'$ , it is natural to assume that its relaxation is given through an integrand

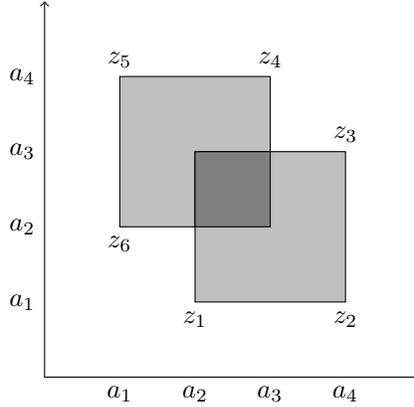


FIGURE 1. Corresponding to Example 36: in light grey, the separately convex hull of  $\{z_i\}_{i=1}^6$ ; in dark grey, the square  $[a_2, a_3] \times [a_2, a_3]$ .

$g : \mathbb{R} \rightarrow \mathbb{R}$  also depending on  $y - y'$ , which can be assumed to be even because of the symmetry. To fix ideas, we consider homogeneous Neumann boundary conditions, i.e., we set  $\mathcal{A} = L_0^p(\Omega, \mathbb{R}^d)$ .

**Example 37.** We consider  $f$ ,  $w$  and  $I$  as in Example 31. Let  $p = 4$ . Let  $I^*$  be the lower semicontinuous envelope of  $I$  in the space  $L_0^p(\Omega)$  with respect to the weak topology. Then any  $u \in L_0^p(\Omega)$  such that  $|u| \leq \sqrt{\alpha}/2$  is a minimizer of  $I^*$ . Moreover, if there exists a continuous even  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$I^*(u) = \int_{\Omega} \int_{\Omega} g(u(x) - u(x')) \, dx \, dx' \quad \text{for all } u \in L_0^p(\Omega)$$

then  $g$  is convex and  $g(t) = -\alpha^2/2$  for all  $t \in [-\sqrt{\alpha}, \sqrt{\alpha}]$ .

*Proof.* Thanks to Example 31, we have the bound

$$-|\Omega|^2 \frac{\alpha^2}{2} \leq I^*.$$

Let  $u \in L_0^p(\Omega)$  be such that  $|u| \leq \sqrt{\alpha}/2$ . Define  $\lambda \in L^\infty(\Omega)$  and  $\nu \in \mathcal{Y}(\Omega)$  as

$$\lambda(x) = \frac{1}{2} \left( 1 - \frac{2}{\sqrt{\alpha}} u(x) \right), \quad \nu_x = \lambda(x) \delta_{-\sqrt{\alpha}/2} + (1 - \lambda(x)) \delta_{\sqrt{\alpha}/2}, \quad \text{a.e. } x \in \Omega.$$

As  $u \in L_0^p(\Omega)$ , it is easy to check that  $\nu \in \mathcal{Y}_0^p(\Omega)$ . Moreover,

$$\begin{aligned} \bar{I}(\nu) &= \int_{\Omega \times \mathbb{R}} \int_{\Omega \times \mathbb{R}} f(y - y') \, d\nu(x, y) \, d\nu(x', y') \\ &= \int_{\Omega} \int_{\Omega} [\lambda(x) \lambda(x') f(0) + 2\lambda(x) (1 - \lambda(x')) f(\sqrt{\alpha}) + (1 - \lambda(x)) (1 - \lambda(x')) f(0)] \, dx \, dx' \\ &= 2f(\sqrt{\alpha}) \int_{\Omega} \lambda(x) \, dx \int_{\Omega} (1 - \lambda(x)) \, dx = -|\Omega|^2 \frac{\alpha^2}{2}. \end{aligned}$$

Let  $\{u_j\}_{j \in \mathbb{N}}$  be a bounded sequence in  $L_0^p(\Omega)$  generating  $\nu$  such that  $\{|u_j|^p\}_{j \in \mathbb{N}}$  is equiintegrable; this sequence was shown to exist in Proposition 25. Then

$$-|\Omega|^2 \frac{\alpha^2}{2} = \bar{I}(\nu) = \lim_{j \rightarrow \infty} I(u_j)$$

and  $u_j \rightharpoonup u$  in  $L^p(\Omega, \mathbb{R}^d)$ , since  $u$  is the first moment of  $\nu$  (recall Lemma 13), so

$$I^*(u) \leq \lim_{j \rightarrow \infty} I(u_j) = -|\Omega|^2 \frac{\alpha^2}{2}.$$

Hence,  $u$  is a minimizer of  $I^*$ .

Now, suppose that there exists a  $g$  as in the statement. Proposition 15 and Theorem 17 show that  $g$  must be convex. Now, given  $t \in \mathbb{R}$  and any measurable set  $A \subset \Omega$  with  $\mathcal{L}^n(A) = \frac{1}{2}\mathcal{L}^n(\Omega)$  we apply the inequalities  $-|\Omega|^2 \frac{\alpha^2}{2} \leq I^* \leq I$  to

$$u = \frac{t}{2}(\chi_A - \chi_{A^c}) \in L_0^p(\Omega)$$

and obtain

$$-|\Omega|^2 \frac{\alpha^2}{2} \leq \mathcal{L}^n(A)^2 g(0) + 2\mathcal{L}^n(A)\mathcal{L}^n(A^c)g(t) + \mathcal{L}^n(A^c)^2 g(0) \leq \mathcal{L}^n(A)^2 f(0) + 2\mathcal{L}^n(A)\mathcal{L}^n(A^c)f(t) + \mathcal{L}^n(A^c)^2 f(0),$$

which reduce to

$$-\alpha^2 \leq g(0) + g(t) \leq f(t).$$

As  $g$  is convex, we deduce that

$$-\alpha^2 \leq g(0) + g(t) \leq f^c(t),$$

where  $f^c : \mathbb{R} \rightarrow \mathbb{R}$  is the *convexification* of  $f$ . Since  $f$  attains its minimum at  $t = \pm\sqrt{\alpha}$  and equals  $-\alpha^2$ , we infer that  $f^c(t) = -\alpha^2$  for all  $t \in [-\sqrt{\alpha}, \sqrt{\alpha}]$ . Therefore,

$$-\alpha^2 \leq g(0) + g(t) \leq -\alpha^2 \quad \text{for all } t \in [-\sqrt{\alpha}, \sqrt{\alpha}].$$

which shows that  $g(t) = -\alpha^2/2$  for all  $t \in [-\sqrt{\alpha}, \sqrt{\alpha}]$ . □

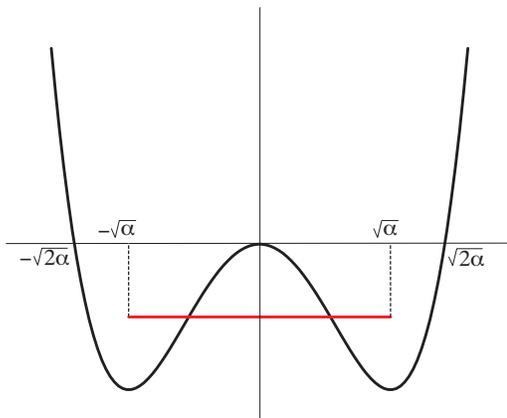


FIGURE 2. Functions  $f$  (black) and  $g$  (red) of Example 37.

This example shows the difficulty of computing  $I^*$ , since there is no natural candidate. In particular, it is still possible that  $I^*$  is given through the integration of a  $g$  as in the statement, despite the fact that  $g$  takes values both below and above  $f$  (see Figure 2).

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