QUASISTATIC EVOLUTION OF CAVITIES IN NONLINEAR ELASTICITY

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Abstract. We show the well-posedness of a free-discontinuity model in nonlinear elasticity allowing for cavitation. The model, based on global minimization, takes into account the non-interpenetration of matter and the inhomogeneties of the material. Then we prove the existence of a quasistatic evolution corresponding to this variational model that takes into account the irreversibility of the process of the cavity formation.

Key words. Cavitation, Quasiestatic evolution, Nonlinear elasticity

AMS subject classifications. 49J45, 74B20, 74C15, 74G65, 74H20, 74R99

1. Introduction. In elasticity theory, cavitation is the process of sudden formation of voids in a nonlinear elastic material, typically near-incompressible, that is subjected to a relatively large triaxial tension. It can be considered as a micro-crack, and plays an important role in the initiation of fracture in rubber and ductile metals through void growth and coalescence (see [61, 55, 34, 31, 60, 28, 54]). Cavitation, which has mainly been observed in elastomers and metals, also appears in biological tissues [32].

The engineering and experimental literature on cavitation is rich, starting with the paper by Gent & Lindley [29], who brought the phenomenom of cavitation to the attention of the scientific community. In contrast, rigourous mathematical analyses for cavitation are relatively few. It was Ball [4] who presented the first well-posed mathematical model for cavitation in nonlinear elasticity. His model assumed radial symmetry and reduced the study to a singular ordinary differential equation. In this way, he interpreted cavitation as a bifurcation phenomenon. Based on his work, many studies in radial cavitation appeared covering not only the isotropic case of [4] but also other possible symmetries of the material; we refer to the review paper [40].

The assumption of radial symmetry was removed in the free-discontinuity model of Müller & Spector [51] based on global minimization, where the location of the cavity points and the shape of the cavity were not prescribed, but rather selected as an energetic competition between bulk energy and energy due to cavitation. This new cavitation energy, which is essential in order for the well-posedness of the problem, was chosen to be proportional to the perimeter of the image, hence related to the new surface area created by the deformation.

An alternative well-posed model in cavitation was given by Sivaloganathan & Spector [57]: they prescribed the possible cavity points, and asigned either no energy due to cavitation, or an energy depending on the volume of the cavity formed.

More recently, Henao & Mora-Corral [36, 37] proposed a model allowing for both cavitation and fracture, again in the variational context of nonlinear elasticity. Using tools from geometric measure theory, they defined a concept of new surface created by the deformation, and showed that the corresponding energy due to cavitation or fracture was proportional to the area of the created surface. Moreover, using the work by Conti & De Lellis [14] to cover the critical exponent, they showed in [38]

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that, when restricted to Sobolev deformations (so that cavitation was permitted but not fracture), the model turned out to be essentially equivalent to that of [51].

We refer to the review papers [27, 40, 23] and the recent works [43, 62, 39] for further motivation and references.

In this paper we prove the well-posednees of a time-dependent mathematical model for the void creation and growth, which is a preliminary step in order to study the subsequent coalescence and eventual fracture.

Time-dependent problems in cavitation have been studied in the contexts of elasticity and plasticity, with or without viscous effects. In particular, radial symmetry was assumed in [53, 12, 2, 9] and other prescribed geometries in [13]. We believe, however, that this is the first rigourous mathematical analysis for the time-dependent problem of cavitation that does not prescribe the location or shape of the cavity.

Our starting point is the above-mentioned static model of [38]. In this context, deformations $\mathbf{u}$ are assumed to be in the Sobolev space $W^{1,n-1}(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ represents the reference configuration of the body, and n is the spatial dimension. They are also assumed to satisfy condition INV (see [51]) and be orientation preserving, i.e., $\det D\mathbf{u} > 0$ a.e.; these two conditions make up a version of the non-interpenetration of matter requirement that is specially suited for Sobolev deformations. In addition, an L^∞ a priori bound is also assumed.

The total energy in [38] was considered to be the sum of the elastic energy of the deformation $\mathbf{u}$ plus a term accounting for the process of cavity formation, which was called surface energy because it is proportional to the area of the surface created by the deformation. The choice of a cavitation energy as proportional to the area of the created surface is essential to make the problem well-posed (see [51] and the counterexample of Ball & Murat [7]). In addition, experimental, mechanical and thermodynamical considerations [61, 30, 13, 31, 41, 28, 11, 63, 8] show the consistency of this choice, which is reminiscent of both Griffith's [33] theory on fracture mechanics and the concept of surface tension in fluids.

We could also consider in the total energy that due to external body and surface forces, but since their treatment is standard, we prefer to ignore them in order to simplify the exposition.

As for the cavitation energy, the model of this paper makes two deviations from other surface energies defined in the literature (in particular [51, 36, 38]). While the proposed cavitation energy is still proportional to the area of the surface created, in this paper we allow for inhomogeneities of the material, so that the constant of proportionality may vary from point to point where the cavity appears.

More importantly, we add a new source of cavitation energy, which is the energy of initiation of a cavity. Cavitation is an irreversible process (see [61, 23, 21] and in particular the loading-unloading diagram of [29]). Indeed, once a cavity has been formed, the shape and size of the cavity can change in time and even disappear macroscopically, but the cavity point in the reference configuration will ever remain as a flaw point. The initiation energy that we propose is capital in order to model the irreversibility of the cavity formation, and presupposes a fixed amount of energy spent on the mere process of cavity formation. Said in energetic terms, the initiation energy is the only source of dissipation of energy, whereas all remaining energies involved (elastic, surface of cavitation) are conservative. We have not found in the literature an analogue of this initiation energy, but it is compatible with irreversibility, dissipativity and the fact that after the first loading producing the cavity and a subsequent unloading, the second loading requires less energy to achieve the same strain (see the diagrams in [29, 21]).

Thus, given a family of deformations $\{\mathbf{u}(t)\}_{t\in[0,1]}$ parametrized by the time t, we will define the set $S(t)$ as the union of the cavity points of $\mathbf{u}(s)$ when s runs over $[0,t]$. The energy of initiation of cavities will be expressed as an additive function $\mathcal{S}_0$ depending on $S(t)$, so that the function $\mathcal{S}_0(S(\cdot))$ is increasing in time. As a matter of fact, our model allows only finitely many cavities, so that the function $\mathcal{S}_0(S(\cdot))$ takes only finitely many values for a given family $\{\mathbf{u}(t)\}_{t\in[0,1]}$. This is still a free-discontinuity problem, since the location of the cavities is not prescribed, neither is their shape. The conservative part of the energy at time t will only depend on $\mathbf{u}(t)$ and on the boundary conditions at time t, but not on their history.

In this paper we prove the well-posedness of the quasistatic evolution problem associated to the above-mentioned static variational model. Our work is inspired by the quasistatic analyses for crack growth of [26, 16, 18, 42], which were themselves adaptations (or, sometimes, earlier particular cases) of the general scheme of rate-independent processes developed by Mielke and coworkers [49, 45, 48, 46, 47, 44, 25] to the case of fracture mechanics. As in those papers, our approach to quasistatic evolution is based on three premises: global minimization, irreversibility and energy balance. Global minimization states that for each time t, the deformation $\mathbf{u}(t)$ is a minimizer of the total energy among all admissible deformations. Irreversibility is expressed in the fact that the set S of cavity or flaw points is increasing in time. Finally, energy balance states that the total energy of the state $(\mathbf{u}(t), S(t))$ is an absolutely continuous function with respect to time, whose a.e. derivative can be computed through $\mathbf{u}(t)$ and the boundary conditions at time t. In fact, that derivative can be given a meaning in terms of the work done by the surface forces (see [44, 16]), which in our setting that work presents a peculiarity involving also the new surface created by cavitation and its mean curvature.

Apart from being a useful mathematical tool, quasistatic evolution is compatible with the experiments in [15], in which the creation of a cavity is almost instantaneous (less than 0.04 seconds), while the subsequent growth of the cavity is relatively slow. Indeed, quasistatic evolutions, much like in the context of fracture mechanics, can model both instantaneous creation of cavities and time-continuous growth.

The general scheme of the proof is standard and relies on solving a sequence of time-discrete problems and passing to the limit as the partition of the time interval becomes finer and finer. In particular, the general method and techniques of the proof are borrowed from Dal Maso, Francfort & Toader [16]. Since, in addition, we deal with deformations that are one-to-one a.e. and orientation preserving, the techniques of Dal Maso & Lazzaroni [18] will also be needed. As in [16, 17], it is convenient to pass to the limit independently in the deformation $\mathbf{u}(t)$ and in the set $S(t)$, and later prove that $S(t)$ equals the union of the cavity points of $\mathbf{u}(s)$ for $s \in [0,t]$.

The outline of the paper is as follows. In Section 2 we present the general notation, and recall the results of the static model for cavitation of [38] that will be used throughout the paper. Sections 3–4 study the static model. Precisely, Section 3 explains how to modify the model of [38] in order to allow for inhomogeneties in the material, and proves the corresponding lower semicontinuity result for the new functional. Section 4 adapts the static model of Section 3 so that irreversibility can be considered, via the definition of an initiation energy for cavitation. We also prove the corresponding lower semicontinuity result.

The quasistatic model is analyzed in Sections 5–13. Section 5 lists the assumptions on the stored-energy function, the cavitation energy and the boundary condition. It also states the main result of the paper (Theorem 5.2), which is the existence of a quasistatic evolution $\{\mathbf{u}(t)\}_{t\in[0,1]}$ starting at a given initial datum. Section 6 explains the reformulation of the problem to one with time-independent boundary conditions via composition with a diffeomorphism: $\mathbf{u}(t) = \boldsymbol{\psi}(t) \circ \mathbf{v}(t)$. The function $\boldsymbol{\psi}(t)$ is the boundary data, and for the rest of the paper the new deformation unknown will be $\mathbf{v}(t)$ instead of $\mathbf{u}(t)$. The energy functionals are redefined accordingly, and we show the properties that the new energy functionals inherit from the old ones. Section 7 defines the notions of convergence for sets and for functions that will be used in the subsequent analysis. We also prove the corresponding compactness and lower semicontinuity results, which will be used in Section 8 to show the existence of minimizers for a given time. Section 9 presents the time discretization, while in Section 10 we pass to the limit in the time-discrete minimizers as the partition becomes finer and finer, and thus obtain a family of deformations $\{\mathbf{v}(t)\}_{t\in[0,1]}$. While the limit passage is standard, the key property for showing afterwards the stability of minimizers is that no cavities are lost in this limit passage. Even though one can easily construct a sequence of deformations whose limit has fewer cavity points than the sequence itself, we show that this cannot be the case for a sequence of time-discrete minimizers. This is, in fact, a qualitative result on the time-discrete minimizers that is inside the proof of Proposition 10.3. This property allows us to prove easily in Section 11 the so-called 'transfer property' (see [26]) of Proposition 11.1, with which the stability of minimizers follows. This stability property (Proposition 11.2) states that $\mathbf{v}(t)$ is a minimizer of the energy at time t. Section 12 establishes the energy balance, following the standard method (see [16, 18]). The proof of Theorem 5.2, done in Section 13, easily follows and concludes the paper.

2. Notation and preliminary results. In this section we explain the general notation and preliminary results to be used throughout the paper. Special attention will be paid to the results on cavitation that follow from [36, 37, 38].

2.1. General notation. We will work in dimension $n \geq 2$. Our basic object will be the deformation, which is a Sobolev map $\mathbf{u} : \Omega \to \mathbb{R}^n$ satisfying certain conditions. Throughout the paper, Ω is a bounded open set of $\mathbb{R}^n$ with a Lipschitz boundary representing the reference configuration of the body. Vector-valued and matrix-valued quantities will be written in boldface. Coordinates in the reference configuration will generically be denoted by $\mathbf{x}$, while coordinates in the deformed configuration by $\mathbf{y}$.

The closure of a set A is denoted by $\bar{A}$, and its boundary by ∂A. Given two open sets U, V of $\mathbb{R}^n$, we will write $U \subset\subset V$ if U is bounded and $\bar{U} \subset V$. The open ball of radius $r > 0$ centred at $\mathbf{x} \in \mathbb{R}^n$ is denoted by $B(\mathbf{x}, r)$.

The identity matrix in $\mathbb{R}^{n\times n}$ is denoted by $\mathbf{1}$. Given a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, its transpose is denoted by $\mathbf{A}^T$, and its determinant by $\det \mathbf{A}$. The cofactor matrix of $\mathbf{A}$, denoted by $\operatorname{cof} \mathbf{A}$, is the matrix that satisfies $(\det \mathbf{A})\mathbf{1} = \mathbf{A}^T \operatorname{cof} \mathbf{A}$. The transpose of $\operatorname{cof} \mathbf{A}$ is $\operatorname{adj} \mathbf{A}$. The inner (dot) product of vectors and of matrices will be denoted by $\cdot$. The associated norm is denoted by $|\cdot|$. The set $GL_+(n)$ is the set of $\mathbf{A} \in \mathbb{R}^{n\times n}$ such that $\det \mathbf{A} > 0$, while $SO(n)$ denotes the set of $\mathbf{R} \in GL_+(n)$ such that $\mathbf{R}\mathbf{R}^T = \mathbf{1}$. The set $\mathbb{S}^{n-1}$ denotes the set of unit vectors of $\mathbb{R}^n$.

The abreviation *a.e.* for *almost everywhere* or *almost every* will be widely used. Unless otherwise stated, expressions like *a.e.*, *measurable* or *negligible* refer to the Lebesgue measure in $\mathbb{R}^n$, which is denoted by $\mathcal{L}^n$. The $(n-1)$-dimensional Hausdorff measure will be indicated by $\mathcal{H}^{n-1}$, while the counting measure is represented by $\mathcal{H}^0$. For a given set A and a measurable subset B of $\mathbb{R}^m$, we will use expressions like "for all $a \in A$ and a.e. $b \in B$, property (P) holds". This means that for each $a \in A$

there exists an $\mathcal{L}^m$-negligible set N_a such that property (P) holds for all $b \in B \setminus N_a$. Likewise, the expression "for a.e. $b \in B$ and all $a \in A$, property (P) holds" means that there exists an $\mathcal{L}^m$-negligible set N such that for all $b \in B \setminus N$ and all $a \in A$, property (P) holds.

The Lebesgue L^p and Sobolev $W^{1,p}$ spaces are defined in the usual way. So are the set of functions of class C^k for some integer k or infinity. The set $C^k_c(\Omega, \mathbb{R}^n)$ denotes the space of C^k functions with compact support in Ω. We will always indicate the domain and target space, as in, for example, $L^p(\Omega, \mathbb{R}^n)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^p(\Omega)$. If $K \subset \mathbb{R}^n$ is compact then $W^{1,p}(\Omega, K)$ indicates the set of $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^n)$ such that $\mathbf{u}(\mathbf{x}) \in K$ for a.e. $\mathbf{x} \in \Omega$, and analogously with other function spaces. Sometimes we will use Lebesgue spaces in $(n-1)$-dimensional sets; for example, if Ω is a set with a Lipschitz boundary, then $L^1(\partial\Omega)$ denotes the Lebesgue L^1 space on $\partial\Omega$ with respect to the $\mathcal{H}^{n-1}$ measure. The distributional derivative of a Sobolev function $\mathbf{u}$ is denoted by $D\mathbf{u}$. The space of finite Radon measures in Ω is called $\mathcal{M}(\Omega)$.

With $\rightharpoonup$ we indicate weak convergente in the relevant space, and with $\to$, strong (norm) convergence or a.e. convergence. We also use a.e. convergence of sets, meaning that their characteristic functions converge a.e. Finally, we will use weak* convergence in $\mathcal{M}(\Omega)$, for which we use the symbol $\overset{*}{\rightharpoonup}$.

With $\langle \cdot, \cdot \rangle$ we will indicate the duality product between a measure and a continuous function, or between a distribution and a smooth function of compact support. If μ is a measure, $|\mu|$ denotes its total variation measure. For each $\mathbf{x} \in \Omega$, the measure $\delta_{\mathbf{x}}$ indicates the Dirac delta at $\mathbf{x}$.

The identity function in $\mathbb{R}^n$ is denoted by $\mathbf{id}$. By a *modulus of continuity* we mean an increasing function $\omega : [0, 1] \to [0, \infty)$ such that $\omega(h) \to 0$ as $h \searrow 0$.

Given two sets A, B of $\mathbb{R}^n$, we write $A = B$ a.e. if $\mathcal{L}^n(A \setminus B) = \mathcal{L}^n(B \setminus A) = 0$, and analogously when we write that $A = B$ holds $\mathcal{H}^{n-1}$-a.e.

Given a measurable set $A \subset \mathbb{R}^n$, its characteristic function will be denoted by χ_A. The perimeter of A is defined as

$$\operatorname{Per} A := \sup \left\{ \int_A \operatorname{div} \mathbf{g}(\mathbf{y}) \, \mathrm{d}\mathbf{y} : \ \mathbf{g} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \ \|\mathbf{g}\|_\infty \leq 1 \right\},$$

where div denotes the divergence operator (in the deformed configuration). If $\operatorname{Per} A < \infty$ then $\operatorname{Per} A = \mathcal{H}^{n-1}(\partial^* A)$, where $\partial^* A$ denotes the measure-theoretic boundary of A (see, e.g., [1, Sect. 3.5] for the definitions and proofs).

2.2. Degree, topological image and condition INV in $W^{1,n-1} \cap L^\infty$. We quickly review the definition and properties of the degree in $W^{1,n-1} \cap L^\infty$, which in fact is a particular case of the Brezis & Nirenberg [10] degree. We refer to Conti & De Lellis [14, Sect. 3] for the proof (see also [51, 38]).

Proposition 2.1. *Let $K \subset \mathbb{R}^n$ be a compact set, and let $\mathbf{u} \in W^{1,n-1}(\Omega, K)$. Let $U \subset\subset \Omega$ be a nonempty open set with a C^2 boundary, and suppose that $\mathbf{u} \in W^{1,n-1}(\partial U, K)$. Then there exists a unique $L^1(\mathbb{R}^n)$ function, denoted by* $\deg(\mathbf{u}, \partial U, \cdot)$, *such that*

$$\int_{\partial U} \mathbf{g}(\mathbf{u}(\mathbf{x})) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x}) \, \boldsymbol{\nu}(\mathbf{x})) \, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{x}) = \int_{\mathbb{R}^n} \deg(\mathbf{u}, \partial U, \mathbf{y}) \operatorname{div} \mathbf{g}(\mathbf{y}) \, \mathrm{d}\mathbf{y}$$

for all $\mathbf{g} \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, where $\boldsymbol{\nu}$ denotes the unit exterior normal to U. Moreover, it satisfies the following properties:

i) $\deg(\mathbf{u},\partial U,\cdot)\in BV(\mathbb{R}^n,\mathbb{Z})$.
ii) $\deg(\mathbf{u},\partial U,\mathbf{y})=0$ *for a.e.* $\mathbf{y}\in\mathbb{R}^n\setminus K$.
iii) If $\mathbf{u}$ *is continuous then* $\deg(\mathbf{u},\partial U,\cdot)$ *coincides with the Brouwer degree a.e. in* $\mathbb{R}^n\setminus\mathbf{u}(\partial U)$.

The concept of topological image was introduced by Šverák [59] (see also [51]); we follow here the extension made in [14, Def. 3.5].

Definition 2.2. *Let* $U\subset\subset\mathbb{R}^n$ *be a nonempty open set with a* C^2 *boundary, and suppose that* $\mathbf{u}\in W^{1,n-1}(\partial U,\mathbb{R}^n)\cap L^\infty(\partial U,\mathbb{R}^n)$. *We define the topological image* $\operatorname{im_T}(\mathbf{u},U)$ *of* U *under* $\mathbf{u}$ *as the set of* $\mathbf{y}\in\mathbb{R}^n$ *such that*

$$\lim_{r\searrow 0}\frac{\mathcal{L}^n(B(\mathbf{y},r)\cap A_{\mathbf{u},U})}{\mathcal{L}^n(B(\mathbf{y},r))}=1,$$

where $A_{\mathbf{u},U}:=\{\mathbf{y}\in\mathbb{R}^n:\deg(\mathbf{u},\partial U,\mathbf{y})\neq 0\}$.

A geometric interpretation of this definition is as follows (see [51]). Suppose that $\mathbf{u}\in W^{1,p}(\Omega,\mathbb{R}^n)$ with $p>n-1$, and fix $\mathbf{x}\in\Omega$. Then, for a.e. $r\in(0,\operatorname{dist}(\mathbf{x},\partial\Omega))$, the restriction $\mathbf{u}|_{\partial B(\mathbf{x},r)}$ is in $W^{1,p}(\partial B(\mathbf{x},r),\mathbb{R}^n)$, and, by the Sobolev embedding because $p>n-1$, it is continuous on $\partial B(\mathbf{x},r)$, provided the precise representative of $\mathbf{u}$ was chosen. Hence $\mathbf{u}|_{\partial B(\mathbf{x},r)}$ has a well-defined Brouwer degree. The set

$$\{\mathbf{y}\in\mathbb{R}^n\setminus\mathbf{u}(\partial B(\mathbf{x},r)):\deg(\mathbf{u},\partial B(\mathbf{x},r),\mathbf{y})\neq 0\}$$

is then the topological image of $\mathbf{u}$ and of any continuous extension $\tilde{\mathbf{u}}$ of $\mathbf{u}|_{\partial B(\mathbf{x},r)}$ to $\bar{B}(\mathbf{x},r)$. The natural generalization for $W^{1,n-1}\cap L^\infty$ maps is then given in Definition 2.2.

Condition INV (see [14, Def. 3.6]) is defined as follows.

Definition 2.3. *Let* $\mathbf{u}\in W^{1,n-1}(\Omega,\mathbb{R}^n)\cap L^\infty(\Omega,\mathbb{R}^n)$. *We say that* $\mathbf{u}$ *satisfies condition INV if for every* $\mathbf{x}_0\in\Omega$ *and a.e.* $r\in(0,\operatorname{dist}(\mathbf{x}_0,\partial\Omega))$, *the following conditions hold:*
i) $\mathbf{u}(\mathbf{x})\in\operatorname{im_T}(\mathbf{u},B(\mathbf{x}_0,r))$ *for a.e.* $\mathbf{x}\in B(\mathbf{x}_0,r)$.
ii) $\mathbf{u}(\mathbf{x})\notin\operatorname{im_T}(\mathbf{u},B(\mathbf{x}_0,r))$ *for a.e.* $\mathbf{x}\in\Omega\setminus B(\mathbf{x}_0,r)$.

Intuitively (see [51]), a map satisfying condition INV and creating a hole cannot send matter from outside the hole to inside, or vice versa. In fact, maps satisfying condition INV enjoy a higher degree of regularity than the typical Sobolev function.

If $\mathbf{u}\in W^{1,n-1}(\Omega,\mathbb{R}^n)\cap L^\infty(\Omega,\mathbb{R}^n)$ satisfies $\det D\mathbf{u}>0$ a.e., and $\mathbf{x}\in\Omega$, the topological image $\operatorname{im_T}(\mathbf{u},\mathbf{x})$ of $\mathbf{x}$ under $\mathbf{u}$ is defined as the intersection of $\operatorname{im_T}(\mathbf{u},B(\mathbf{x},r))$ for r in a certain set $R_{\mathbf{x}}$ of full measure in $(0,\operatorname{dist}(\mathbf{x},\partial\Omega))$ (see [38] or else [59, 51, 14]). Since that intersection is decreasing as r decreases, the precise form of $R_{\mathbf{x}}$ is not relevant in the applications. It was shown in [51] (see also [14, 38] as well as Lemma 2.7 below) that $\operatorname{im_T}(\mathbf{u},\mathbf{x})$ is the 'hole' (cavity volume) in the deformed configuration created by the cavity point $\mathbf{x}$.

The concept of *geometric image* (see [51] for the original definition, and [14, 38] for slight adaptations) is also central in the study of cavitation. Given $\mathbf{u}\in W^{1,n-1}(\Omega,\mathbb{R}^n)$ and a measurable set A of Ω, the geometric image $\operatorname{im_G}(\mathbf{u},A)$ of A under $\mathbf{u}$ is defined as $\mathbf{u}(A\cap\Omega_d)$, where Ω_d is the set of approximate differentiability points of $\mathbf{u}$. This definition is based of Federer's [22] area formula, according to which $\mathbf{u}|_{\Omega_d}$ sends sets of measure zero to sets of measure zero.

2.3. Distributional determinant and cavity points. We present the definition of distributional determinant (see [3] or [50]).

Definition 2.4. *Let* $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$. *The distribution* $\operatorname{Det} D\mathbf{u}$ *in* Ω *is defined as*

$$\langle \operatorname{Det} D\mathbf{u}, \phi \rangle := -\frac{1}{n} \int_\Omega \mathbf{u}(\mathbf{x}) \cdot (\operatorname{cof} D\mathbf{u}(\mathbf{x})\, D\phi(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \qquad \phi \in C_c^\infty(\Omega),$$

and called the distributional determinant of $\mathbf{u}$. *If*

$$\sup \{ \langle \operatorname{Det} D\mathbf{u}, \phi \rangle : \ \phi \in C_c^\infty(\Omega), \ \|\phi\|_\infty \le 1 \} < \infty$$

then $\operatorname{Det} D\mathbf{u}$ *can be extended uniquely to a finite Radon measure in* Ω, *and this extension will also be called* $\operatorname{Det} D\mathbf{u}$. *If this is the case, we define* $C(\mathbf{u})$ *as the set of* $\mathbf{x} \in \Omega$ *such that* $\operatorname{Det} D\mathbf{u}(\{\mathbf{x}\}) \neq 0$.

It is immediate to check that if $\operatorname{Det} D\mathbf{u}$ is a measure, then the set $C(\mathbf{u})$ is countable (that is, it is either countably infinite, finite or empty). While classic existence theorems in nonlinear elasticity (see [3, 50, 52]) are based on the equality $\operatorname{Det} D\mathbf{u} = \det D\mathbf{u}\, \mathcal{L}^n$, this equality is no longer true when cavitation occurs, and in fact $\operatorname{Det} D\mathbf{u}$ detects the cavity points (see Lemma 2.7 below).

2.4. Surface energy for cavitation. In [36] a surface energy $\mathcal{E}(\mathbf{u})$ was introduced for maps $\mathbf{u}$ of bounded variation, so in particular they could exhibit both cavitation and fracture. The definition is as follows. For each $\mathbf{f} \in C_c^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, set

$$\mathcal{E}(\mathbf{u}, \mathbf{f}) := \int_\Omega [D\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \cdot \operatorname{cof} D\mathbf{u}(\mathbf{x}) + \operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))\, \det D\mathbf{u}(\mathbf{x})]\, \mathrm{d}\mathbf{x}. \tag{2.1}$$

Here, $D\mathbf{f}(\mathbf{x}, \mathbf{y})$ denotes the derivative of $\mathbf{f}(\cdot, \mathbf{y})$ at $\mathbf{x} \in \Omega$, while $\operatorname{div} \mathbf{f}(\mathbf{x}, \mathbf{y})$ is the divergence of $\mathbf{f}(\mathbf{x}, \cdot)$ at $\mathbf{y} \in \mathbb{R}^n$. The surface energy is then defined as

$$\mathcal{E}(\mathbf{u}) := \sup \left\{ \mathcal{E}(\mathbf{u}, \mathbf{f}) : \ \mathbf{f} \in C_c^1(\Omega \times \mathbb{R}^n, \mathbb{R}^n), \ \|\mathbf{f}\|_\infty \le 1 \right\}. \tag{2.2}$$

The full proof that $\mathcal{E}(\mathbf{u})$ measures the area of the surface created by the deformation $\mathbf{u}$ came in [37]. When restricted to $W^{1,n-1} \cap L^\infty$ maps satisfying $\mathcal{E}(\mathbf{u}) < \infty$, condition INV and $\det D\mathbf{u} > 0$ a.e. (as in this paper), it was shown in [38, Th. 4.6] that the energy $\mathcal{E}(\mathbf{u})$ has a clear geometrical and physical interpretation:

$$\mathcal{E}(\mathbf{u}) = \sum_{\mathbf{a} \in C(\mathbf{u})} \operatorname{Per} \operatorname{im_T}(\mathbf{u}, \mathbf{a}). \tag{2.3}$$

In other words, $\mathcal{E}(\mathbf{u})$ is a true surface energy and measures the sum of the perimeters of each hole (cavity volume) produced in the deformation.

The functional setting for cavitation is the Sobolev space $W^{1,p}$ with $n-1 < p < n$ (see [51, 57]) or, for the critical exponent, $W^{1,n-1} \cap L^\infty$ (see [14, 38]); in addition, deformations are required to satisfy condition INV and $\det D\mathbf{u} > 0$ a.e. It was proved in [38, Th. 4.8] that the only process of creation of new surface for these maps is precisely the cavitation. In this paper we have chosen the setting $W^{1,n-1} \cap L^\infty$, although the results are also true in the case $W^{1,p}$ with the same (or sometimes easier) proofs.

2.5. Some results on cavitation. The purpose of this subsection is to gather some results in the mathematical theory of cavitation that will be used throughout the paper. They are taken from [51, 14, 36, 38] or are small adaptations of results therein. For the sake of completeness, we provide brief proofs when a specific reference is not available.

LEMMA 2.5. *Let $K \subset \mathbb{R}^n$ be compact. Let $\mathbf{u} \in W^{1,n-1}(\Omega, K)$ satisfy INV and $\det D\mathbf{u} > 0$ a.e. Then the following properties hold:*

i) The restriction of $\mathbf{u}$ to a set of full measure in Ω is injective.

ii) $\det D\mathbf{u} \in L^1(\Omega)$.

iii) For all $\mathbf{x} \in \Omega$ and a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$, the set $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, B(\mathbf{x}, r))$ has finite perimeter and

$$\deg(\mathbf{u}, \partial B(\mathbf{x}, r), \cdot) = \chi_{\operatorname{im}_{\mathrm{T}}(\mathbf{u}, B(\mathbf{x}, r))} \quad a.e.$$

iv) For each $\mathbf{x} \in \Omega$ and a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$, we have $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, B(\mathbf{x}, r)) \subset K$ a.e., and $\operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{x}) \subset K$ a.e.

Proof. Statement *i)* was proved in [14, Lemma 3.9] (see also [51, Lemma 3.4]).

Now, Sobolev maps are approximately differentiable a.e. (see, e.g., [51, Prop. 2.4]). Therefore, thanks to *i)* and the assumptions, there exists a set $\Omega' \subset \Omega$ with $\mathcal{L}^n(\Omega \setminus \Omega') = 0$ such that $\mathbf{u}|_{\Omega'}$ is injective, $\det D\mathbf{u} > 0$ in Ω', $\mathbf{u}$ is approximately differentiable at each point of Ω', and $\mathbf{u}(\mathbf{x}) \in K$ for all $\mathbf{x} \in \Omega'$. We then apply Federer's [22] area formula (see also [51, Prop. 2.6]) to obtain that

$$\|\det D\mathbf{u}\|_{L^1(\Omega)} = \int_{\Omega'} \det D\mathbf{u}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \mathcal{H}^0\left(\mathbf{u}^{-1}(\mathbf{y}) \cap \Omega'\right) \mathrm{d}\mathbf{y} \leq \mathcal{L}^n(K),$$

which shows *ii)*.

Statement *iii)* was proved in [14, Lemma 3.10] (see also [38, Prop. 2.17]).

Statement *iv)* follows from *iii)* and Proposition 2.1 ii). □

LEMMA 2.6. *Let $K \subset \mathbb{R}^n$ be compact. For each $j \in \mathbb{N}$, let $\mathbf{u}_j, \mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n)$. Assume that*

$$\mathbf{u}_j \to \mathbf{u} \quad a.e., \qquad \operatorname{cof} D\mathbf{u}_j \rightharpoonup \operatorname{cof} D\mathbf{u} \quad in\ L^1(\Omega, \mathbb{R}^{n\times n})$$

as $j \to \infty$. Then, for all $\phi \in C_c^\infty(\Omega)$,

$$\langle \operatorname{Det} D\mathbf{u}_j, \phi \rangle \to \langle \operatorname{Det} D\mathbf{u}, \phi \rangle \tag{2.4}$$

as $j \to \infty$. Moreover, for each $\mathbf{x} \in \Omega$ and a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$,

$$\deg(\mathbf{u}_j, \partial B(\mathbf{x}, r), \cdot) \to \deg(\mathbf{u}, \partial B(\mathbf{x}, r), \cdot) \quad a.e.$$

as $j \to \infty$.

Proof. A standard convergence result (see, e.g., [57, Lemma 6.7]) concerning the product of a weakly convergent sequence in L^1 by a bounded sequence in L^∞ converging a.e. shows that

$$(\operatorname{adj} D\mathbf{u}_j)\, \mathbf{u}_j \rightharpoonup (\operatorname{adj} D\mathbf{u})\, \mathbf{u} \qquad \text{in } L^1(\Omega, \mathbb{R}^n)$$

as $j \to \infty$. This readily implies convergence (2.4), due to Definition 2.4.

For the second part, note that by the coarea formula, for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$,

$$\sup_{j\in\mathbb{N}} \|\mathbf{u}_j\|_{L^\infty(\partial B(\mathbf{x}, r), \mathbb{R}^n)} < \infty$$

and $\mathbf{u}_j \to \mathbf{u}$ $\mathcal{H}^{n-1}$-a.e. in $\partial B(\mathbf{x}, r)$ as $j \to \infty$. Moreover, it was shown in [38, Lemma 8.2] that for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$ there exists a subsequence (not relabelled) such that

$$(\operatorname{cof} D\mathbf{u}_j)\, \boldsymbol{\nu}_r \rightharpoonup (\operatorname{cof} D\mathbf{u})\, \boldsymbol{\nu}_r \quad \text{in } L^1(\partial B(\mathbf{x}, r), \mathbb{R}^n),$$

as $j \to \infty$, where $\boldsymbol{\nu}_r$ is the unit exterior normal to $\partial B(\mathbf{x}, r)$. The result follows by the continuity of the degree proved in [38, Lemma 8.1]. □

LEMMA 2.7. *Let $K \subset \mathbb{R}^n$ be compact. Suppose $\mathbf{u} \in W^{1,n-1}(\Omega, K)$ satisfies condition INV and is such that $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$. Then the following properties are satisfied:*

i) $\operatorname{Det} D\mathbf{u}$ *is a positive measure that can be represented as*

$$\operatorname{Det} D\mathbf{u} = (\det D\mathbf{u})\, \mathcal{L}^n + \sum_{\mathbf{a} \in C(\mathbf{u})} \operatorname{Det} D\mathbf{u}(\{\mathbf{a}\})\, \delta_{\mathbf{a}}. \tag{2.5}$$

Moreover, $C(\mathbf{u}) = \{\mathbf{x} \in \Omega : \mathcal{L}^n(\operatorname{im_T}(\mathbf{u}, \mathbf{x})) > 0\}$, and for all $\mathbf{a} \in C(\mathbf{u})$,

$$\operatorname{Per} \operatorname{im_T}(\mathbf{u}, \mathbf{a}) < \infty \quad \textit{and} \quad \operatorname{Det} D\mathbf{u}(\{\mathbf{a}\}) = \mathcal{L}^n(\operatorname{im_T}(\mathbf{u}, \mathbf{a})).$$

Furthermore, there exists a dimensional constant $c_n > 0$ such that

$$\sum_{\mathbf{a} \in C(\mathbf{u})} \operatorname{Det} D\mathbf{u}(\{\mathbf{a}\}) \le c_n\, \mathcal{E}(\mathbf{u})^{\frac{n}{n-1}}. \tag{2.6}$$

ii) For each $\mathbf{x} \in \Omega$ and a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$,

$$\operatorname{im_T}(\mathbf{u}, B(\mathbf{x}, r)) = \operatorname{im_G}(\mathbf{u}, B(\mathbf{x}, r)) \cup \bigcup_{\mathbf{a} \in C(\mathbf{u}) \cap B(\mathbf{x}, r)} \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \quad \textit{a.e.},$$

where the union in the right-hand side is pairwise disjoint, up to sets of measure zero.

iii) $\partial^* \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \subset K$ $\mathcal{H}^{n-1}$-a.e., for each $\mathbf{a} \in C(\mathbf{u})$.

Proof. From Lemma 2.5 we find that $\det D\mathbf{u} \in L^1(\Omega)$. Thanks to [38, Thms. 4.6 and 6.2], the distributional determinant $\operatorname{Det} D\mathbf{u}$ is a measure, and representation (2.5) holds. Furthermore, by [14, Th. 4.2] (see also [38, Th. 3.2]), $C(\mathbf{u})$ coincides with the set of $\mathbf{x} \in \Omega$ such that $\mathcal{L}^n(\operatorname{im_T}(\mathbf{u}, \mathbf{x})) > 0$, and $\operatorname{Det} D\mathbf{u}(\{\mathbf{a}\}) = \mathcal{L}^n(\operatorname{im_T}(\mathbf{u}, \mathbf{a}))$ for all $\mathbf{a} \in C(\mathbf{u})$. Moreover, by [38, Th. 4.6], $\operatorname{im_T}(\mathbf{u}, \mathbf{a})$ has finite perimeter for all $\mathbf{a} \in C(\mathbf{u})$. Now, taking from [38] both the formula after Prop. 6.1, and Th. 4.6, we find that, for an increasing sequence $\{U_k\}_{k \in \mathbb{N}}$ of subsets of Ω such that $\Omega = \bigcup_{k \in \mathbb{N}} U_k$, the inequalities

$$\sum_{\mathbf{a} \in C(\mathbf{u}) \cap U_k} \operatorname{Det} D\mathbf{u}\,(\{\mathbf{a}\}) \le c_n \left(\sum_{\mathbf{a} \in C(\mathbf{u}) \cap U_k} \operatorname{Per} \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \right)^{\frac{n}{n-1}} \le c_n\, \mathcal{E}(\mathbf{u})^{\frac{n}{n-1}},$$

hold for some dimensional constant $c_n > 0$. Letting $k \to \infty$, we conclude the proof of *i)*.

Statement *ii)* is a consequence of *i)* and [38, Th. 3.2].

As for *iii)*, we notice that thanks to *i)*, for each $\mathbf{a} \in C(\mathbf{u})$, the set $\operatorname{im_T}(\mathbf{u}, \mathbf{a})$ has finite perimeter, hence $\partial^* \operatorname{im_T}(\mathbf{u}, \mathbf{a}) = \partial^- \operatorname{im_T}(\mathbf{u}, \mathbf{a})$ $\mathcal{H}^{n-1}$-a.e., where ∂^- denotes the essential boundary (see, e.g., [1, Th. 3.61] for the definitions and proofs). Now, by

Lemma 2.5 iv), $\operatorname{im_T}(\mathbf{u}, \mathbf{a}) \subset K$ a.e., which implies that $\partial^- \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \subset K$ because K is compact. ∎

The following is a particular case of [36, Thms. 2 and 3].

LEMMA 2.8. *For each* $j \in \mathbb{N}$, *let* $\mathbf{u}_j, \mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ *satisfy* $\det D\mathbf{u}_j \in L^1(\Omega)$. *Assume that there exists* $\theta \in L^1(\Omega)$ *such that*

$$\mathbf{u}_j \to \mathbf{u} \quad \text{a.e.}, \quad \operatorname{cof} D\mathbf{u}_j \rightharpoonup \operatorname{cof} D\mathbf{u} \quad \text{in } L^1(\Omega, \mathbb{R}^{n\times n}), \quad \det D\mathbf{u}_j \rightharpoonup \theta \quad \text{in } L^1(\Omega) \tag{2.7}$$

as $j \to \infty$. *Suppose that* $\sup_{j\in\mathbb{N}} \mathcal{E}(\mathbf{u}_j) < \infty$. *Then* $\theta = \det D\mathbf{u}$ *a.e. and*

$$\mathcal{E}(\mathbf{u}) \le \liminf_{j\to\infty} \mathcal{E}(\mathbf{u}_j).$$

Moreover, for every $\mathbf{x} \in \Omega$ *and a.e.* $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$, *there exists a subsequence (not relabelled) such that*

$$\operatorname{im_G}(\mathbf{u}_j, B(\mathbf{x}, r)) \to \operatorname{im_G}(\mathbf{u}, B(\mathbf{x}, r)) \qquad \text{a.e.}$$

as $j \to \infty$.

LEMMA 2.9. *For each* $j \in \mathbb{N}$, *let* $\mathbf{u}_j, \mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ *satisfy INV,* $\det D\mathbf{u}_j > 0$ *a.e. and* $\det D\mathbf{u} > 0$ *a.e. Suppose*

$$\sup_{j\in\mathbb{N}} \left[\mathcal{E}(\mathbf{u}_j) + \|\mathbf{u}_j\|_{L^\infty(\Omega,\mathbb{R}^n)}\right] < \infty.$$

Assume that there exists $\theta \in L^1(\Omega)$ *such that* $\theta > 0$ *a.e. and the convergences* (2.7) *hold as* $j \to \infty$. *Then* $\mathbf{u}$ *satisfies INV. Moreover,*

$$\sum_{\mathbf{a}\in C(\mathbf{u}_j)} \operatorname{Det} D\mathbf{u}_j(\{\mathbf{a}\})\, \delta_{\mathbf{a}} \overset{*}{\rightharpoonup} \sum_{\mathbf{a}\in C(\mathbf{u})} \operatorname{Det} D\mathbf{u}(\{\mathbf{a}\})\, \delta_{\mathbf{a}} \quad \text{in } \mathcal{M}(\Omega), \tag{2.8}$$

as $j \to \infty$.

Proof. Clearly, $\mathbf{u} \in L^\infty(\Omega, \mathbb{R}^n)$ That $\mathbf{u}$ satisfies INV was shown in [38, Prop. 8.4].

Now, for each $j \in \mathbb{N}$, thanks to Lemma 2.7, the distributional determinants $\operatorname{Det} D\mathbf{u}_j$ and $\operatorname{Det} D\mathbf{u}$ are positive measures,

$$\begin{aligned} \operatorname{Det} D\mathbf{u}_j &= \det D\mathbf{u}_j\, \mathcal{L}^n + \sum_{\mathbf{a}\in C(\mathbf{u}_j)} \operatorname{Det} D\mathbf{u}_j(\{\mathbf{a}\})\, \delta_{\mathbf{a}}, \\ \operatorname{Det} D\mathbf{u} &= \det D\mathbf{u}\, \mathcal{L}^n + \sum_{\mathbf{a}\in C(\mathbf{u})} \operatorname{Det} D\mathbf{u}(\{\mathbf{a}\})\, \delta_{\mathbf{a}}, \end{aligned} \tag{2.9}$$

$\operatorname{Det} D\mathbf{u}_j(\{\mathbf{a}\}) > 0$ for all $\mathbf{a} \in C(\mathbf{u}_j)$, and

$$\sum_{\mathbf{a}\in C(\mathbf{u}_j)} \operatorname{Det} D\mathbf{u}_j\, (\{\mathbf{a}\}) \le c_n\, \mathcal{E}(\mathbf{u}_j)^{\frac{n}{n-1}}, \tag{2.10}$$

for some dimensional constant $c_n > 0$.

The convergence of $\{\det D\mathbf{u}_j\}_{j\in\mathbb{N}}$ yields the bound

$$\sup_{j\in\mathbb{N}} \|\det D\mathbf{u}_j\|_{L^1(\Omega)} < \infty. \tag{2.11}$$

Bounds (2.10) and (2.11) and representation (2.9) imply that

$$\sup_{j\in\mathbb{N}} \operatorname{Det} D\mathbf{u}_j(\Omega) < \infty. \tag{2.12}$$

Now, bound (2.12) and the convergence (2.4) of Lemma 2.6 show that $\operatorname{Det} D\mathbf{u}_j \overset{*}{\rightharpoonup} \operatorname{Det} D\mathbf{u}$ in $\mathcal{M}(\Omega)$, as $j \to \infty$. Because of the representation (2.9), this reads as

$$\det D\mathbf{u}_j \, \mathcal{L}^n + \sum_{\mathbf{a} \in C(\mathbf{u}_j)} \operatorname{Det} D\mathbf{u}_j(\{\mathbf{a}\}) \, \delta_{\mathbf{a}} \overset{*}{\rightharpoonup} \det D\mathbf{u} \, \mathcal{L}^n + \sum_{\mathbf{a} \in C(\mathbf{u})} \operatorname{Det} D\mathbf{u}(\{\mathbf{a}\}) \, \delta_{\mathbf{a}} \quad (2.13)$$

in $\mathcal{M}(\Omega)$, as $j \to \infty$. Now, it was shown in Lemma 2.8 that $\theta = \det D\mathbf{u}$ a.e., so actually (2.13) implies the convergence (2.8). □

3. An inhomogeneous surface energy for cavitation. Just as the stored energy function of the body may depend on $\mathbf{x} \in \Omega$ so as to model inhomogeneities of the material, the surface energy may well depend on the cavity point in the reference configuration that produces the hole in the deformed configuration. In view (2.3), a natural candidate is

$$\mathcal{S}_1(\mathbf{u}) := \sum_{\mathbf{a} \in C(\mathbf{u})} \kappa_1(\mathbf{a}) \operatorname{Per} \operatorname{im_T}(\mathbf{u}, \mathbf{a}) \quad (3.1)$$

for some function $\kappa_1 : \Omega \to [0, \infty]$. The function κ_1 indicates the proportionality constant of the energetic cost of creating a cavity with a given perimeter. In an extreme case, κ_1 may take just the two values 0 and ∞, thus indicating that the material can only present cavitation at some prescribed points with no cost of energy. This is, in fact, the model of Sivaloganathan & Spector [57].

Under an a priori bound on the number of cavities, we present a proof of the lower semicontinuity of $\mathcal{S}_1$ by showing that, for a sequence of deformations with bounded surface energy, the cavity volume converges to the cavity volume. The proof of Proposition 3.1 below uses the techniques of [57] and [38].

PROPOSITION 3.1. *For each $j \in \mathbb{N}$, let $\mathbf{u}_j \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ satisfy condition INV,* $\det D\mathbf{u}_j > 0$ *a.e. and*

$$\sup_{j \in \mathbb{N}} \left[\|\mathbf{u}_j\|_{L^\infty(\Omega, \mathbb{R}^n)} + \mathcal{E}(\mathbf{u}_j) + \mathcal{H}^0(C(\mathbf{u}_j)) \right] < \infty. \quad (3.2)$$

Let $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n)$ satisfy $\det D\mathbf{u} > 0$ *a.e. Assume that there exists $\theta \in L^1(\Omega)$ such that the convergences* (2.7) *hold as $j \to \infty$. Let $\kappa_1 : \Omega \to [0, \infty]$ be lower semicontinuous, and consider the functional $\mathcal{S}_1$ of* (3.1)*. Then*

$$\mathcal{S}_1(\mathbf{u}) \leq \liminf_{j \to \infty} \mathcal{S}_1(\mathbf{u}_j). \quad (3.3)$$

Proof. Recall from Lemma 2.5 that $\det D\mathbf{u}_j \in L^1(\Omega)$ for each $j \in \mathbb{N}$, from Lemma 2.8 that $\theta = \det D\mathbf{u}$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$, and from Lemma 2.9 that $\mathbf{u}$ satisfies condition INV. By passing to a subsequence (not relabelled), we can assume that the limit $\lim_{j \to \infty} \mathcal{S}_1(\mathbf{u}_j)$ exists.

Fix $\mathbf{x} \in \Omega$. By Lemma 2.6, there exists a subsequence (not relabelled) such that for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$,

$$\deg(\mathbf{u}_j, \partial B(\mathbf{x}, r), \cdot) \to \deg(\mathbf{u}, \partial B(\mathbf{x}, r), \cdot) \quad \text{a.e.} \quad (3.4)$$

as $j \to \infty$. By Lemma 2.5, for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$,

$$\deg(\mathbf{u}_j, \partial B(\mathbf{x}, r), \cdot) = \chi_{\operatorname{im_T}(\mathbf{u}_j, B(\mathbf{x}, r))} \quad \text{and} \quad \deg(\mathbf{u}, \partial B(\mathbf{x}, r), \cdot) = \chi_{\operatorname{im_T}(\mathbf{u}, B(\mathbf{x}, r))} \quad (3.5)$$

for all $j \in \mathbb{N}$. By Lemma 2.8, for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$ and for a further subsequence (not relabelled),

$$\operatorname{im}_{\mathrm{G}}(\mathbf{u}_j, B(\mathbf{x}, r)) \to \operatorname{im}_{\mathrm{G}}(\mathbf{u}, B(\mathbf{x}, r)) \quad \text{a.e.} \tag{3.6}$$

as $j \to \infty$. Putting together (3.4), (3.5), (3.6) and Lemma 2.7, we find that for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$,

$$\bigcup_{\mathbf{a} \in C(\mathbf{u}_j) \cap B(\mathbf{x}, r)} \operatorname{im}_{\mathrm{T}}(\mathbf{u}_j, \mathbf{a}) \to \bigcup_{\mathbf{a} \in C(\mathbf{u}) \cap B(\mathbf{x}, r)} \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{a}) \quad \text{a.e.} \tag{3.7}$$

as $j \to \infty$.

As we have a uniform bound on the number of cavity points, there exists $M \in \mathbb{N}$ such that, for a further subsequence (not relabelled), $\mathcal{H}^0(C(\mathbf{u}_j)) = M$, $C(\mathbf{u}_j) = \{\mathbf{a}_{1j}, \ldots, \mathbf{a}_{Mj}\}$ for all $k \in \mathbb{N}$, and

$$\lim_{j \to \infty} \mathbf{a}_{ij} = \mathbf{a}_i, \quad \lim_{j \to \infty} \operatorname{Det} D\mathbf{u}_j(\{\mathbf{a}_{ij}\}) = \alpha_i, \qquad 1 \leq i \leq M, \tag{3.8}$$

for some $\mathbf{a}_{ij} \in \Omega$, $\mathbf{a}_i \in \bar{\Omega}$ and $\alpha_i \in [0, \infty)$; the fact that $\alpha_i < \infty$ is due to (2.10). Let S be the set of $i \in \{1, \ldots, M\}$ such that $\mathbf{a}_i \in \Omega$ and $\alpha_i > 0$. From (3.8) it is immediate to see that

$$\sum_{\mathbf{a} \in C(\mathbf{u}_j)} \operatorname{Det} D\mathbf{u}_j(\{\mathbf{a}\})\, \delta_{\mathbf{a}} \overset{*}{\rightharpoonup} \sum_{i \in S} \alpha_i\, \delta_{\mathbf{a}_i} \quad \text{in} \;\; \mathcal{M}(\Omega) \tag{3.9}$$

as $j \to \infty$. Comparing (3.9) with the convergence (2.8) of Lemma 2.9 we conclude that

$$\sum_{\mathbf{a} \in C(\mathbf{u})} \operatorname{Det} D\mathbf{u}(\{\mathbf{a}\})\, \delta_{\mathbf{a}} = \sum_{i \in S} \alpha_i\, \delta_{\mathbf{a}_i} \tag{3.10}$$

and $C(\mathbf{u}) = \{\mathbf{a}_i : i \in S\}$.

For each $\mathbf{a} \in C(\mathbf{u})$, let $S_{\mathbf{a}}$ be the set of $i \in \{1, \ldots, M\}$ such that $\lim_{j \to \infty} \mathbf{a}_{ij} = \mathbf{a}$. Then, taking in (3.7) an $r > 0$ sufficiently small, we find that

$$\bigcup_{i \in S_{\mathbf{a}}} \operatorname{im}_{\mathrm{T}}(\mathbf{u}_i, \mathbf{a}_{ij}) \to \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{a}) \quad \text{a.e.}$$

as $j \to \infty$, and, consequently, by the lower semicontinuity of the perimeter,

$$\operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{a}) \leq \liminf_{j \to \infty} \operatorname{Per} \bigcup_{i \in S_{\mathbf{a}}} \operatorname{im}_{\mathrm{T}}(\mathbf{u}_j, \mathbf{a}_{ij}) \leq \liminf_{j \to \infty} \sum_{i \in S_{\mathbf{a}}} \operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}_j, \mathbf{a}_{ij}).$$

Now $\kappa_1(\mathbf{a}) \leq \liminf_{j \to \infty} \kappa_1(\mathbf{a}_{ij})$ for all $i \in S_{\mathbf{a}}$, due to the lower semicontinuity of κ_1. Therefore,

$$\kappa_1(\mathbf{a}) \operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{a}) \leq \liminf_{j \to \infty} \sum_{i \in S_{\mathbf{a}}} \kappa_1(\mathbf{a}_{ij}) \operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}_j, \mathbf{a}_{ij}),$$

whence we conclude that

$$\begin{aligned} \mathcal{S}_1(\mathbf{u}) &\leq \liminf_{j \to \infty} \sum_{\mathbf{a} \in C(\mathbf{u})} \sum_{i \in S_{\mathbf{a}}} \kappa_1(\mathbf{a}_{ij}) \operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}_j, \mathbf{a}_{ij}) \\ &\leq \liminf_{j \to \infty} \sum_{i=1}^{M} \kappa_1(\mathbf{a}_{ij}) \operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}_j, \mathbf{a}_{ij}) = \liminf_{j \to \infty} \mathcal{S}_1(\mathbf{u}_j), \end{aligned}$$

which is the desired inequality. $\square$

4. Irreversibility and initiation energy for cavitation. We model the irreversibility of the cavitation process by postulating that the energy spent on the creation of a cavity will be the sum of a fixed amount of energy accounting for the mere process of cavity formation, plus a term depending on the perimeter of the surface created. To be precise, given a function $\kappa_0 : \Omega \to [0, \infty]$, a deformation $\mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n) \cap L^\infty(\Omega, \mathbb{R}^n)$ satisfying INV and $\det D\mathbf{u} > 0$ a.e., and a finite set $S \subset \Omega$ containing $C(\mathbf{u})$, we let

$$\mathcal{S}_0(S) := \sum_{\mathbf{a} \in S} \kappa_0(\mathbf{a}), \qquad \mathcal{S}(\mathbf{u}, S) := \mathcal{S}_0(S) + \mathcal{S}_1(\mathbf{u}) \tag{4.1}$$

where $\mathcal{S}_1(\mathbf{u})$ is as in (3.1), and define the cavitation energy associated to the pair $(\mathbf{u}, S)$ as $\mathcal{S}(\mathbf{u}, S)$. Intuitively, the set S is the union of the past and present cavity points of $\mathbf{u}$. This distinction between past and present cavity points, and between S and $C(\mathbf{u})$ will be clearer in Section 9.

The following result shows the lower semicontinuity of $\mathcal{S}_0$.

PROPOSITION 4.1. *For each $j \in \mathbb{N}$, let $\mathbf{u}_j, \mathbf{u} \in W^{1,n-1}(\Omega, \mathbb{R}^n)$. Let $\mathbf{u}_j$ satisfy INV,* $\det D\mathbf{u}_j > 0$ *a.e. and* $\det D\mathbf{u} > 0$ *a.e. Suppose* (3.2). *Assume that there exists $\theta \in L^1(\Omega)$ such that the convergences* (2.7) *hold as $j \to \infty$. Let $\kappa_0 : \Omega \to [0, \infty]$ be lower semicontinuous, and let S be a finite subset of Ω. Then*

$$\mathcal{S}_0(C(\mathbf{u}) \cup S) \leq \liminf_{j \to \infty} \mathcal{S}_0(C(\mathbf{u}_j) \cup S).$$

Proof. By passing to a subsequence, we can assume, without loss of generality, that $\lim_{j \to \infty} \mathcal{S}_0(C(\mathbf{u}_j) \cup S)$ exists.

Lemma 2.8 shows that $\theta = \det D\mathbf{u}$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$, while Lemma 2.9 shows that $\mathbf{u}$ satisfies condition INV, and that convergence (2.8) holds as $j \to \infty$.

Now we argue as in Proposition 3.1. Accordingly, there exists $M \in \mathbb{N}$ such that, for a further subsequence (not relabelled), $\mathcal{H}^0(C(\mathbf{u}_j)) = M$ and $C(\mathbf{u}_j) = \{\mathbf{a}_{1j}, \ldots, \mathbf{a}_{Mj}\}$ for all $k \in \mathbb{N}$, and equation (3.8) holds for some $\mathbf{a}_{ij} \in \Omega$, $\mathbf{a}_i \in \bar{\Omega}$ and $\alpha_i \in [0, \infty)$. Let S be the set of $i \in \{1, \ldots, M\}$ such that $\mathbf{a}_i \in \Omega$ and $\alpha_i > 0$. Then, equalities (3.10) and $C(\mathbf{u}) = \{\mathbf{a}_i : i \in S\}$ hold.

For each $\mathbf{a} \in C(\mathbf{u})$, let $S_{\mathbf{a}}$ be the set of $i \in \{1, \ldots, M\}$ such that $\lim_{j \to \infty} \mathbf{a}_{ij} = \mathbf{a}$. The lower semicontinuity of κ_0 assures that $\kappa_0(\mathbf{a}) \leq \liminf_{j \to \infty} \kappa_0(\mathbf{a}_{ij})$ for each $\mathbf{a} \in C(\mathbf{u})$ and $i \in S_{\mathbf{a}}$. In particular,

$$\kappa_0(\mathbf{a}) \leq \sum_{i \in S_{\mathbf{a}}} \liminf_{j \to \infty} \kappa_0(\mathbf{a}_{ij})$$

for all $\mathbf{a} \in C(\mathbf{u}) \setminus S$. Therefore,

$$\begin{aligned} \sum_{\mathbf{a} \in C(\mathbf{u}) \setminus S} \kappa_0(\mathbf{a}) &\leq \sum_{\mathbf{a} \in C(\mathbf{u}) \setminus S} \sum_{i \in S_{\mathbf{a}}} \liminf_{j \to \infty} \kappa_0(\mathbf{a}_{ij}) \\ &\leq \liminf_{j \to \infty} \sum_{\mathbf{a} \in C(\mathbf{u}) \setminus S} \sum_{i \in S_{\mathbf{a}}} \kappa_0(\mathbf{a}_{ij}) = \liminf_{j \to \infty} \mathcal{S}_0(C(\mathbf{u}_j) \setminus S). \end{aligned}$$

Adding $\sum_{\mathbf{a} \in S} \kappa_0(\mathbf{a})$ to both sides, we conclude the desired inequality. ∎

5. Formulation of the problem of quasistatic evolution. In this section we present the full model for cavitation and list the hypotheses that will allow us to prove the existence of a quasistatic evolution. Most of the assumptions are standard in the context of fracture mechanics, even though the functional setting is different to that of cavitation; see, in particular, [16, 25, 18, 42] for a motivation and a physical interpretation of the assumptions.

5.1. Reference configuration and time interval. The reference configuration is represented by an open bounded subset Ω of $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$. We fix an $\mathcal{H}^{n-1}$-rectifiable (following the terminology of [1, Def. 2.57]) subset Γ_D of $\partial\Omega$, on which the boundary deformation will be prescribed. Usually, the condition $\mathcal{H}^{n-1}(\Gamma_D) > 0$ is required in order to apply a version of Poincaré's inequality. In our setting, this is not necessary because of the a priori bound given by K (see Subsection 5.2).

Without loss of generality, the time interval is set to be $[0, 1]$.

5.2. Body deformations and elastic energy. A body deformation is any function $\mathbf{u} \in W^{1,n-1}(\Omega, K)$ satisfying INV, $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$. Here K is a fixed compact subset of $\mathbb{R}^n$ with a Lipschitz boundary, and plays the role of a container.

As explained in Subsection 2.4, we have chosen to work in the space $W^{1,n-1} \cap L^\infty$, which is the critical case for cavitation. All the work presented here is valid for $W^{1,p}$ with $n - 1 < p < n$, in which case no L^∞ bound is needed since it is implied (locally) by condition INV (see [51, Sect. 8] or [57, Sect. 2]).

The body is assumed to be hyperelastic, so that the elastic bulk energy of a deformation $\mathbf{u}$ can be written as

$$\mathcal{W}(\mathbf{u}) := \int_\Omega W(\mathbf{x}, D\mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x},$$

where $W : \Omega \times \mathbb{R}^{n\times n} \to \mathbb{R} \cup \{\infty\}$ satisfies the following conditions:

(W0) $W(\mathbf{x}, \mathbf{F}) = W(\mathbf{x}, \mathbf{RF})$ for a.e. $\mathbf{x} \in \Omega$, all $\mathbf{F} \in \mathbb{R}^{n\times n}$ and all $\mathbf{R} \in SO(n)$.

(W1) There exists a function $\tilde{W} : \Omega \times \mathbb{R}^\tau \to \mathbb{R} \cup \{\infty\}$ such that the function $\tilde{W}(\cdot, \xi)$ is measurable for every $\xi \in \mathbb{R}^\tau$, the function $\tilde{W}(\mathbf{x}, \cdot)$ is continuous (in the order topology of $\mathbb{R} \cup \{\infty\}$) and convex for a.e. $\mathbf{x} \in \Omega$, and

$$W(\mathbf{x}, \mathbf{F}) = \tilde{W}(\mathbf{x}, \boldsymbol{\mu}(\mathbf{F})), \quad \text{for a.e. } \mathbf{x} \in \Omega \text{ and all } \mathbf{F} \in \mathbb{R}^{n\times n},$$

where $\boldsymbol{\mu}(\mathbf{F}) \in \mathbb{R}^\tau$ is the vector composed by all minors of $\mathbf{F}$, and τ is the number of all such minors.

(W2) For a.e. $\mathbf{x} \in \Omega$,

$$W(\mathbf{x}, \mathbf{F}) < \infty \quad \text{if and only if} \quad \mathbf{F} \in GL_+(n).$$

Moreover, for a.e. $\mathbf{x} \in \Omega$, the function $W(\mathbf{x}, \cdot)$ is of class C^1 on $GL_+(n)$.

(W3) $W(\cdot, \mathbf{1}) \in L^1(\Omega)$.

(W4) There exist $a \in L^1(\Omega)$, a constant $c > 0$, an increasing function $h_1 : (0, \infty) \to [0, \infty)$ and a convex function $h_2 : (0, \infty) \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{h_1(t)}{t} = \lim_{t\to\infty} \frac{h_2(t)}{t} = \lim_{t\searrow 0} h_2(t) = \infty \tag{5.1}$$

and

$$W(\mathbf{x}, \mathbf{F}) \geq a(\mathbf{x}) + c\,|\mathbf{F}|^{n-1} + h_1\left(|\mathrm{cof}\, \mathbf{F}|\right) + h_2(\det \mathbf{F})$$

for a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{F} \in GL_+(n)$.

(W5) There exist $c_W^1 > 0$ and a function $c_W^0 \in L^1(\Omega)$ with $c_W^0 \geq 0$ such that for a.e. $\mathbf{x} \in \Omega$ and every $\mathbf{F} \in GL_+(n)$,

$$\left|\mathbf{F}^T D_2 W(\mathbf{x}, \mathbf{F})\right| \leq c_W^1 \left(W(\mathbf{x}, \mathbf{F}) + c_W^0(\mathbf{x})\right).$$

Here $D_2 W(\mathbf{x}, \mathbf{F})$ denotes the derivative of $W(\mathbf{x}, \cdot)$ at $\mathbf{F}$.

(W6) There exists a function $c_W^0 \in L^1(\Omega)$ with $c_W^0 \geq 0$ satisfying that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{F}, \mathbf{G} \in GL_+(n)$ with $|\mathbf{G} - \mathbf{1}| < \delta$,

$$\left|D_2 W(\mathbf{x}, \mathbf{G}\mathbf{F})(\mathbf{G}\mathbf{F})^T - D_2 W(\mathbf{x}, \mathbf{F})\mathbf{F}^T\right| \leq \varepsilon \left(W(\mathbf{x}, \mathbf{F}) + c_W^0(\mathbf{x})\right).$$

Except for (W4), this set of assumptions coincides with that of [42] (see also [18]). The growth condition (W4) is the natural one in the case of cavitation, according to [38, Th. 8.5].

The orientation preserving condition $\det D\mathbf{u} > 0$ and, as a result, the coercivity assumption (W4) and the multiplicative decomposition estimates (W5) and (W6) (which also appear in [5, 25]) are sometimes regarded as a refinement of the theory, which in other contexts is previously proved for stored energy functions W with polynomial growth (see, e.g., [16]). However, in this case, the orientation preserving and the INV conditions are essential in the theory of cavitation, as shown in [51], since otherwise no existence theory or interpretation of the topological image as a cavity volume would be possible (see [38] for a counterexample). As a matter of fact, the set of admissible deformations is multiplicative, as will be shown in Propositions 6.1 and 6.2, since both the restriction $\det D\mathbf{u} > 0$ and condition INV are multiplicative.

5.3. Cavity points and cavitation energy. A cavity set is represented in the reference configuration as a finite subset S of Ω containing $C(\mathbf{u})$. Given $\mathbf{u} \in W^{1,n-1}(\Omega, K)$ satisfying INV, $\det D\mathbf{u} > 0$ a.e. and $\mathcal{E}(\mathbf{u}) < \infty$, and given two continuous functions $\kappa_0, \kappa_1 : \Omega \to (0, \infty)$ such that

(K1) $\inf \kappa_0 > 0$ and $\inf \kappa_1 > 0$,

we define, as in (3.1) and (4.1),

$$\mathcal{S}_0(S) := \sum_{\mathbf{a} \in S} \kappa_0(\mathbf{a}), \quad \mathcal{S}_1(\mathbf{u}) := \sum_{\mathbf{a} \in C(\mathbf{u})} \kappa_1(\mathbf{a}) \operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{a}), \quad \mathcal{S}(\mathbf{u}, S) := \mathcal{S}_0(S) + \mathcal{S}_1(\mathbf{u}).$$

The cavity energy of the configuration $(\mathbf{u}, S)$ is set to be $\mathcal{S}(\mathbf{u}, S)$.

Recalling (2.3), we find that

$$(\inf \kappa_1) \, \mathcal{E}(\mathbf{u}) \leq \mathcal{S}_1(\mathbf{u}) \leq (\sup \kappa_1) \, \mathcal{E}(\mathbf{u}). \tag{5.2}$$

In the general theory of cavitation in the static case (see [51, 14, 38]), any deformation $\mathbf{u}$ with finite energy many have a countable set $C(\mathbf{u})$ of cavities. This is because no amount of energy is spent on the process of initiation of cavities; in other words, in the notation of this section, those papers set $\kappa_0 = 0$. In our case, a cavity at a point $\mathbf{a} \in \Omega$ is penalized energetically with $\kappa_0(\mathbf{a})$, as well as with $\kappa_1(\mathbf{a}) \operatorname{Per} \operatorname{im}_{\mathrm{T}}(\mathbf{u}, \mathbf{a})$. The fact that $\inf \kappa_0 > 0$ implies that any configuration $(\mathbf{u}, S)$ with finite energy has a finite set of cavities. As a matter of fact, the finiteness of S simplifies the development of the theory, as will be seen, for example, in Propositions 10.3 and 11.1.

We also mention that the theory of Sivaloganathan & Spector [57] can also handle with a term similar to $\mathcal{S}_0$, but not with an energy depending on the perimeter of the cavities but on their volume. In addition, in their theory, the set of cavities is contained in a prescribed finite set.

5.4. Body and surface forces. One could add body and surface forces to the model, the inclusion of which is by now a standard technique. In particular, one would take from [42] the set of minimal assumptions compatible with the model. However, for the sake of clarity and in order to underline the main novelty of the paper (i.e., the cavitation energy $\mathcal{S}$), we set external body and surface forces to be zero.

5.5. Prescribed boundary deformations. For each $t \in [0,1]$ any deformation $\mathbf{u}$ must satisfy the boundary condition $\mathbf{u}|_{\Gamma_D} = \boldsymbol{\psi}(t)|_{\Gamma_D}$ in the sense of traces. The assumptions on the prescribed deformation $\boldsymbol{\psi}$ are the following:

(B1) There exist $\boldsymbol{\psi}, \boldsymbol{\phi} \in C^1([0,1], C^1(K,K))$ such that for every $(t,\mathbf{x}) \in [0,1] \times K$,

$$\boldsymbol{\psi}(t)\left(\boldsymbol{\phi}(t)(\mathbf{x})\right) = \mathbf{x} = \boldsymbol{\phi}(t)\left(\boldsymbol{\psi}(t)(\mathbf{x})\right).$$

(B2) $\sup_{t\in[0,1]} \mathcal{W}(\boldsymbol{\psi}(t)) < \infty$.

This set of assumptions is a slight adaptation of Dal Maso & Lazzaroni [18] (see also [25]). We remark that Lazzaroni [42] has developed a quasistatic evolution theory for fracture with weaker regularity assumptions (namely, Lipschitz) on the boundary conditions. We could adopt that setting as well, with proofs following the same lines, although more technical. Nevertheless, for the sake of clarity we assume the C^1 regularity of the boundary conditions.

The space $C^1(K,K)$ is, of course, the set of continuous functions from K to K that are differentiable in the interior of K, and the derivative is uniformly continuous, hence can be extended uniquely to K by continuity.

Conditions (W2) and (B2) imply that $\det D(\boldsymbol{\psi}(t)) > 0$ for every $t \in [0,1]$. Together with condition (B1), we obtain that the function $\boldsymbol{\psi}(t)|_\Omega$ is a C^1 diffeomorphism onto its image. Therefore, by [38, Lemma 5.2], $\boldsymbol{\psi}(t)$ satisfies condition INV and has no cavities.

The following estimates will be useful in the development of the theory.

LEMMA 5.1. *There exists $\alpha \geq 1$, depending on $\boldsymbol{\psi}$, such that for any orthonormal set of vectors $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_{n-1}$ of $\mathbb{R}^n$, and calling $\boldsymbol{\tau} := (\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_{n-1}) \in \mathbb{R}^{n\times(n-1)}$, we have that for every $t \in [0,1]$,*

$$\frac{1}{\alpha} \leq \det D(\boldsymbol{\psi}(t)) \leq \alpha, \qquad \frac{1}{\alpha} \leq \det\Big((D(\boldsymbol{\psi}(t))\boldsymbol{\tau})^T \, (D(\boldsymbol{\psi}(t))\boldsymbol{\tau}) \Big) \leq \alpha.$$

Proof. The function $\boldsymbol{\psi}$, when considered from $[0,1] \times K$ to K, is of class C^1. In particular, the function $D_2\boldsymbol{\psi} : [0,1] \times K \to \mathbb{R}^{n\times n}$ is continuous, where D_2 indicates derivative with respect to the second variable $\mathbf{x} \in K$. By compactness, it suffices to show that

$$\det D_2\boldsymbol{\psi}(t,\mathbf{x}) > 0, \quad \det\Big((D_2\boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\tau})^T \, (D_2\boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\tau}) \Big) > 0, \qquad (t,\mathbf{x}) \in [0,1] \times K. \tag{5.3}$$

The first inequality of (5.3) was shown before.

As for the second, we recall that, as a consequence of the Cauchy–Binet formula (e.g., [1, Prop. 2.68]), for any $\mathbf{L} \in \mathbb{R}^{n\times(n-1)}$, we have that $\det(\mathbf{L}^T\mathbf{L}) \geq 0$, with equality if and only if the rank of $\mathbf{L}$ is less than $n-1$. As $D_2\boldsymbol{\psi}(t,\mathbf{x})$ is invertible for all $[0,1]\times K$, and $\boldsymbol{\tau}$ has rank $n-1$, then the second equality of (5.3) holds true. ∎

5.6. Admissible configurations and total energy. An *admissible deformation* at time $t \in [0,1]$ is a function $\mathbf{u} \in W^{1,n-1}(\Omega, K)$ satisfying INV, $\det D\mathbf{u} > 0$ a.e.,

$\mathcal{E}(\mathbf{u}) < \infty$, $\mathcal{H}^0(C(\mathbf{u})) < \infty$, and $\mathbf{u}|_{\Gamma_D} = \boldsymbol{\psi}(t)|_{\Gamma_D}$ in the sense of traces. The set of admissible deformations at time $t \in [0,1]$ is called $\mathfrak{A}(t)$. An *admissible configuration* at time $t \in [0,1]$ is a pair $(\mathbf{u}, S)$ where $\mathbf{u} \in \mathfrak{A}(t)$ and S is a finite subset of Ω containing $C(\mathbf{u})$. The set $\mathfrak{B}(t)$ denotes the set of admissible configurations at time $t \in [0,1]$. Given a finite subset S_0 of Ω, the set $\mathfrak{B}_{S_0}(t)$ denotes the set of $(\mathbf{u}, S) \in \mathfrak{B}(t)$ such that $S \supset S_0$. The remarks in Subsection 5.5 show in particular that $(\boldsymbol{\psi}(t), S) \in \mathfrak{B}_{S_0}(t)$ for any finite $S \supset S_0$.

For each $t \in [0,1]$ and $(\mathbf{u}, S) \in \mathfrak{B}(t)$, we define

$$\mathcal{I}(\mathbf{u}, S) := \mathcal{W}(\mathbf{u}) + \mathcal{S}(\mathbf{u}, S), \qquad \mathcal{I}^c(\mathbf{u}) := \mathcal{W}(\mathbf{u}) + \mathcal{S}_1(\mathbf{u}).$$

The quantity $\mathcal{I}(\mathbf{u}, S)$ is the total energy of $(\mathbf{u}, S)$, while the quantity $\mathcal{I}^c(\mathbf{u})$ is the conservative part of the energy of $\mathbf{u}$. That $\mathcal{I}^c(\mathbf{u})$ is indeed conservative will become apparent in Proposition 12.1. As mentioned in Subsection 5.4, one could add body and surface forces to the model, which would be part of the conservative energy.

5.7. Initial data and quasistatic evolution. As an initial data, we take any configuration $(\mathbf{u}^0, S^0)$ that is a minimizer of $\mathcal{I}$ in $\mathfrak{B}_{S^0}(0)$. We will see in Proposition 8.1 that such a minimizer exists.

The main result of the paper establishes the existence of a quasistatic evolution starting from any initial data.

THEOREM 5.2. *For each $t \in [0,1]$ there exists $\mathbf{u}(t) \in \mathfrak{A}(t)$ such that, when one defines $S(t) := S^0 \cup \bigcup_{s\in[0,t]} C(\mathbf{u}(s))$, the following conditions hold:*
(a) $(\mathbf{u}(0), S(0)) = (\mathbf{u}^0, S^0)$.
(b) For every $t \in [0,1]$, the pair $(\mathbf{u}(t), S(t))$ is a minimizer of $\mathcal{I}$ in $\mathfrak{B}_{S(t)}(t)$.
(c) The function $t \mapsto I(t) := \mathcal{I}(\mathbf{u}(t), S(t))$ is absolutely continuous on $[0,1]$, and for a.e. $t \in [0,1]$,

$$I'(t) = \int_\Omega D_2W\left(\mathbf{x}, D\mathbf{u}(t)\right) \cdot D\left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t) \circ \mathbf{u}(t)\right) \mathrm{d}\mathbf{x} + \sum_{\mathbf{a}\in C(\mathbf{u}(t))} \kappa_1(\mathbf{a}) \int_{\partial^* \operatorname{im_T}(\mathbf{u}(t),\mathbf{a})} \operatorname{div}^{\partial^* \operatorname{im_T}(\mathbf{u}(t),\mathbf{a})} \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t)\right) \mathrm{d}\mathcal{H}^{n-1}.$$

In the formula above, $\operatorname{div}^{\partial^* \operatorname{im_T}(\mathbf{u}(t),\mathbf{a})}$ denotes the operator of tangential divergence (in the deformed configuration) on $\partial^* \operatorname{im_T}(\mathbf{u}(t), \mathbf{a})$.

The rest of the paper is devoted to the proof of Theorem 5.2.

In Theorem 5.2, no regularity is claimed of the map $t \mapsto \mathbf{u}(t)$; as in [17, Th. 3.5] (see also [18, Th. 6.1]), one may show, with minor changes in the proof, that it can be chosen to be measurable from $[0,1]$ to $W^{1,n-1}(\Omega, \mathbb{R}^n)$. Similarly, the function S can be chosen to be left-continuous; the proof of this fact would be easier that the analogue of [17, Prop. 3.1], since $S(t)$ is finite, and S takes only finitely many values.

6. Reformulation with time-independent Dirichlet data. In this section we follow the general scheme of [25] in order to turn the problem into a quasistatic evolution with time-independent prescribed boundary deformations, but with time-dependent elastic and cavitation energy.

We thus look for deformations $\mathbf{u} \in \mathfrak{A}(t)$ of the form $\mathbf{u} = \boldsymbol{\psi}(t) \circ \mathbf{v}$ for some $\mathbf{v} \in W^{1,n-1}(\Omega, K)$. The set, now independent of time, of admissible deformations must be accordingly redefined as follows: $\mathfrak{A}$ is the set of $\mathbf{v} \in W^{1,n-1}(\Omega, K)$ satisfying INV, $\det D\mathbf{v} > 0$ a.e., $\mathcal{E}(\mathbf{v}) < \infty$ and $\mathbf{v}|_{\Gamma_D} = \mathbf{id}|_{\Gamma_D}$ in the sense of traces. Likewise, we define $\mathfrak{B}$ as the set of pairs $(\mathbf{v}, S)$ such that $\mathbf{v} \in \mathfrak{A}$ and S is a finite subset of Ω

containing $C(\mathbf{v})$. Finally, given a finite subset S_0 of Ω, the set $\mathfrak{B}_{S_0}$ denotes the set of $(\mathbf{v}, S) \in \mathfrak{B}$ such that $S \supset S_0$.

The corresponding energy functionals are defined as follows: for each $t \in [0,1]$ and $(\mathbf{v}, S) \in \mathfrak{B}$,

$$\begin{aligned}&\bar{\mathcal{W}}(t)(\mathbf{v}) := \mathcal{W}(\boldsymbol{\psi}(t)\circ\mathbf{v}), \quad \bar{\mathcal{S}}_1(t)(\mathbf{v}) := \mathcal{S}_1(\boldsymbol{\psi}(t)\circ\mathbf{v}), \quad \bar{\mathcal{S}}(t)(\mathbf{v},S) := \mathcal{S}(\boldsymbol{\psi}(t)\circ\mathbf{v},S),\\ &\bar{\mathcal{I}}(t)(\mathbf{v},S) := \mathcal{I}(\boldsymbol{\psi}(t)\circ\mathbf{v},S), \quad \bar{\mathcal{I}}^c(t)(\mathbf{v}) := \mathcal{I}^c(\boldsymbol{\psi}(t)\circ\mathbf{v}).\end{aligned} \tag{6.1}$$

Note that

$$\begin{aligned}&\bar{\mathcal{S}}(t)(\mathbf{v},S) = \bar{\mathcal{S}}_1(t)(\mathbf{v}) + \mathcal{S}_0(S), \qquad \bar{\mathcal{I}}(t)(\mathbf{v},S) = \bar{\mathcal{W}}(t)(\mathbf{v}) + \bar{\mathcal{S}}(t)(\mathbf{v},S),\\ &\bar{\mathcal{I}}^c(t)(\mathbf{v}) = \bar{\mathcal{W}}(t)(\mathbf{v}) + \bar{\mathcal{S}}_1(t)(\mathbf{v}).\end{aligned}$$

The following result shows that this reformulation of the problem is indeed equivalent to the original one.

Proposition 6.1. *Let $t \in [0,1]$, and let S_0 be a finite subset of Ω. Then the following three maps are bijections:*

$$\begin{array}{rlrlrl}\mathfrak{A} &\to \mathfrak{A}(t), & \mathfrak{B} &\to \mathfrak{B}(t), & \mathfrak{B}_{S_0} &\to \mathfrak{B}_{S_0}(t)\\ \mathbf{v} &\mapsto \boldsymbol{\psi}(t)\circ\mathbf{v} & (\mathbf{v},S) &\mapsto (\boldsymbol{\psi}(t)\circ\mathbf{v},S) & (\mathbf{v},S) &\mapsto (\boldsymbol{\psi}(t)\circ\mathbf{v},S).\end{array}$$

Moreover, for every $\mathbf{v} \in \mathfrak{A}$,

$$C(\boldsymbol{\psi}(t)\circ\mathbf{v}) = C(\mathbf{v}) \text{ and } \operatorname{im_T}(\boldsymbol{\psi}(t)\circ\mathbf{v},\mathbf{a}) = \boldsymbol{\psi}(t)\left(\operatorname{im_T}(\mathbf{v},\mathbf{a})\right) \text{ for all } \mathbf{a} \in C(\mathbf{v}). \tag{6.2}$$

Proof. For simplicity, we call $\bar{\boldsymbol{\psi}}$ the deformation $\boldsymbol{\psi}(t)$, and analogously for $\bar{\boldsymbol{\phi}}$.

Let $\mathbf{v} \in \mathfrak{A}$ and call $\mathbf{u} := \bar{\boldsymbol{\psi}} \circ \mathbf{v}$. The chain rule (see, e.g., [64, Th. 2.1.11]) shows that $\mathbf{u} \in W^{1,n-1}(\Omega, K)$ and $D\mathbf{u}(\mathbf{x}) = D\bar{\boldsymbol{\psi}}(\mathbf{v}(\mathbf{x}))D\mathbf{v}(\mathbf{x})$ for a.e. $\mathbf{x} \in \Omega$. In particular, $\det D\mathbf{u} > 0$ a.e.

We now prove that $\mathbf{u}|_{\Gamma_D} = \bar{\boldsymbol{\psi}}|_{\Gamma_D}$ in the sense of traces. Only for this proof, let us call Tr the trace operator between the relevant spaces. Let $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ be a sequence in $C^\infty_c(\mathbb{R}^n, \mathbb{R}^n)$ such that $\mathbf{v}_j \to \mathbf{v}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ and a.e. as $j \to \infty$. By trace theory, $\mathbf{v}_j \to \operatorname{Tr}\mathbf{v}$ in $L^{n-1}(\partial\Omega, \mathbb{R}^n)$ as $j \to \infty$, and, for a subsequence (not relabelled), the convergence also holds $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$. In particular, $\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j$ converges $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$ to $\bar{\boldsymbol{\psi}} \circ \operatorname{Tr}\mathbf{v}$ as $j \to \infty$. Using the chain rule we find that $\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j \rightharpoonup \mathbf{u}$ in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ as $j \to \infty$, as can be seen, for example, because they converge a.e., and the sequence $\{\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j\}_{j\in\mathbb{N}}$ is bounded in $W^{1,n-1}(\Omega, \mathbb{R}^n)$. By trace theory, $\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j \to \operatorname{Tr}\mathbf{u}$ in $L^{n-1}(\partial\Omega, \mathbb{R}^n)$ as $j \to \infty$, and, for a subsequence (not relabelled), also $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$. Therefore, the equality $\operatorname{Tr}\mathbf{u} = \bar{\boldsymbol{\psi}} \circ \operatorname{Tr}\mathbf{v}$ holds $\mathcal{H}^{n-1}$-a.e. in $\partial\Omega$, which implies that $\mathbf{u}|_{\Gamma_D} = \bar{\boldsymbol{\psi}}|_{\Gamma_D}$ in the sense of traces.

We pass to the inequality $\mathcal{E}(\mathbf{u}) < \infty$. Let $\mathbf{g} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ and define $\mathbf{f} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ as

$$\mathbf{f}(\mathbf{x},\mathbf{y}) := \operatorname{adj} D\bar{\boldsymbol{\psi}}(\mathbf{y})\, \mathbf{g}(\mathbf{x}, \bar{\boldsymbol{\psi}}(\mathbf{y})), \qquad (\mathbf{x},\mathbf{y}) \in \Omega \times \mathbb{R}^n. \tag{6.3}$$

This definition is made so that $\mathcal{E}(\mathbf{u},\mathbf{g}) = \mathcal{E}(\mathbf{v},\mathbf{f})$. Indeed, we have for a.e. $\mathbf{x} \in \Omega$,

$$\operatorname{cof} D\mathbf{u}(\mathbf{x}) = \operatorname{cof} D\bar{\boldsymbol{\psi}}(\mathbf{v}(\mathbf{x}))\, \operatorname{cof} D\mathbf{v}(\mathbf{x}), \qquad \det D\mathbf{u}(\mathbf{x}) = \det D\bar{\boldsymbol{\psi}}(\mathbf{v}(\mathbf{x}))\, \det D\mathbf{v}(\mathbf{x}),$$

and, consequently,

$$\operatorname{cof} D\mathbf{u}(\mathbf{x}) \cdot D\mathbf{g}(\mathbf{x},\mathbf{u}(\mathbf{x})) = \operatorname{cof} D\mathbf{v}(\mathbf{x}) \cdot \left(\operatorname{adj} D\bar{\boldsymbol{\psi}}(\mathbf{v}(\mathbf{x}))\, D\mathbf{g}(\mathbf{x},\mathbf{u}(\mathbf{x}))\right).$$

Since

$$D\mathbf{f}(\mathbf{x},\mathbf{y}) = \operatorname{adj} D\bar{\boldsymbol{\psi}}(\mathbf{y})\, D\mathbf{g}(\mathbf{x},\bar{\boldsymbol{\psi}}(\mathbf{y})), \qquad (\mathbf{x},\mathbf{y}) \in \Omega \times \mathbb{R}^n.$$

we find that

$$\operatorname{cof} D\mathbf{v}(\mathbf{x}) \cdot D\mathbf{f}(\mathbf{x},\mathbf{v}(\mathbf{x})) = \operatorname{cof} D\mathbf{u}(\mathbf{x}) \cdot D\mathbf{g}(\mathbf{x},\mathbf{u}(\mathbf{x})), \qquad \text{a.e. } \mathbf{x} \in \Omega. \tag{6.4}$$

Moreover, using Piola's identity

$$\sum_{i=1}^{n} \frac{\partial}{\partial y^i} \left(\operatorname{adj} D\bar{\boldsymbol{\psi}}\right)_{ij} = 0, \qquad j \in \{1,\dots,n\}$$

as well as the property $(\det \mathbf{A})\mathbf{1} = \mathbf{A}(\operatorname{adj}\mathbf{A})$ for each $\mathbf{A} \in \mathbb{R}^{n\times n}$, we find that

$$\operatorname{div} \mathbf{f}(\mathbf{x},\mathbf{y}) = \det D\bar{\boldsymbol{\psi}}(\mathbf{y})\, \operatorname{div} \mathbf{g}(\mathbf{x},\bar{\boldsymbol{\psi}}(\mathbf{y})), \qquad (\mathbf{x},\mathbf{y}) \in \Omega \times \mathbb{R}^n,$$

so in particular

$$\operatorname{div} \mathbf{f}(\mathbf{x},\mathbf{v}(\mathbf{x})) = \det D\bar{\boldsymbol{\psi}}(\mathbf{v}(\mathbf{x}))\, \operatorname{div} \mathbf{g}(\mathbf{x},\mathbf{u}(\mathbf{x})), \qquad \text{a.e. } \mathbf{x} \in \Omega. \tag{6.5}$$

Equalities (6.4) and (6.5) show by the definition (2.1) that $\mathcal{E}(\mathbf{u},\mathbf{g}) = \mathcal{E}(\mathbf{v},\mathbf{f})$. Now, definition (6.3) implies that

$$\|\mathbf{f}\|_{L^\infty(\Omega\times\mathbb{R}^n,\mathbb{R}^n)} \le \left\|\operatorname{cof} D\bar{\boldsymbol{\psi}}\right\|_{L^\infty(K,\mathbb{R}^{n\times n})} \|\mathbf{g}\|_{L^\infty(\Omega\times\mathbb{R}^n,\mathbb{R}^n)} \,.$$

As a consequence, using the definition (2.2), we find that

$$\mathcal{E}(\mathbf{u}) \le \left\|\operatorname{cof} D\bar{\boldsymbol{\psi}}\right\|_{L^\infty(K,\mathbb{R}^{n\times n})} \mathcal{E}(\mathbf{v}) < \infty.$$

Now we show that $\mathbf{u}$ satisfies INV and that (6.2) holds. Fix $\mathbf{x}_0 \in \Omega$. For this proof we assume, without loss of generality, that the set K of Subsection 5.2 is a closed ball. As before, let $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^n,\mathbb{R}^n)$ converging to $\mathbf{v}$ in $W^{1,n-1}(\Omega,\mathbb{R}^n)$ and a.e., and satisfying

$$\mathbf{v}_j(\Omega) \subset \left\{ y \in \mathbb{R}^n : \operatorname{dist}(y,K) \le \frac{1}{j} \right\} \qquad \text{for each } j \in \mathbb{N}. \tag{6.6}$$

Then $\operatorname{cof} D\mathbf{v}_j \to \operatorname{cof} D\mathbf{v}$ in $L^1(\Omega,\mathbb{R}^{n\times n})$ as $j\to\infty$. Thanks to Lemma 2.6,

$$\deg(\mathbf{v}_j,\partial B(\mathbf{x},r),\cdot) \to \deg(\mathbf{v},\partial B(\mathbf{x},r),\cdot) \quad \text{a.e.} \tag{6.7}$$

As $\bar{\boldsymbol{\phi}}$ maps negligible sets into negligible sets, we have that

$$\deg(\mathbf{v}_j,\partial B(\mathbf{x},r),\bar{\boldsymbol{\psi}}(\mathbf{y})) \to \deg(\mathbf{v},\partial B(\mathbf{x},r),\bar{\boldsymbol{\psi}}(\mathbf{y})) \quad \text{a.e. } \mathbf{y} \in K$$

as $j\to\infty$. On the other hand, for a.e. $\mathbf{y} \in \mathbb{R}^n \setminus K$ we have $\deg(\mathbf{v}_j,\partial B(\mathbf{x},r),\mathbf{y}) \to 0$ as $j\to\infty$, thanks to (6.6), whereas $\deg(\mathbf{v},\partial B(\mathbf{x},r),\mathbf{y}) = 0$ for a.e. $\mathbf{y} \in \mathbb{R}^n \setminus K$ (see Proposition 2.1). We obtain in total that

$$\deg(\mathbf{v}_j,\partial B(\mathbf{x},r),\bar{\boldsymbol{\psi}}(\cdot)) \to \deg(\mathbf{v},\partial B(\mathbf{x},r),\bar{\boldsymbol{\psi}}(\cdot)) \quad \text{a.e.}$$

As before, the following convergences hold

$$\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j \rightharpoonup \mathbf{u} \text{ in } W^{1,n-1}(\Omega,\mathbb{R}^n) \text{ and a.e.}, \qquad \operatorname{cof} D(\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j) \rightharpoonup \operatorname{cof} D\mathbf{u} \text{ in } L^1(\Omega,\mathbb{R}^{n\times n})$$

as $j \to \infty$, and, moreover, $\sup_{j\in\mathbb{N}} \|\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j\|_{L^\infty(\Omega,\mathbb{R}^n)} < \infty$. Therefore, again because of Lemma 2.6, for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$,

$$\deg(\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j, \partial B(\mathbf{x}, r), \cdot) \to \deg(\mathbf{u}, \partial B(\mathbf{x}, r), \cdot) \quad \text{a.e.}$$

As before, this implies that

$$\deg(\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j, \partial B(\mathbf{x}, r), \bar{\boldsymbol{\psi}}(\cdot)) \to \deg(\mathbf{u}, \partial B(\mathbf{x}, r), \bar{\boldsymbol{\psi}}(\cdot)) \quad \text{a.e.} \tag{6.8}$$

The multiplicative property of the Brouwer degree together with the fact that $\bar{\boldsymbol{\psi}}$ is an orientation-preserving diffeomorphism implies (see, e.g., [24, Th. 2.10 and Eq. (0.1)]) that for each $j \in \mathbb{N}$,

$$\deg\left(\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j, \partial B(\mathbf{x}, r), \bar{\boldsymbol{\psi}}(\mathbf{y})\right) = \deg\left(\mathbf{v}_j, \partial B(\mathbf{x}, r), \mathbf{y}\right), \qquad \mathbf{y} \in \mathbb{R}^n \setminus \mathbf{v}_j(\partial B(\mathbf{x}, r)).$$

As before, the set $\mathbf{v}_j(\partial B(\mathbf{x}, r))$ is negligible for all $j \in \mathbb{N}$ and all $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$. Therefore, for all $j \in \mathbb{N}$,

$$\deg\left(\bar{\boldsymbol{\psi}} \circ \mathbf{v}_j, \partial B(\mathbf{x}, r), \bar{\boldsymbol{\psi}}(\cdot)\right) = \deg\left(\mathbf{v}_j, \partial B(\mathbf{x}, r), \cdot\right) \quad \text{a.e.} \tag{6.9}$$

From (6.7), (6.8) and (6.9) we conclude that

$$\deg(\mathbf{v}, \partial B(\mathbf{x}, r), \cdot) = \deg(\mathbf{u}, \partial B(\mathbf{x}, r), \bar{\boldsymbol{\psi}}(\cdot)) \qquad \text{a.e.}$$

This readily implies (see Definition 2.2) that $\operatorname{im_T}(\mathbf{u}, B(\mathbf{x}, r)) = \bar{\boldsymbol{\psi}}\left(\operatorname{im_T}(\mathbf{v}, B(\mathbf{x}, r))\right)$ for a.e. $r \in (0, \operatorname{dist}(\mathbf{x}, \partial\Omega))$. Consequently, $\mathbf{u}$ satisfies condition INV (see Definition 2.3), and equalities (6.2) hold (see Lemma 2.7). Therefore, $\mathbf{u} \in \mathfrak{A}(t)$.

As a consequence of the equality $C(\mathbf{u}) = C(\mathbf{v})$, if now $(\mathbf{v}, S) \in \mathfrak{B}$ then $(\mathbf{u}, S) \in \mathfrak{B}(t)$, and if $(\mathbf{v}, S) \in \mathfrak{B}_{S_0}$ then $(\mathbf{u}, S) \in \mathfrak{B}_{S_0}(t)$. Similarly, one proves that for any $\mathbf{u} \in \mathfrak{A}(t)$, the map $\bar{\boldsymbol{\phi}} \circ \mathbf{u}$ is in $\mathfrak{A}$. Hence the first map is a bijection, and, consequently, so are the other two. □

The analogue of Proposition 6.1 was proved in [51, Th. 6.1] under the assumption that $\mathbf{v} \in W^{1,p}(\Omega, \mathbb{R}^n)$ with $p > n - 1$.

Although it is not used in the reformulation of the problem presented in this section, we show now the dual counterpart of Proposition 6.1, in which the invariance of condition INV under right composition is proved. It will be used in proof of the existence of quasistatic evolutions. It is the analogue of [51, Th. 9.1] but now covering the critical exponent $p = n - 1$.

PROPOSITION 6.2. *Let S_0 be a finite subset of Ω. Let $\boldsymbol{\eta}$ be a C^2 diffeomorphism from Ω onto itself that coincides with the identity in a neigbourhood of $\partial\Omega$. Then the following three maps are bijections:*

$$\begin{array}{rlcrlcrl} \mathfrak{A} & \to \mathfrak{A}, & \qquad & \mathfrak{B} & \to \mathfrak{B}, & \qquad & \mathfrak{B}_{S_0} & \to \mathfrak{B}_{\boldsymbol{\eta}^{-1}(S_0)} \\ \mathbf{v} & \mapsto \mathbf{v} \circ \boldsymbol{\eta} & & (\mathbf{v}, S) & \mapsto (\mathbf{v} \circ \boldsymbol{\eta}, \boldsymbol{\eta}^{-1}(S)) & & (\mathbf{v}, S) & \mapsto (\mathbf{v} \circ \boldsymbol{\eta}, \boldsymbol{\eta}^{-1}(S)). \end{array}$$

Moreover, for every $\mathbf{v} \in \mathfrak{A}$,

$$C(\mathbf{v} \circ \boldsymbol{\eta}) = \boldsymbol{\eta}^{-1}(C(\mathbf{v})) \quad \textit{and} \quad \operatorname{im_T}(\mathbf{v} \circ \boldsymbol{\eta}, \boldsymbol{\eta}^{-1}(\mathbf{a})) = \operatorname{im_T}(\mathbf{v}, \mathbf{a}) \ \textit{for all } \mathbf{a} \in C(\mathbf{v}).$$

Proof. Let $\mathbf{v} \in \mathfrak{A}$ and call $\mathbf{u} = \mathbf{v} \circ \boldsymbol{\eta}$. By the chain rule, $\mathbf{u} \in W^{1,n-1}(\Omega, K)$ and $\det D\mathbf{u} > 0$ a.e. As $\boldsymbol{\eta}$ coincides with the identity in a neigbourhood of $\partial\Omega$, then $\mathbf{u}|_{\partial\Omega} = \mathbf{v}|_{\partial\Omega}$ in the sense of traces, and, hence, $\mathbf{u}|_{\Gamma_D} = \mathbf{id}|_{\Gamma_D}$.

To show that $\mathcal{E}(\mathbf{u}) < \infty$, similarly to Proposition 6.1, for each $\mathbf{g} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ we define $\mathbf{f} \in C^1_c(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$ as

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) := \mathbf{g}(\boldsymbol{\eta}^{-1}(\mathbf{x}), \mathbf{y}), \qquad (\mathbf{x}, \mathbf{y}) \in \Omega \times \mathbb{R}^n.$$

Then it is easy to show that $\mathcal{E}(\mathbf{u}, \mathbf{g}) = \mathcal{E}(\mathbf{v}, \mathbf{f})$ and $\|\mathbf{f}\|_{L^\infty(\Omega\times\mathbb{R}^n,\mathbb{R}^n)} = \|\mathbf{g}\|_{L^\infty(\Omega\times\mathbb{R}^n,\mathbb{R}^n)}$, which implies $\mathcal{E}(\mathbf{u}) = \mathcal{E}(\mathbf{v}) < \infty$.

Finally, a similar argument to that of Proposition 6.1 (in fact, slightly easier) shows that

$$\deg\left(\mathbf{u}, \boldsymbol{\eta}^{-1}(\partial U), \cdot\right) = \deg\left(\mathbf{v}, \partial U, \cdot\right) \quad \text{a.e.}$$

for any open set $U \subset\subset \Omega$ with a C^2 boundary. This implies that $\operatorname{im_T}(\mathbf{u}, \boldsymbol{\eta}^{-1}(U)) = \operatorname{im_T}(\mathbf{v}, U)$ (see [38, Sect. 2] for an explanation of why it is convenient to use C^2 open sets). As a consequence, $\mathbf{u}$ satisfies condition INV, $\operatorname{im_T}(\mathbf{u}, \boldsymbol{\eta}^{-1}(\mathbf{a})) = \operatorname{im_T}(\mathbf{v}, \mathbf{a})$ for all $\mathbf{a} \in \Omega$, and $C(\mathbf{u}) = \boldsymbol{\eta}^{-1}(C(\mathbf{v}))$.

We thus infer that the three maps of the statement are well defined. Arguing with $\mathbf{v} \circ \boldsymbol{\eta}^{-1}$, we conclude that they are bijections. $\square$

Naturally, properties (W0)–(W6) of the function W (see Subsection 5.2) induce analogous properties for the function $\bar{W} : [0,1] \times \Omega \times K \times GL_+(n) \to \mathbb{R}$ defined as

$$\bar{W}(t, \mathbf{x}, \mathbf{y}, \mathbf{F}) := W(\mathbf{x}, D\boldsymbol{\psi}(t)(\mathbf{y})\, \mathbf{F}),$$

all of which are listed in [18] (see also [42]). The definition of $\bar{W}$ is made so that for all $t \in [0,1]$ and $\mathbf{v} \in \mathfrak{A}$,

$$\bar{\mathcal{W}}(t)(\mathbf{v}) = \int_\Omega \bar{W}(t, \mathbf{x}, \mathbf{v}(\mathbf{x}), D\mathbf{v}(\mathbf{x}))\, \mathrm{d}\mathbf{x}. \tag{6.10}$$

We state here those properties $\bar{W}$ of that are explicitly used in our construction. There exist $\gamma \in (0,1)$, $b \in L^1(\Omega)$ with $b \geq 0$ and $c > 0$ such that:

($\bar{\text{W}}$3) $\sup\limits_{t\in[0,1]} \bar{\mathcal{W}}(t)(\mathbf{id}) < \infty$.

($\bar{\text{W}}$4) There exist an increasing function $\bar{h}_1 : (0,\infty) \to [0,\infty)$ and a convex function $\bar{h}_2 : (0,\infty) \to \mathbb{R}$ such that

$$\lim_{t\to\infty} \frac{\bar{h}_1(t)}{t} = \lim_{t\to\infty} \frac{\bar{h}_2(t)}{t} = \lim_{t\searrow 0} \bar{h}_2(t) = \infty \tag{6.11}$$

and

$$\bar{W}(t, \mathbf{x}, \mathbf{y}, \mathbf{F}) \geq b(\mathbf{x}) + c\, |\mathbf{F}|^{n-1} + \bar{h}_1\left(|\operatorname{cof} \mathbf{F}|\right) + \bar{h}_2(\det \mathbf{F})$$

for a.e. $\mathbf{x} \in \Omega$ and all $(t, \mathbf{y}, \mathbf{F}) \in [0,1] \times K \times GL_+(n)$.

($\bar{\text{W}}$5) For a.e. $\mathbf{x} \in \Omega$ and every $t \in [0,1]$, $\mathbf{y} \in K$ and $\mathbf{F}, \mathbf{G} \in GL_+(n)$ with $|\mathbf{G} - \mathbf{1}| < \gamma$,

$$\bar{W}(t, \mathbf{x}, \mathbf{y}, \mathbf{F}\mathbf{G}) + b(\mathbf{x}) \leq c\left(\bar{W}(t, \mathbf{x}, \mathbf{y}, \mathbf{F}) + b(\mathbf{x})\right).$$

($\bar{\text{W}}$6) For a.e. $\mathbf{x} \in \Omega$ and every $t_1, t_2 \in [0,1]$, $\mathbf{y} \in K$ and $\mathbf{F} \in GL_+(n)$ with $|t_1 - t_2| < \gamma$,

$$\left|\bar{W}(t_1, \mathbf{x}, \mathbf{y}, \mathbf{F}) - \bar{W}(t_2, \mathbf{x}, \mathbf{y}, \mathbf{F})\right| \leq c\left(\bar{W}(t_1, \mathbf{x}, \mathbf{y}, \mathbf{F}) + b(\mathbf{x})\right) |t_1 - t_2|.$$

($\bar{\mathrm{W}}$7) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for a.e. $\mathbf{x} \in \Omega$ and every $t_1, t_2 \in [0,1]$, $\mathbf{y} \in K$ and $\mathbf{F} \in GL_+(n)$ with $|t_1 - t_2| < \delta$,

$$\left| D_1 \bar{W}(t_1, \mathbf{x}, \mathbf{y}, \mathbf{F}) - D_1 \bar{W}(t_2, \mathbf{x}, \mathbf{y}, \mathbf{F}) \right| \leq \varepsilon \left(\bar{W}(t_1, \mathbf{x}, \mathbf{y}, \mathbf{F}) + b(\mathbf{x}) \right),$$

where D_1 indicates derivative with respect to the first variable.

($\bar{\mathrm{W}}$8) For a.e. $\mathbf{x} \in \Omega$ and every $t \in [0,1]$, $\mathbf{y}_1, \mathbf{y}_2 \in K$ and $\mathbf{F} \in GL_+(n)$,

$$\bar{W}(t, \mathbf{x}, \mathbf{y}_2, \mathbf{F}) + b(\mathbf{x}) \leq c \left(\bar{W}(t, \mathbf{x}, \mathbf{y}_1, \mathbf{F}) + b(\mathbf{x}) \right).$$

Properties ($\bar{\mathrm{W}}$3) and ($\bar{\mathrm{W}}$5)–($\bar{\mathrm{W}}$8) are proved in [18]. As short proof of ($\bar{\mathrm{W}}$4) goes as follows. Consider the funtions h_1 and h_2 of (W4), let

$$m := \min \left\{ \inf_{t \in [0,1]} \inf \left| \operatorname{cof} D\boldsymbol{\phi}(t) \right|^{-1}, \inf_{t \in [0,1]} \inf \det D\boldsymbol{\psi}(t) \right\},$$

note that $0 < m < 1$, and define $\bar{h}_1(\xi) := h_1(m\xi)$ for all $\xi \in (0,\infty)$. Now, let $\xi_0 \in (0,\infty)$ be a point where h_2 is minimum, and define

$$\bar{h}_2(\xi) := \begin{cases} h_2(\xi) & \text{if } 0 < \xi < \xi_0, \\ h_2(\xi_0) & \text{if } \xi_0 \leq \xi \leq \frac{\xi_0}{m}, \\ h_2(m\xi) & \text{if } \xi > \frac{\xi_0}{m}. \end{cases}$$

It is easy to check that the functions $\bar{h}_1$ and $\bar{h}_2$ satisfy the required properties.

The following result shows that the function $\bar{\mathcal{S}}_1(\cdot)(\mathbf{v})$ is of class C^1. We will use the operator div^M of tangential divergence over an $\mathcal{H}^{n-1}$-rectifiable manifold M, the definition of which can be found, e.g., in [1, Def. 7.27].

Proposition 6.3. *For each $\mathbf{v} \in \mathfrak{A}$, the function $\bar{\mathcal{S}}_1(\cdot)(\mathbf{v})$ is of class C^1 in $[0,1]$, and for each $t \in [0,1]$,*

$$\bar{\mathcal{S}}_1'(t)(\mathbf{v}) = \sum_{\mathbf{a} \in C(\mathbf{v})} \kappa_1(\mathbf{a}) \int_{\boldsymbol{\psi}(t)(\partial^* \operatorname{im_T}(\mathbf{v},\mathbf{a}))} \operatorname{div}^{\boldsymbol{\psi}(t)(\partial^* \operatorname{im_T}(\mathbf{v},\mathbf{a}))} \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t) \right) \mathrm{d}\mathcal{H}^{n-1}. \tag{6.12}$$

Moreover, there exist $c, c_1, c_2 > 0$, depending only on $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$, such that for all $t \in [0,1]$,

$$\left| \bar{\mathcal{S}}_1'(t)(\mathbf{v}) \right| \leq c\, \mathcal{S}_1(\mathbf{v}), \qquad c_1 \bar{\mathcal{S}}_1(t)(\mathbf{v}) \leq \mathcal{S}_1(\mathbf{v}) \leq c_2 \bar{\mathcal{S}}_1(t)(\mathbf{v}).$$

Furthermore, there exists a modulus of continuity $\omega : [0,1] \to [0,\infty)$ such that for all $t_1, t_2 \in [0,1]$ and all $\mathbf{v} \in \mathfrak{A}$, we have

$$\left| \bar{\mathcal{S}}_1'(t_2)(\mathbf{v}) - \bar{\mathcal{S}}_1'(t_1)(\mathbf{v}) \right| \leq \mathcal{S}_1(\mathbf{v})\, \omega\left(|t_2 - t_1| \right).$$

Proof. From (6.2) we infer that for each $t \in [0,1]$,

$$\begin{aligned} \bar{\mathcal{S}}_1(t)(\mathbf{v}) = \mathcal{S}_1(\boldsymbol{\psi}(t) \circ \mathbf{v}) &= \sum_{\mathbf{a} \in C(\mathbf{v})} \kappa_1(\mathbf{a}) \operatorname{Per}\left(\boldsymbol{\psi}(t) \left(\operatorname{im_T}(\mathbf{v},\mathbf{a}) \right) \right) \\ &= \sum_{\mathbf{a} \in C(\mathbf{v})} \kappa_1(\mathbf{a})\, \mathcal{H}^{n-1}\left(\boldsymbol{\psi}(t) \left(\partial^* \operatorname{im_T}(\mathbf{v},\mathbf{a}) \right) \right). \end{aligned} \tag{6.13}$$

Fix $t_0 \in [0,1]$ and $\mathbf{a} \in C(\mathbf{v})$, and call $M := \partial^* \operatorname{im_T}(\mathbf{v}, \mathbf{a})$. Thanks to Lemma 2.7 iii), we have that $M \subset K$ $\mathcal{H}^{n-1}$-a.e. Define the set $M_0 := \boldsymbol{\psi}(t_0)(M)$ and the function

$$\mathbf{F} : [0,1] \to C^1(K, \mathbb{R}^n), \qquad \mathbf{F}(t) := \boldsymbol{\psi}(t) \circ \boldsymbol{\phi}(t_0), \quad t \in [0,1].$$

Clearly,

$$\boldsymbol{\psi}(t)(M) = \mathbf{F}(t)(M_0), \qquad \mathbf{F}(t_0) = \mathbf{id}, \qquad \mathbf{F}'(t_0) = \boldsymbol{\psi}'(t_0) \circ \boldsymbol{\phi}(t_0),$$

where the last equality is easily seen using the regularity of $\boldsymbol{\psi}$. The first variation area formula (see, e.g. [56, §9] or [1, Sect. 7.3]) shows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \mathcal{H}^{n-1}(\mathbf{F}(t)(M_0)) = \int_{M_0} \operatorname{div}^{M_0} \mathbf{F}'(t_0)\, \mathrm{d}\mathcal{H}^{n-1}.$$

Indeed, although it is customary to impose in the proof of the first variation of the area that $\mathbf{F}$ is a compactly supported perturbation of the identity, the same proof is valid for a general family of diffeomorphisms, as is $\mathbf{F}$ in our case (see, if necessary, e.g., [58, Th. 9.11] or [20, Sect. 2.1]). Repeating this argument for any $\mathbf{a} \in C(\mathbf{v})$ and any $t_0 \in [0,1]$, we obtain formula (6.12).

We pass now to prove the existence of the modulus of continuity. As before, fix $\mathbf{a} \in C(\mathbf{v})$ and call $M = \partial^* \operatorname{im_T}(\mathbf{v}, \mathbf{a})$. Define now, for each $t \in [0,1]$ and $\mathcal{H}^{n-1}$-a.e. $\mathbf{y} \in M$,

$$H(t) := \operatorname{div}^{\boldsymbol{\psi}(t)(M)} \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t)\right), \quad \bar{H}(t)(\mathbf{y}) := H(t)(\boldsymbol{\psi}(t)(\mathbf{y}))\, |\operatorname{cof} D\boldsymbol{\psi}(t)(\mathbf{y})\, \boldsymbol{\nu}_M(\mathbf{y})| .$$

where $\boldsymbol{\nu}_M(\mathbf{y})$ is the unit normal of M at $\mathbf{y}$. Thus, $H(t)$ is a function from K to $\mathbb{R}$, and $\bar{H}(t)$ is a function defined $\mathcal{H}^{n-1}$-a.e. from M to $\mathbb{R}$.

Let $t_1, t_2 \in [0,1]$. By Federer's [22] area formula for surfaces (see also [38, Prop. 2.9]), for $i = 1, 2$ we have

$$\int_{\boldsymbol{\psi}(t_i)(M)} H(t_i)(\mathbf{y})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}) = \int_M \bar{H}(t_i)(\mathbf{y})\, \mathrm{d}\mathcal{H}^{n-1}(\mathbf{y}). \tag{6.14}$$

The regularity of $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ shows, successively, that

the function $t \mapsto \boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t)$	is in $C([0,1], C^1(K, \mathbb{R}^n))$,
the function $t \mapsto \operatorname{div}\left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t)\right)$	is in $C([0,1], C(K))$,
the function $t \mapsto \operatorname{cof} D\boldsymbol{\psi}(t)$	is in $C^1([0,1], C(K, \mathbb{R}^{n\times n}))$.

Assuming for a moment that M were a C^1 manifold, we then would have that the functions

$$t \mapsto \operatorname{div}^M \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t)\right)$$

and then H would be in $C([0,1], C(K))$. Therefore, by compactness and continuity, the function

$$(t, \mathbf{y}, \boldsymbol{\nu}) \mapsto H(t)(\boldsymbol{\psi}(t)(\mathbf{y}))\, |\operatorname{cof} D\boldsymbol{\psi}(t)(\mathbf{y})\, \boldsymbol{\nu}|$$

would be uniformly continuous in $[0,1] \times K \times \mathbb{S}^{n-1}$. Therefore, there would exist a modulus of continuity ω, depending on $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$, but not on M or $\mathbf{v}$, such that

$$\left\|\bar{H}(t_2) - \bar{H}(t_1)\right\|_{C(M)} \leq \omega\left(|t_2 - t_1|\right), \qquad t_1, t_2 \in [0,1].$$

Hence,

$$\left| \int_M \bar{H}(t_2) \, \mathrm{d}\mathcal{H}^{n-1} - \int_M \bar{H}(t_1) \, \mathrm{d}\mathcal{H}^{n-1} \right| \leq \mathcal{H}^{n-1}(M) \, \omega \left(|t_2 - t_1| \right). \tag{6.15}$$

In truth, M is only $\mathcal{H}^{n-1}$-rectifiable, so it is a subset, up to $\mathcal{H}^{n-1}$-null sets, of a C^1 manifold. Standard arguments for rectifiable sets, together with the fact that ω does not depend on M, show that (6.15) is indeed true without the assumption of C^1 regularity of M. Therefore, using (6.14) we find that

$$\left| \int_{\boldsymbol{\psi}(t_2)(M)} H(t_2) \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\boldsymbol{\psi}(t_1)(M)} H(t_1) \, \mathrm{d}\mathcal{H}^{n-1} \right| \leq \mathcal{H}^{n-1}(M) \, \omega \left(|t_2 - t_1| \right).$$

Repeating this argument for any $\mathbf{a} \in C(\mathbf{v})$, having in mind formulas (6.12) and (6.13), we obtain that

$$\begin{aligned} \left| \bar{\mathcal{S}}_1'(t_2)(\mathbf{v}) - \bar{\mathcal{S}}_1'(t_1)(\mathbf{v}) \right| &\leq \sum_{\mathbf{a} \in C(\mathbf{v})} \kappa_1(\mathbf{a}) \, \mathcal{H}^{n-1}(\partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{v}, \mathbf{a})) \, \omega \left(|t_2 - t_1| \right) \\ &= \mathcal{S}_1(\mathbf{v}) \, \omega \left(|t_2 - t_1| \right). \end{aligned}$$

Finally, to give a bound on $\bar{\mathcal{S}}_1'(t)$, we just observe that

$$\left| \operatorname{div}^{\boldsymbol{\psi}(t)(\partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{v}, \mathbf{a}))} \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t) \right) \right| \leq C_n \left| D \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t) \right) \right|,$$

for some dimensional constant C_n, and that by a standard property of the Hausdorff measure (see, e.g., [1, Prop. 2.49]),

$$\mathcal{H}^{n-1} \left(\boldsymbol{\psi}(t)(\partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{v}, \mathbf{a})) \right) \leq c_1(t) \, \mathcal{H}^{n-1} \left(\partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{v}, \mathbf{a}) \right),$$

where $c_1(t) > 0$ is the Lipschitz constant of $\boldsymbol{\psi}(t)$. Recalling (6.12) and (6.13), we find that there exists a constant $c > 0$, depending only on $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$, such that

$$\left| \bar{\mathcal{S}}_1'(t)(\mathbf{v}) \right| \leq c \sum_{\mathbf{a} \in C(\mathbf{v})} \kappa_1(\mathbf{a}) \mathcal{H}^{n-1} \left(\partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{v}, \mathbf{a}) \right) = c \, \mathcal{S}_1(\mathbf{v}).$$

A similar argument starting again from (6.13) shows that

$$c_1 \, \bar{\mathcal{S}}_1(t)(\mathbf{v}) \leq \mathcal{S}_1(\mathbf{v}) \leq c_2 \, \bar{\mathcal{S}}_1(t)(\mathbf{v}),$$

where c_1 is the maximum in $t \in [0,1]$ of the Lipschitz constant of $\boldsymbol{\psi}(t)$, and c_2 is the maximum in $t \in [0,1]$ of the Lipschitz constant of $\boldsymbol{\phi}(t)$. $\square$

COROLLARY 6.4. *For each $\mathbf{v} \in \mathfrak{A}$, the function $\bar{\mathcal{I}}^c(\cdot)(\mathbf{v})$ is of class C^1 in $[0,1]$ and for each $t \in [0,1]$,*

$$\begin{aligned} \left(\bar{\mathcal{I}}^c \right)' (t)(\mathbf{v}) = & \int_\Omega D_2 W \left(\mathbf{x}, D \left(\boldsymbol{\psi}(t) \circ \mathbf{v} \right) \right) \cdot D \left(\boldsymbol{\psi}'(t) \circ \mathbf{v} \right) \mathrm{d}\mathbf{x} \\ & + \sum_{\mathbf{a} \in C(\mathbf{v})} \kappa_1(\mathbf{a}) \int_{\boldsymbol{\psi}(t)(\partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{v}, \mathbf{a}))} \operatorname{div}^{\boldsymbol{\psi}(t)(\partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{v}, \mathbf{a}))} \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t) \right) \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

Moreover, for every $M > 0$ there exists a modulus of continuity $\omega_M : [0,1] \to [0,\infty)$ such that for every $\mathbf{v} \in \mathfrak{A}$ with $\bar{\mathcal{I}}^c(0)(\mathbf{v}) \leq M$, we have

$$\left| (\bar{\mathcal{I}}^c)'(t_1)(\mathbf{v}) - (\bar{\mathcal{I}}^c)'(t_2)(\mathbf{v}) \right| \leq \omega_M \left(|t_1 - t_2| \right), \qquad t_1, t_2 \in [0,1].$$

Proof. The function $\bar{\mathcal{W}}(\cdot)(\mathbf{v})$ is of class C^1 thanks to property ($\bar{\mathrm{W}}$6) (see, if necessary [18, Rk. 2.8]), and its derivative was calculated in [18, Eq. (2.26)]. The existence of the modulus of continuity is a consequence of ($\bar{\mathrm{W}}$7). The corresponding properties for $\bar{\mathcal{S}}_1(\cdot)(\mathbf{v})$ follow from Proposition 6.3. □

7. Convergence in the sense of cavitation. The following two definitions explain the notions of convergence of sets and of functions that will be needed in the sequel.

DEFINITION 7.1. *Let $\{S_j\}_{j\in\mathbb{N}}$ be a sequence of finite subsets of Ω, and let $S \subset \bar{\Omega}$. We say that $\{S_j\}_{j\in\mathbb{N}}$ converges to S componentwise when there exist $M \in \mathbb{N}$, points $\mathbf{a}_1, \dots, \mathbf{a}_M \in \bar{\Omega}$ and, for each $j \in \mathbb{N}$, points $\mathbf{a}_1(j), \dots, \mathbf{a}_M(j) \in \Omega$ such that*
1) $S_j = \{\mathbf{a}_1(j), \dots, \mathbf{a}_M(j)\}$ and $\mathcal{H}^0(S_j) = M$ for each $j \in \mathbb{N}$.
2) $\lim_{j\to\infty} \mathbf{a}_i(j) = \mathbf{a}_i$ for each $i \in \{1, \dots, M\}$, and $S = \{\mathbf{a}_1, \dots, \mathbf{a}_M\}$.

DEFINITION 7.2. *Let $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ be a sequence of functions in $W^{1,n-1}(\Omega, K)$, and let $\mathbf{u} \in W^{1,n-1}(\Omega, K)$. We say that $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ converges to $\mathbf{u}$ in the sense of cavitation when*

$$\begin{aligned}
&\mathbf{u}_j \to \mathbf{u} \textit{ a.e. }, \qquad \mathbf{u}_j \rightharpoonup \mathbf{u} \textit{ in } W^{1,n-1}(\Omega, \mathbb{R}^n),\\
&\operatorname{cof} D\mathbf{u}_j \rightharpoonup \operatorname{cof} D\mathbf{u} \textit{ in } L^1(\Omega, \mathbb{R}^{n\times n}), \qquad \det D\mathbf{u}_j \rightharpoonup \det D\mathbf{u} \textit{ in } L^1(\Omega),\\
&\sum_{\mathbf{a}\in C(\mathbf{u}_j)} \mathcal{L}^n(\operatorname{im_T}(\mathbf{u}_j, \mathbf{a}))\, \delta_{\mathbf{a}} \overset{*}{\rightharpoonup} \sum_{\mathbf{a}\in C(\mathbf{u})} \mathcal{L}^n(\operatorname{im_T}(\mathbf{u}, \mathbf{a}))\, \delta_{\mathbf{a}} \quad \textit{in } \mathcal{M}(\Omega),
\end{aligned}$$

as $j \to \infty$.

These definitions enjoy good compactness a lower semicontinuity properties, as shown in the following results.

LEMMA 7.3. *Let S_0 be a finite subset of Ω, and let $\{S_j\}_{j\in\mathbb{N}}$ be a sequence of finite subsets of Ω such that*

$$\sup_{j\in\mathbb{N}} \mathcal{H}^0(S_j) < \infty.$$

and $S_0 \subset S_j$ for all $j \in \mathbb{N}$. Then there exists a subsequence (not relabelled) and a finite set $S \subset \bar{\Omega}$ such that $S_j \to S$ componentwise, as $j \to \infty$. Moreover, for any such subsequences and any such limits, we have that $S_0 \subset S$ and

$$\mathcal{S}_0(S \cap \Omega) \le \liminf_{j\to\infty} \mathcal{S}_0(S_j). \tag{7.1}$$

Proof. By selecting a subsequence we can assume, without loss of generality, that the sequence $\{\mathcal{H}^0(S_j)\}_{j\in\mathbb{N}}$ is constant, say $M \in \mathbb{N}$, and $\lim_{j\to\infty} \mathcal{S}_0(S_j)$ exists. For each $j \in \mathbb{N}$, choose an ordering of the elements of S_j so that its first elements are precisely those of S_0, and identify this ordered S_j with an element of $\mathbb{R}^M$. For a further subsequence, $S_j \to S$ as $j \to \infty$ in the topology of $\mathbb{R}^M$, for some $S \subset \bar{\Omega}$ with $S_0 \subset S$. Then $S_j \to S$ as $j \to \infty$ componentwise. Using the notation of Definition 7.1, we have that

$$\mathcal{S}_0(S \cap \Omega) \le \sum_{i=1}^M \kappa_0(\mathbf{a}_i) = \sum_{i=1}^M \lim_{j\to\infty} \kappa_0(\mathbf{a}_i(j)) = \lim_{j\to\infty} \mathcal{S}_0(S_j),$$

due to the continuity of κ_0. The result is proved. □

We now show that natural bounds on the deformations provide compactness in the sense of Definition 7.2. The following result is in fact part of the proof of [38, Th. 8.5], but we write it out for the sake of completeness.

PROPOSITION 7.4. *Let $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ be a sequence of functions in $W^{1,n-1}(\Omega,K)$. Assume that $\mathbf{v}_j$ satisfies INV and $\det D\mathbf{v}_j > 0$ a.e. for each $j \in \mathbb{N}$. Suppose that there exist an increasing function $\bar{h}_1 : (0,\infty) \to [0,\infty)$ and a convex function $\bar{h}_2 : (0,\infty) \to \mathbb{R}$ satisfying (6.11) and*

$$\sup_{j\in\mathbb{N}} \left[\|D\mathbf{v}_j\|_{L^{n-1}(\Omega,\mathbb{R}^{n\times n})} + \left\|\bar{h}_1\left(|\mathrm{cof}\, D\mathbf{v}_j|\right)\right\|_{L^1(\Omega)} + \left\|\bar{h}_2(\det D\mathbf{v}_j)\right\|_{L^1(\Omega)} + \mathcal{E}(\mathbf{v}_j) \right] < \infty. \tag{7.2}$$

Then there exist $\mathbf{v} \in W^{1,n-1}(\Omega,K)$ satisfying INV, $\det D\mathbf{v} > 0$ a.e. and $\mathcal{E}(\mathbf{v}) < \infty$, and a subsequence (not relabelled) such that $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ converges to $\mathbf{v}$ in the sense of cavitation.

Proof. The sequence $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ is bounded in $W^{1,n-1}(\Omega,\mathbb{R}^n)$, because of the bound of the gradient given by (7.2) and of the L^∞ bound given by the set K. Therefore, there exists $\mathbf{v} \in W^{1,n-1}(\Omega,K)$ and a subsequence such that

$$\mathbf{v}_j \rightharpoonup \mathbf{v} \text{ in } W^{1,n-1}(\Omega,\mathbb{R}^n) \quad \text{and} \quad \mathbf{v}_j \to \mathbf{v} \text{ a.e.}$$

as $j \to \infty$. Thanks to bounds (7.2) and De La Vallée Poussin's criterion, there exist $\boldsymbol{\vartheta} \in L^1(\Omega,\mathbb{R}^{n\times n})$ and $\theta \in L^1(\Omega)$ such that, for a subsequence (not relabelled),

$$\mathrm{cof}\, D\mathbf{v}_j \rightharpoonup \boldsymbol{\vartheta} \;\text{ in }\; L^1(\Omega,\mathbb{R}^{n\times n}), \qquad \det D\mathbf{v}_j \rightharpoonup \theta \;\text{ in }\; L^1(\Omega).$$

as $j \to \infty$. As $\{\mathrm{cof}\, D\mathbf{v}_j\}_{j\in\mathbb{N}}$ converges weakly to $\boldsymbol{\vartheta}$ in $L^1(\Omega,\mathbb{R}^{n\times n})$, and the sequence $\{D\mathbf{v}_j\}_{j\in\mathbb{N}}$ is bounded in $L^{n-1}(\Omega,\mathbb{R}^{n\times n})$, by a standard result on weak continuity of minors (see, e.g., [6, Th. 4.11]), we obtain $\boldsymbol{\vartheta} = \mathrm{cof}\, D\mathbf{v}$ a.e. Thanks to Lemma 2.8, we have that $\theta = \det D\mathbf{v}$ a.e. and $\mathcal{E}(\mathbf{v}) < \infty$.

Clearly, $\theta \geq 0$ a.e. If θ were zero in a set $A \subset \Omega$ of positive measure, then we would have (for a subsequence) $\det D\mathbf{v}_j \to 0$ a.e. in A; hence $\bar{h}_2(\det D\mathbf{v}_j) \to \infty$ a.e. in A, as $j \to \infty$, due to the growth condition (6.11). Thus, thanks to Fatou's lemma, $\left\|\bar{h}_2(\det D\mathbf{v}_j)\right\|_{L^1(\Omega)} \to \infty$, as $j \to \infty$, which is a contradiction with (7.2). Therefore, $\theta > 0$ a.e.

By Lemma 2.9, $\mathbf{v}$ satisfies INV and the convergence (2.8) holds. □

The lower semicontinuity property of Definition 7.2 is as follows.

PROPOSITION 7.5. *For each $j \in \mathbb{N}$, let $\mathbf{v}_j, \mathbf{v} \in W^{1,n-1}(\Omega,K)$ satisfy INV and*

$$\det D\mathbf{v}_j > 0 \;\; \textit{a.e.}, \qquad \det D\mathbf{v} > 0 \;\textit{a.e.}, \qquad \sup_{j\in\mathbb{N}} \mathcal{E}(\mathbf{v}_j) < \infty, \qquad \mathcal{E}(\mathbf{v}) < \infty.$$

Suppose that $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ converges to $\mathbf{v}$ in the sense of cavitation. Fix $t \in [0,1]$. Then

$$\bar{\mathcal{W}}(t)(\mathbf{v}) \leq \liminf_{j\to\infty} \bar{\mathcal{W}}(t)(\mathbf{v}_j) \quad \textit{and} \quad \bar{\mathcal{S}}_1(t)(\mathbf{v}) \leq \liminf_{j\to\infty} \bar{\mathcal{S}}_1(t)(\mathbf{v}_j).$$

Proof. For simplicity, call $\bar{\boldsymbol{\psi}} := \boldsymbol{\psi}(t)$. For each $j \in \mathbb{N}$ define $\mathbf{u}_j := \bar{\boldsymbol{\psi}} \circ \mathbf{v}_j$ and $\mathbf{u} := \bar{\boldsymbol{\psi}} \circ \mathbf{v}$. From Proposition 6.1 we find that $\mathbf{u}_j, \mathbf{u} \in W^{1,n-1}(\Omega,K)$ satisfy INV,

$$\det D\mathbf{u}_j > 0 \;\text{ a.e.}, \qquad \det D\mathbf{u} > 0 \text{ a.e.}, \qquad \mathcal{E}(\mathbf{u}_j) < \infty, \qquad \mathcal{E}(\mathbf{u}) < \infty.$$

The a.e. convergence of $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ given by Definition 7.2, together with the continuity of $\bar{\boldsymbol{\psi}}$ show that $\mathbf{u}_j \to \mathbf{u}$ a.e. as $j \to \infty$. Moreover, the chain rule and the inequality

$$|D\mathbf{u}_j| \leq \left\|D\bar{\boldsymbol{\psi}}\right\|_{L^\infty(K,\mathbb{R}^{n\times n})} |D\mathbf{v}_j|, \qquad j \in \mathbb{N}$$

show that, for a subsequence, $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ converges weakly in $W^{1,n-1}(\Omega,\mathbb{R}^n)$. The a.e. convergence of $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ implies that in fact, for the whole sequence, $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{1,n-1}(\Omega,\mathbb{R}^n)$, as $j\to\infty$. On the other hand, thanks to the chain rule and the multiplicativity of the cofactor and of the determinant, we can apply a standard convergence result (see, e.g., [57, Lemma 6.7]) concerning the product of a weakly convergent sequence in L^1 by a bounded sequence in L^∞ converging a.e., so as to conclude that

$$\operatorname{cof} D\mathbf{u}_j \rightharpoonup \operatorname{cof} D\mathbf{u} \text{ in } L^1(\Omega,\mathbb{R}^{n\times n}), \qquad \det D\mathbf{u}_j \rightharpoonup \det D\mathbf{u} \text{ in } L^1(\Omega),$$

as $j\to\infty$. Finally, Lemma 2.9 shows that the convergence (2.8) holds as $j\to\infty$. Thus, $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ tends to $\mathbf{u}$ in the sense of cavitation. In fact, standard theorems on weak continuity of minors (see, e.g., [3, Cor. 6.2.2]) show that, not only $\operatorname{cof} D\mathbf{u}_j$ and $\det D\mathbf{u}_j$, but also all the minors of $D\mathbf{u}_j$ converge to the corresponding minors of $D\mathbf{u}$ weakly in L^1 as $j\to\infty$. Using the polyconvexity property (W4) and the standard lower semicontinuity theorem of [6, Th. 5.4], we obtain that

$$\mathcal{W}(\mathbf{u}) \le \liminf_{j\to\infty} \mathcal{W}(\mathbf{u}_j),$$

whereas Proposition 3.1 shows inequality (3.3). These are actually the inequalities of the statement, thanks to the definition (6.1). □

We finally show how the two convergences of Definitions 7.1 and 7.2 are related.

Lemma 7.6. *For each $j\in\mathbb{N}$, let $(\mathbf{v}_j,S_j)\in\mathfrak{B}$. Let $\mathbf{v}\in\mathfrak{A}$ and let S be a finite set of $\bar\Omega$. Suppose $\mathbf{v}_j\to\mathbf{v}$ in the sense of cavitation, and $S_j\to S$ componentwise, as $j\to\infty$. Then $C(\mathbf{v})\subset S$.*

Proof. Let $\mathbf{a}_0\in C(\mathbf{v})$. By Lemma 2.7, $\mathcal{L}^n(\operatorname{im_T}(\mathbf{v},\mathbf{a}_0))>0$. From the weak* convergence in $\mathcal{M}(\Omega)$ of Definition 7.2, and using a version of the Portmanteau theorem (see, e.g. [1, Ex. 1.63]), we find that for all $r\in(0,\operatorname{dist}(\mathbf{a}_0,\partial\Omega))$,

$$0 < \sum_{\mathbf{a}\in C(\mathbf{v})\cap B(\mathbf{a}_0,r)} \mathcal{L}^n(\operatorname{im_T}(\mathbf{v},\mathbf{a})) \le \liminf_{j\to\infty} \sum_{\mathbf{a}\in C(\mathbf{v}_j)\cap B(\mathbf{a}_0,r)} \mathcal{L}^n(\operatorname{im_T}(\mathbf{v}_j,\mathbf{a})). \tag{7.3}$$

Choose any sequence $\{r_j\}_{j\in\mathbb{N}}$ in $(0,\operatorname{dist}(\mathbf{a}_0,\partial\Omega))$ tending to zero. Thanks to (7.3), for each $j\in\mathbb{N}$ we can find a $\mathbf{b}_j\in C(\mathbf{v}_j)\cap B(\mathbf{a}_0,r_j)$. Hence $\mathbf{b}_j\to\mathbf{a}_0$ as $j\to\infty$, and $\mathbf{b}_j\in S_j$ for each $j\in\mathbb{N}$. By Definition 7.1, $\mathbf{a}_0\in S$. □

8. Existence of minimizers. We have now all compactness and lower semicontinuity properties so as to prove the existence of minimizers for the total energy.

Proposition 8.1. *Let $t\in[0,1]$ and let S_0 be a finite subset of Ω. Then there exists a minimizer of $\bar{\mathcal{I}}(t)$ in $\mathfrak{B}_{S_0}$.*

Proof. It was noted in Subsection 5.6 that $\mathfrak{B}_{S_0}(t)\neq\varnothing$, and in Subsection 5.5 that $\mathcal{I}$ is not identically $+\infty$ in $\mathfrak{B}_{S_0}(t)$. Hence, by Proposition 6.1, $\mathfrak{B}_{S_0}\neq\varnothing$, and $\bar{\mathcal{I}}(t)$ is not identically $+\infty$ in $\mathfrak{B}_{S_0}$; in fact $\bar{\mathcal{I}}(t)(\mathbf{id},S_0)<\infty$. In addition, condition ($\bar{\text{W}}$4) shows that $\bar{\mathcal{W}}(t)$ and hence $\bar{\mathcal{I}}(t)$ are bounded below in $\mathfrak{B}_{S_0}$.

Let $\{(\mathbf{v}_j,S_j)\}_{j\in\mathbb{N}}$ be a minimizing sequence of $\bar{\mathcal{I}}(t)$ in $\mathfrak{B}_{S_0}$. Thanks to (K1) we clearly have $S_j=C(\mathbf{v}_j)\cup S_0$. As shown before,

$$\lim_{j\to\infty} \bar{\mathcal{I}}(t)(\mathbf{v}_j,S_j) \le \bar{\mathcal{I}}(t)(\mathbf{id},S_0) < \infty,$$

whereas assumptions ($\bar{\text{W}}$4) and (K1) and inequality (5.2) show that the bounds (7.2) hold, where $\bar h_1$ and $\bar h_2$ are as in assumption ($\bar{\text{W}}$4).

By Proposition 7.4, there exists $\mathbf{v} \in W^{1,n-1}(\Omega, K)$ satisfying INV, $\det D\mathbf{v} > 0$ a.e. and $\mathcal{E}(\mathbf{v}) < \infty$, and a subsequence (not relabelled) such that $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ converges to $\mathbf{v}$ in the sense of cavitation. The continuity of the traces shows that $\mathbf{v}|_{\Gamma_D} = \mathbf{id}|_{\Gamma_D}$ in the sense of traces. Therefore, $\mathbf{v} \in \mathfrak{A}$.

From Lemma 7.3, we can find a subsequence and a finite subset S of $\bar{\Omega}$ containing S_0 such that $S_j \to S$ componentwise, as $j \to \infty$, and inequality (7.1) holds. Thanks to Lemma 7.6, we have $C(\mathbf{v}) \subset S \cap \Omega$. Thus, $(\mathbf{v}, S \cap \Omega) \in \mathfrak{B}_{S_0}$. Proposition 7.5 then shows that

$$\bar{\mathcal{I}}^c(t)(\mathbf{v}) \le \liminf_{j\to\infty} \bar{\mathcal{I}}^c(t)(\mathbf{v}_j),$$

which, together with inequality (7.1), yields

$$\bar{\mathcal{I}}(t)(\mathbf{v}, S \cap \Omega) \le \liminf_{j\to\infty} \bar{\mathcal{I}}(t)(\mathbf{v}_j, S_j),$$

and, hence, $(\mathbf{v}, S \cap \Omega)$ is a minimizer of $\bar{\mathcal{I}}(t)$ in $\mathfrak{B}_{S_0}$. □

9. Time discretization. In this section we start the analysis of the quasistatic evolution of the reformulation of the problem given in Section 6. As common in the literature (e.g., [44, 16]), we adopt the scheme of time discretization to approximate the evolution of the problem.

For each $k \in \mathbb{N}$, we let the numbers $0 = t_k^0 < \cdots < t_k^k = 1$ satisfy

$$\lim_{k\to\infty} \max_{1\le i\le k} (t_k^i - t_k^{i-1}) = 0.$$

Let $(\mathbf{v}^0, S^0) \in \mathfrak{B}$ be a minimizer of $\bar{\mathcal{I}}(0)$ in $\mathfrak{B}_{S^0}$, the existence of which is guaranteed by Proposition 8.1. For every $k \in \mathbb{N}$ and $0 \le i \le k$ we define $(\mathbf{v}_k^i, S_k^i)$ by induction as follows: $(\mathbf{v}_k^0, S_k^0) := (\mathbf{v}^0, S^0)$, and for $i = 1, \dots, k$, let $(\mathbf{v}_k^i, S_k^i)$ be a minimizer of $\bar{\mathcal{I}}(t_k^i)$ in $\mathfrak{B}_{S_k^{i-1}}$, the existence of which is again a consequence of Proposition 8.1.

For every $k \in \mathbb{N}$ and every $t \in [0,1)$, let $0 \le i \le k-1$ satisfy $t \in [t_k^i, t_k^{i+1})$; for $t = 1$ take $i = k$. Consider

$$\mathbf{v}_k(t) := \mathbf{v}_k^i, \qquad S_k(t) := S_k^i, \qquad \tau_k(t) := t_k^i, \qquad \theta_k(t) := (\bar{\mathcal{I}}^c)'(t)(\mathbf{v}_k^i). \tag{9.1}$$

Thanks to Corollary 6.4, the function θ_k is well defined. Note that by construction,

$$\lim_{k\to\infty} \tau_k(t) = t. \tag{9.2}$$

The following result (which is the analogue of [18, Prop. 3.10]) provides us with enough a priori bounds for the discrete evolution (9.1) to pass to the limit. In addition, when passing to the limit, it will yield one of the inequalities of the energy balance (Proposition 12.1).

Proposition 9.1. *For each $k \in \mathbb{N}$ and $t \in [0,1]$,*

$$\bar{\mathcal{I}}(\tau_k(t))(\mathbf{v}_k(t), S_k(t)) \le \bar{\mathcal{I}}(0)(\mathbf{v}^0, S^0) + \int_0^{\tau_k(t)} \theta_k(s)\,\mathrm{d}s. \tag{9.3}$$

Moreover,

$$\sup_{t\in[0,1]} \sup_{k\in\mathbb{N}} \left[|\theta_k(t)| + \mathcal{H}^0(S_k(t)) + \left|\bar{\mathcal{W}}(t)(\mathbf{v}_k(t))\right| + \mathcal{E}(\mathbf{v}_k(t)) \right] < \infty.$$

Proof. Let $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$. First, as $(\mathbf{id}, S_k^i) \in \mathfrak{B}_{S_k^{i-1}}$ (see Subsection 5.6 and Proposition 6.1) and $(\mathbf{v}_k^i, S_k^i)$ is a minimizer of $\bar{\mathcal{I}}(t_k^i)$ in $\mathfrak{B}_{S_k^{i-1}}$, we obtain that $\bar{\mathcal{I}}^c(t_k^i)(\mathbf{v}_k^i) \leq \bar{\mathcal{I}}^c(t_k^i)(\mathbf{id})$; hence, using estimates ($\bar{\mathrm{W}}3$) and ($\bar{\mathrm{W}}4$),

$$\sup_{k\in\mathbb{N}} \max_{0\leq i\leq k} \left[\left| \bar{\mathcal{W}}(t_k^i)(\mathbf{v}_k^i) \right| + \bar{\mathcal{S}}_1(t_k^i)(\mathbf{v}_k^i) \right] < \infty, \tag{9.4}$$

where the case $i = 0$ has been deduced separately. Likewise, as $(\mathbf{v}_k^{i-1}, S_k^{i-1}) \in \mathfrak{B}_{S_k^{i-1}}$, we obtain that $\bar{\mathcal{I}}(t_k^i)(\mathbf{v}_k^i, S_k^i) \leq \bar{\mathcal{I}}(t_k^i)(\mathbf{v}_k^{i-1}, S_k^{i-1})$. Since, by Corollary 6.4,

$$\bar{\mathcal{I}}^c(t_k^i)(\mathbf{v}_k^{i-1}) = \bar{\mathcal{I}}^c(t_k^{i-1})(\mathbf{v}_k^{i-1}) + \int_{t_k^{i-1}}^{t_k^i} \theta_k(s)\, \mathrm{d}s,$$

we find that

$$\bar{\mathcal{I}}(t_k^i)(\mathbf{v}_k^i, S_k^i) \leq \bar{\mathcal{I}}(t_k^{i-1})(\mathbf{v}_k^{i-1}, S_k^{i-1}) + \int_{t_k^{i-1}}^{t_k^i} \theta_k(s)\, \mathrm{d}s.$$

Applying iteratively this inequality, we conclude that

$$\bar{\mathcal{I}}(t_k^i)(\mathbf{v}_k^i, S_k^i) \leq \bar{\mathcal{I}}(0)(\mathbf{v}^0, S^0) + \int_0^{t_k^i} \theta_k(s)\, \mathrm{d}s.$$

Since this inequality is also true when $i = 0$, the proof of (9.3) is done.

Bounds (9.4), (5.2) and (K1) show that

$$\sup_{t\in[0,1]} \sup_{k\in\mathbb{N}} \mathcal{E}(\mathbf{v}_k(t)) < \infty, \tag{9.5}$$

while estimate ($\bar{\mathrm{W}}6$) and again (9.4) yield

$$\sup_{t\in[0,1]} \sup_{k\in\mathbb{N}} \left[\left| \bar{\mathcal{W}}(t)(\mathbf{v}_k(t)) \right| + \left| \bar{\mathcal{W}}'(t)(\mathbf{v}_k(t)) \right| \right] < \infty. \tag{9.6}$$

Now, Proposition 6.3 and inequalities (5.2) and (9.5) imply that

$$\sup_{t\in[0,1]} \sup_{k\in\mathbb{N}} \left| \bar{\mathcal{S}}_1'(t)(\mathbf{v}_k(t)) \right| < \infty. \tag{9.7}$$

Estimates (9.6) and (9.7) thus show that

$$\sup_{t\in[0,1]} \sup_{k\in\mathbb{N}} |\theta_k(t)| < \infty.$$

This inequality and (9.3) imply that

$$\sup_{t\in[0,1]} \sup_{k\in\mathbb{N}} \left| \bar{\mathcal{I}}(\tau_k(t))(\mathbf{v}_k(t), S_k(t)) \right| < \infty,$$

which thanks to (9.4) shows that

$$\sup_{t\in[0,1]} \sup_{k\in\mathbb{N}} \mathcal{S}_0(S_k(t)) < \infty,$$

which, in turn, due to (K1) yields

$$\sup_{t\in[0,1]} \sup_{k\in\mathbb{N}} \mathcal{H}^0(S_k(t)) < \infty.$$

This last inequality concludes the proof. $\square$

10. Limit passage from discrete to continuum time. In this section we perform the limit of the sequence $\{(\mathbf{v}_k(t), S_k(t))\}_{k\in\mathbb{N}}$ of (9.1).

The following auxiliary result shows the existence of a diffeomorphism prescribing the image of finitely many points. It will be used in Propositions 10.3 and 11.1 whenever we approximate a deformation by another with prescribed cavity points. The construction is analogous to that of Henao [35, Sect. 6.4].

LEMMA 10.1. *Let $M \geq 1$. For each $i \in \{1, \dots, M\}$, let $\{\mathbf{a}_i(k)\}_{k\in\mathbb{N}}$ be a sequence in Ω converging to a certain $\mathbf{a}_i \in \Omega$. Assume that $\mathcal{H}^0(\{\mathbf{a}_1, \dots, \mathbf{a}_M\}) = M$. Then there exist $k_0 \in \mathbb{N}$ and a sequence $\{\mathbf{g}_k\}_{k\geq k_0}$ of C^∞ diffeomorphisms from Ω onto itself such that*

a) $\lim_{k\to\infty} \left[\|\mathbf{g}_k - \mathbf{id}\|_{L^\infty(\Omega,\mathbb{R}^n)} + \|D\mathbf{g}_k - \mathbf{1}\|_{L^\infty(\Omega,\mathbb{R}^{n\times n})} \right] = 0.$

b) $\mathbf{g}_k(\mathbf{a}_i(k)) = \mathbf{a}_i$ *for each* $k \geq k_0$ *and* $i \in \{1, \dots, M\}$.

c) There exists a neigbourhood U of $\partial\Omega$ such that $\mathbf{g}_k|_{U\cap\Omega} = \mathbf{id}|_{U\cap\Omega}$ for all $k \geq k_0$.

Proof. Define

$$m := \frac{1}{2} \min \left\{ \min \left\{ |\mathbf{a}_i - \mathbf{a}_j| : 1 \leq i < j \leq M \right\}, \min \left\{ \operatorname{dist}(\partial\Omega, \mathbf{a}_i) : 1 \leq i \leq M \right\} \right\},$$

and let $\varphi : [0, \infty) \to [0, 1]$ be a C^∞ function such that $\varphi = 1$ in a neigbourhood of 0, and $\varphi = 0$ in $[m, \infty)$. For each $k \in \mathbb{N}$, define $\mathbf{g}_k : \Omega \to \mathbb{R}^n$ as

$$\mathbf{g}_k(\mathbf{x}) := \mathbf{x} + \sum_{i=1}^{M} \varphi\left(|\mathbf{x} - \mathbf{a}_i(k)|\right) \left(\mathbf{a}_i - \mathbf{a}_i(k)\right), \qquad \mathbf{x} \in \Omega.$$

It is easy to see that, for $k_0 \in \mathbb{N}$ large enough, the sequence $\{\mathbf{g}_k\}_{k\geq k_0}$ satisfies the required conditions. □

The next result shows some continuity properties of $\bar{\mathcal{I}}^c$.

LEMMA 10.2. *Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence in $[0, 1]$ converging to some $t \in [0, 1]$. For each $k \in \mathbb{N}$, let $\mathbf{v}_k \in W^{1,n-1}(\Omega, K)$ satisfy INV and $\det D\mathbf{v}_k > 0$ a.e. The following properties hold:*

1) If

$$\sup_{k\in\mathbb{N}} \bar{\mathcal{W}}(t)(\mathbf{v}_k) < \infty. \tag{10.1}$$

then

$$\lim_{k\to\infty} \left[\bar{\mathcal{W}}(t_k)(\mathbf{v}_k) - \bar{\mathcal{W}}(t)(\mathbf{v}_k) \right] = 0. \tag{10.2}$$

If, in addition, there exists $\mathbf{v} \in W^{1,n-1}(\Omega, K)$ such that $\mathbf{v}_k \to \mathbf{v}$ a.e. and $D\mathbf{v}_k \to D\mathbf{v}$ a.e. as $j \to \infty$ then

$$\lim_{k\to\infty} \bar{\mathcal{W}}(t_k)(\mathbf{v}_k) = \bar{\mathcal{W}}(t)(\mathbf{v}). \tag{10.3}$$

2) If $\sup_{k\in\mathbb{N}} \mathcal{S}_1(\mathbf{v}_k) < \infty$ then

$$\lim_{k\to\infty} \left[\bar{\mathcal{S}}_1(t_k)(\mathbf{v}_k) - \bar{\mathcal{S}}_1(t)(\mathbf{v}_k) \right] = 0.$$

Proof. Equality (10.2) follows at once from expression (6.10), bound (10.1) and property ($\bar{\text{W}}$6).

The a.e. convergence of $\{\mathbf{v}_k\}_{k\in\mathbb{N}}$ and $\{D\mathbf{v}_k\}_{k\in\mathbb{N}}$, together with the bounds ($\bar{\text{W}}$5) and ($\bar{\text{W}}$8) allow to apply dominated convergence and conclude, thanks to expression (6.10), that $\lim_{k\to\infty}\bar{\mathcal{W}}(t)(\mathbf{v}_k)=\bar{\mathcal{W}}(t)(\mathbf{v})$. Thus, (10.3) follows from (10.2).

Now, Proposition 6.3 shows that there exists $c>0$ such that for all $k\in\mathbb{N}$,

$$\left|\bar{\mathcal{S}}_1(t_k)(\mathbf{v}_k)-\bar{\mathcal{S}}_1(t)(\mathbf{v}_k)\right|\le c\,\mathcal{S}_1(\mathbf{v}_k)\,|t_k-t|\,,$$

so proving statement 2). □

The following proposition describes the limit of the sequences $\{\mathbf{v}_k(t)\}_{k\in\mathbb{N}}$ and $\{S_k(t)\}_{k\in\mathbb{N}}$ constructed in (9.1). The convergence of $\{S_k(t)\}_{k\in\mathbb{N}}$ is in fact an easy realization of the abstract Helly's theorem (see [44, Th. 3.2] for a precise formulation), for which we follow the proof of [19, Th. 6.3]. The convergence of $\{\mathbf{v}_k(t)\}_{k\in\mathbb{N}}$ follows easily from Propositions 7.4 and 9.1. While the limit passage is straightforward, a crucial property to be proved is that no cavities of $\mathbf{v}_k(t)$ are lost in the limit; in other words, no cavities collapse, heal or escape to the boundary (see Steps 3, 5 and 4, respectively, of the proof below). In the following statement, $\mathcal{P}(\Omega)$ denotes the set of subsets of Ω.

PROPOSITION 10.3. *There exist a subsequence (not relabelled) and an increasing function $S:[0,1]\to\mathcal{P}(\Omega)$ such that for each $t\in[0,1]$ the sequence $\{S_k(t)\}_{k\in\mathbb{N}}$ converges componentwise to $S(t)$. Moreover, for each $t\in[0,1]$ there exists $\mathbf{v}(t)\in\mathfrak{A}$ and a subsequence (not relabelled) such that $\mathbf{v}_k(t)\to\mathbf{v}(t)$ in the sense of cavitation as $k\to\infty$. Furthermore, for any such subsequences and any such limits $\mathbf{v}(t)$ and $S(t)$, we have that*

$$\mathcal{H}^0(S(t))=\mathcal{H}^0(S_k(t)),\qquad k\in\mathbb{N},\tag{10.4}$$

$(\mathbf{v}(t),S(t))\in\mathfrak{B}$ *and*

$$S(t)=S^0\cup\bigcup_{s\in[0,t]}C(\mathbf{v}(s)).$$

Proof. The proof is divided into several steps.

Step 1: convergence of $S_k(t)$. Let D be a countable dense subset of $[0,1]$ containing $\{0,1\}$. Proposition 9.1 provides the bound

$$\sup_{t\in[0,1]}\sup_{k\in\mathbb{N}}\mathcal{H}^0(S_k(t))<\infty.$$

Thus, by Lemma 7.3 and a standard diagonal argument, we can find a subsequence (not relabelled) such that for all $t\in D$, the sequence $\{S_k(t)\}_{k\in\mathbb{N}}$ converges componentwise to a set $S(t)\subset\bar{\Omega}$. By construction, $S(t_1)\subset S(t_2)$ if $t_1,t_2\in D$ with $t_1<t_2$. For each $t\in[0,1]\setminus D$ define

$$S^-(t):=\bigcup_{s\in D\cap[0,t)}S(s),\qquad S^+(t):=\bigcap_{s\in D\cap(t,1]}S(s).$$

Then $S^-(t)\subset S^+(t)$ for all $t\in[0,1]\setminus D$. Let E be the set of $t\in[0,1]\setminus D$ for which $S^-(t)\neq S^+(t)$. Then E is finite, since so is $S(1)$. For each $t\in[0,1]\setminus(D\cup E)$ we define $S(t):=S^-(t)$, and for a further subsequence (not relabelled), for each $t\in E$ the sequence $\{S_k(t)\}_{k\in\mathbb{N}}$ converges componentwise to a set $S(t)\subset\bar{\Omega}$, thanks to Lemma 7.3 again.

We have therefore shown that there exist, for a subsequence, an increasing function $N : [0,1] \to \mathbb{N}$, points

$$\mathbf{a}_1(k), \dots, \mathbf{a}_{N(1)}(k) \in \Omega \qquad \text{for each } k \in \mathbb{N},$$

and points $\mathbf{a}_1, \dots, \mathbf{a}_{N(1)} \in \bar{\Omega}$ such that

$$S_k(t) = \{\mathbf{a}_1(k), \dots, \mathbf{a}_{N(t)}(k)\} \;\text{ and }\; \mathcal{H}^0(S_k(t)) = N(t) \;\text{ for each } t \in [0,1] \text{ and } k \in \mathbb{N}, \tag{10.5}$$

$\lim_{k\to\infty} \mathbf{a}_i(k) = \mathbf{a}_i$ for each $i \in \{1, \dots, N(1)\}$, and $S(t) = \{\mathbf{a}_1, \dots, \mathbf{a}_{N(t)}\}$ for each $t \in [0,1]$.

Step 2: convergence of $\mathbf{v}_k(t)$. Proposition 9.1 and property ($\bar{\mathrm{W}}$4) provides the bounds stated in (7.2). Therefore, by Proposition 7.4 and the continuity of traces, for each $t \in [0,1]$ there exist $\mathbf{v}(t) \in \mathfrak{A}$ and a subsequence (not relabelled) such that $\mathbf{v}_k(t) \to \mathbf{v}(t)$ in the sense of cavitation, as $k \to \infty$. Note that $C(\mathbf{v}(t)) \subset S(t)$ thanks to Lemma 7.6. Now, Proposition 7.5 shows that

$$\bar{\mathcal{I}}^c(t)(\mathbf{v}(t)) \leq \liminf_{k\to\infty} \bar{\mathcal{I}}^c(t)(\mathbf{v}_k(t)).$$

Thanks to Lemma 10.2, using again the bounds of Proposition 9.1, we find that

$$\lim_{k\to\infty} \left[\bar{\mathcal{I}}^c(t)(\mathbf{v}_k(t)) - \bar{\mathcal{I}}^c(\tau_k(t))(\mathbf{v}_k(t))\right] = 0,$$

so we conclude that

$$\bar{\mathcal{I}}^c(t)(\mathbf{v}(t)) \leq \liminf_{k\to\infty} \bar{\mathcal{I}}^c(\tau_k(t))(\mathbf{v}_k(t)).$$

Step 3: $\mathcal{H}^0(S(t)) = N(t)$. Suppose, for a contradiction, that $\mathcal{H}^0(S(t)) < N(t)$ for some $t \in [0,1]$. Then there exist $1 \leq M_1 < M_2 \leq N(t)$ such that $\mathbf{a}_{M_1} = \mathbf{a}_{M_2}$. As $\{\mathbf{a}_{M_1}(k), \mathbf{a}_{M_2}(k)\} \not\subset S^0$ for k large enough, there exists $j_k \in \{1, \dots, k\}$ such that $\{\mathbf{a}_{M_1}(k), \mathbf{a}_{M_2}(k)\} \subset S_k^{j_k}$ and $\mathbf{a}_{M_2}(k) \notin S_k^{j_k-1}$.

Applying the bounds of Proposition 9.1 and property ($\bar{\mathrm{W}}$4), as well as the compactness results of Lemma 7.3 and Proposition 7.4, we find that for a subsequence (not relabelled) there exist $t_0 \in [0,1]$, $\mathbf{v} \in \mathfrak{A}$ and a finite set $S \subset \bar{\Omega}$ such that

$$t_k^{j_k} \to t_0, \quad \mathbf{v}_k^{j_k} \to \mathbf{v} \text{ in the sense of cavitation, and } S_k^{j_k} \to S \text{ componentwise} \tag{10.6}$$

as $k \to \infty$. In addition, by Lemma 7.6, $C(\mathbf{v}) \subset S$. Note, finally, that $\mathbf{a}_{M_1} \in S$.

Let $p := \mathcal{H}^0(C(\mathbf{v}))$, and let $1 \leq N_1 < \dots < N_p \leq N(t)$ satisfy that $M_2 \notin \{N_1, \dots, N_p\}$,

$$C(\mathbf{v}) = \{\mathbf{a}_{N_1}, \dots, \mathbf{a}_{N_p}\} \quad \text{and} \quad \{\mathbf{a}_{N_1}(k), \dots, \mathbf{a}_{N_p}(k)\} \subset S_k^{j_k} \qquad \text{for all } k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$ large enough, let $\mathbf{g}_k$ be a diffeomorphism from Ω onto itself such that $\mathbf{g}_k(\mathbf{a}_{N_i}(k)) = \mathbf{a}_{N_i}$ for each $i \in \{1, \dots, p\}$, and that satisfies the remaining properties described in Lemma 10.1. Define $\tilde{\mathbf{v}}_k := \mathbf{v} \circ \mathbf{g}_k$ and $\tilde{S}_k := S_k^{j_k} \setminus \{\mathbf{a}_{M_2}(k)\}$. By Proposition 6.2, $\tilde{\mathbf{v}}_k \in \mathfrak{A}$ and $C(\tilde{\mathbf{v}}_k) = \{\mathbf{a}_{N_1}(k), \dots, \mathbf{a}_{N_p}(k)\}$. In addition, as $\mathbf{a}_{M_2}(k) \notin S_k^{j_k-1}$ then $(\tilde{\mathbf{v}}_k, \tilde{S}_k) \in \mathfrak{B}_{S_k^{j_k-1}}$. Moreover, by the bounds of Proposition 9.1 and Lemma 10.2, we conclude that

$$\lim_{k\to\infty} \bar{\mathcal{W}}(t_k^{j_k})(\tilde{\mathbf{v}}_k) = \bar{\mathcal{W}}(t_0)(\mathbf{v}), \qquad \lim_{k\to\infty} \left[\bar{\mathcal{I}}^c(t_k^{j_k})(\mathbf{v}_k^{j_k}) - \bar{\mathcal{I}}^c(t_0)(\mathbf{v}_k^{j_k})\right] = 0 \tag{10.7}$$

so we deduce, using Proposition 7.5, that

$$\bar{\mathcal{I}}^c(t_0)(\mathbf{v}) \le \liminf_{k\to\infty} \bar{\mathcal{I}}^c(t_k^{j_k})(\mathbf{v}_k^{j_k}). \tag{10.8}$$

Now, by (6.13), Proposition 6.2, the continuity of κ_1 and Propositions 9.1 and 6.3,

$$\begin{aligned} \lim_{k\to\infty} \bar{\mathcal{S}}_1(t_k^{j_k})(\tilde{\mathbf{v}}_k) &= \lim_{k\to\infty} \sum_{\mathbf{a}\in C(\mathbf{v})} \kappa_1(\mathbf{g}_k^{-1}(\mathbf{a}))\,\mathrm{Per}\left(\boldsymbol{\psi}(t_k^{j_k})\,(\mathrm{im_T}(\mathbf{v},\mathbf{a}))\right) \\ &= \lim_{k\to\infty} \sum_{\mathbf{a}\in C(\mathbf{v})} \kappa_1(\mathbf{a})\,\mathrm{Per}\left(\boldsymbol{\psi}(t_k^{j_k})\,(\mathrm{im_T}(\mathbf{v},\mathbf{a}))\right) \\ &= \lim_{k\to\infty} \bar{\mathcal{S}}_1(t_k^{j_k})(\mathbf{v}) = \bar{\mathcal{S}}_1(t_0)(\mathbf{v}), \end{aligned} \tag{10.9}$$

which, together with (10.7), shows that

$$\lim_{k\to\infty} \bar{\mathcal{I}}^c(t_k^{j_k})(\tilde{\mathbf{v}}_k) = \bar{\mathcal{I}}^c(t_0)(\mathbf{v}). \tag{10.10}$$

On the other hand, for each $k\in\mathbb{N}$,

$$\mathcal{S}_0(\tilde{S}_k) = \mathcal{S}_0(S_k^{j_k}) - \kappa_0(\mathbf{a}_{M_2}(k)) \le \mathcal{S}_0(S_k^{j_k}) - \inf\kappa_0. \tag{10.11}$$

Using now that $(\mathbf{v}_k^{j_k}, S_k^{j_k})$ is a minimizer of $\bar{\mathcal{I}}(t_k^{j_k})$ in $\mathfrak{B}_{S_k^{j_k-1}}$, as well as equations (10.10) and (10.11), we find that

$$\begin{aligned} \liminf_{k\to\infty} \bar{\mathcal{I}}(t_k^{j_k})(\mathbf{v}_k^{j_k}, S_k^{j_k}) &\le \liminf_{k\to\infty} \bar{\mathcal{I}}(t_k^{j_k})(\tilde{\mathbf{v}}_k, \tilde{S}_k) \\ &\le \liminf_{k\to\infty} \left[\bar{\mathcal{I}}^c(t_k^{j_k})(\tilde{\mathbf{v}}_k) + \mathcal{S}_0(S_k^{j_k})\right] - \inf\kappa_0 \\ &= \bar{\mathcal{I}}^c(t_0)(\mathbf{v}) - \inf\kappa_0 + \liminf_{k\to\infty} \mathcal{S}_0(S_k^{j_k}), \end{aligned}$$

which, together with (10.8), shows that

$$\liminf_{k\to\infty} \bar{\mathcal{I}}(t_k^{j_k})(\mathbf{v}_k^{j_k}, S_k^{j_k}) \le \liminf_{k\to\infty} \bar{\mathcal{I}}(t_k^{j_k})(\mathbf{v}_k^{j_k}, S_k^{j_k}) - \inf\kappa_0, \tag{10.12}$$

which is a contradiction due to (K1).

Step 4: $S(t)\subset\Omega$. Suppose, for a contradiction, that there exist $t\in[0,1]$ and $1\le M\le N(t)$ such that $\mathbf{a}_M\in\partial\Omega$. We have that $\mathbf{a}_M(k)\in S_k(t)$ for each $k\in\mathbb{N}$, but $\mathbf{a}_M(k)\notin S^0$ if k is large enough. Therefore, for each $k\in\mathbb{N}$ large enough there exists $j_k\in\{1,\dots,k\}$ such that $\mathbf{a}_M(k)\in S_k^{j_k}\setminus S_k^{j_k-1}$.

As in Step 3, for a subsequence (not relabelled) there exist $t_0\in[0,1]$, $\mathbf{v}\in\mathfrak{A}$ and $S\subset\bar{\Omega}$ such that the convergences (10.6) hold as $k\to\infty$. In addition, $C(\mathbf{v})\subset S$ and $\mathbf{a}_M\in S\setminus C(\mathbf{v})$, since by Definition 2.4, $C(\mathbf{v})\subset\Omega$. Let $p:=\mathcal{H}^0(C(\mathbf{v}))$, and let $1\le N_1<\cdots<N_p\le N(t)$ satisfy that $M\notin\{N_1,\dots,N_p\}$,

$$C(\mathbf{v}) = \{\mathbf{a}_{N_1},\dots,\mathbf{a}_{N_p}\} \quad\text{and}\quad \{\mathbf{a}_{N_1}(k),\dots,\mathbf{a}_{N_p}(k)\}\subset S_k^{j_k} \qquad \text{for all } k\in\mathbb{N}.$$

For each $k\in\mathbb{N}$ large enough, let $\mathbf{g}_k$ be a diffeomorphism from Ω onto itself such that $\mathbf{g}_k(\mathbf{a}_{N_i}(k))=\mathbf{a}_{N_i}$ for each $i\in\{1,\dots,p\}$, and that satisfies the remaining properties described in Lemma 10.1. Define $\tilde{\mathbf{v}}_k:=\mathbf{v}\circ\mathbf{g}_k$ and $\tilde{S}_k:=S_k^{j_k}\setminus\{\mathbf{a}_M(k)\}$. Proceeding as in Step 3, we reach the contradiction (10.12).

Step 5: $S(t) = S^0 \cup \bigcup_{s\in[0,t]} C(\mathbf{v}(s))$. Fix $t \in [0,1]$ and define $\bar{S}(t) := S^0 \cup \bigcup_{s\in[0,t]} C(\mathbf{v}(s))$. By construction, $S^0 \subset S_k(t)$ for all $k \in \mathbb{N}$, so $S^0 \subset S(t)$. Moreover, by Lemma 7.6, we find that $\bar{S}(t) \subset S(t)$. Suppose, for a contradiction, that there exists $1 \le M \le N(t)$ such that $\mathbf{a}_M \notin \bar{S}(t)$. For $k \in \mathbb{N}$ large enough, we have that $\mathbf{a}_M(k) \in S_k(t) \setminus S^0$. Hence, for each $k \in \mathbb{N}$ there exists $j_k \in \{1, \dots, k\}$ such that $\mathbf{a}_M(k) \in S_k^{j_k} \setminus S_k^{j_k-1}$. We arrive at a contradiction by applying the same argument of Step 4 (or Step 3). □

11. Stability of minimizers. In this section we prove the stability of minimizers (Proposition 11.2), which ensures that for each $t \in [0,1]$, the limit of $(\mathbf{v}_k(t), S_k(t))$ is a minimizer of $\bar{\mathcal{I}}(t)$.

The following is the analogue of the so-called jump transfer (see [26, Th. 2.1] in the context of fracture mechanics). The proof is much simpler since we deal with finite sets instead of $\mathcal{H}^{n-1}$-rectifiable sets, and, since, in addition, we took care in Proposition 10.3 in proving that no cavities are lost in the limit passage from $\mathbf{v}_k(t)$ to $\mathbf{v}(t)$.

PROPOSITION 11.1. *Fix $t \in [0,1]$, let $S(t)$ be as constructed in Proposition 10.3, and let $(\mathbf{v}, S) \in \mathfrak{B}_{S(t)}$. Then, for each $k \in \mathbb{N}$ there exists $(\mathbf{v}_k, S_k) \in \mathfrak{B}_{S_k(t)}$ such that*

$$\lim_{k\to\infty} \bar{\mathcal{I}}(\tau_k(t))(\mathbf{v}_k, S_k) = \bar{\mathcal{I}}(t)(\mathbf{v}, S).$$

Proof. As $(\mathbf{v}, S) \in \mathfrak{B}_{S(t)}$, there exist an integer $N \ge 0$ and points $\mathbf{b}_1, \dots, \mathbf{b}_N \in \Omega$ such that

$$S = \left\{\mathbf{b}_1, \dots, \mathbf{b}_N, \mathbf{a}_1, \dots, \mathbf{a}_{N(t)}\right\}, \qquad \mathcal{H}^0(S) = N + N(t).$$

For each $k \in \mathbb{N}$ large enough, we construct a diffeomorphism $\mathbf{g}_k$ from Ω onto itself such that

$$\mathbf{g}_k(\mathbf{a}_i(k)) = \mathbf{a}_i, \quad 1 \le i \le N(t); \qquad \mathbf{g}_k(\mathbf{b}_i) = \mathbf{b}_i, \quad 1 \le i \le N$$

satisfying the remaining properties described in Lemma 10.1 as well.

Define $\mathbf{v}_k := \mathbf{v} \circ \mathbf{g}_k$ and $S_k := \{\mathbf{b}_1, \dots, \mathbf{b}_N, \mathbf{a}_1(k), \dots, \mathbf{a}_{N(t)}(k)\}$. Note that Proposition 10.3 shows that $\mathcal{H}^0(S_k) = \mathcal{H}^0(S)$ for $k \in \mathbb{N}$ large enough. By Proposition 6.2,

$$C(\mathbf{v}_k) = \mathbf{g}_k^{-1}(C(\mathbf{v})) \subset \mathbf{g}_k^{-1}(S) = S_k,$$

so thanks to (10.5) we find that $(\mathbf{v}_k, S_k) \in \mathfrak{B}_{S_k(t)}$. Moreover, Lemma 10.2 shows that

$$\lim_{k\to\infty} \bar{\mathcal{W}}(\tau_k(t))(\mathbf{v}_k) = \bar{\mathcal{W}}(t)(\mathbf{v}), \tag{11.1}$$

whereas the argument of (10.9) yields

$$\lim_{k\to\infty} \bar{\mathcal{S}}_1(\tau_k(t))(\mathbf{v}_k) = \bar{\mathcal{S}}_1(t)(\mathbf{v}), \tag{11.2}$$

while the continuity of κ_0 and the equality $\mathcal{H}^0(S_k) = \mathcal{H}^0(S)$ imply that

$$\lim_{k\to\infty} \mathcal{S}_0(S_k) = \mathcal{S}_0(S). \tag{11.3}$$

Equalities (11.1), (11.2) and (11.3) conclude the proof. □

The stability of minimizers follows now from Propositions 10.3 and 11.1.

Proposition 11.2. *Fix $t \in [0,1]$. The configuration $(\mathbf{v}(t), S(t))$ constructed in Proposition 10.3 is a minimizer of $\bar{\mathcal{I}}(t)$ in $\mathfrak{B}_{S(t)}$. Moreover, for the subsequence chosen,*

$$\bar{\mathcal{I}}^c(t)(\mathbf{v}(t)) = \lim_{k\to\infty} \bar{\mathcal{I}}^c(t)(\mathbf{v}_k(t)), \qquad \mathcal{S}_0(S(t)) = \lim_{k\to\infty} \mathcal{S}_0(S_k(t)). \tag{11.4}$$

Proof. It was shown in Step 2 of the proof of Proposition 10.3 that

$$\bar{\mathcal{I}}^c(t)(\mathbf{v}(t)) \leq \liminf_{k\to\infty} \bar{\mathcal{I}}^c(t)(\mathbf{v}_k(t)), \qquad \lim_{k\to\infty} \left[\bar{\mathcal{I}}^c(t)(\mathbf{v}_k(t)) - \bar{\mathcal{I}}^c(\tau_k(t))(\mathbf{v}_k(t))\right] = 0. \tag{11.5}$$

On the other hand, as $S_k(t) \to S(t)$ componentwise as $k \to \infty$, using (10.4) and the continuity of κ_0 we find that

$$\mathcal{S}_0(S(t)) = \lim_{k\to\infty} \mathcal{S}_0(S_k(t)), \tag{11.6}$$

which is the second equality of (11.4).

Let $(\mathbf{v}, S) \in \mathfrak{B}_{S(t)}$, and let $\{(\mathbf{v}_k, S_k)\}_{k\in\mathbb{N}}$ be the corresponding sequence of Proposition 11.1. By construction, $(\mathbf{v}_k(t), S_k(t))$ is a minimizer of $\bar{\mathcal{I}}(\tau_k(t))$ in $\mathfrak{B}_{S_k(t)}$, and $(\mathbf{v}_k, S_k) \in \mathfrak{B}_{S_k(t)}$. Therefore,

$$\bar{\mathcal{I}}(\tau_k(t))(\mathbf{v}_k(t), S_k(t)) \leq \bar{\mathcal{I}}(\tau_k(t))(\mathbf{v}_k, S_k), \qquad k \in \mathbb{N}.$$

Passing to the limit, and using (11.5), (11.6) and Proposition 11.1, we conclude that $\bar{\mathcal{I}}(t)(\mathbf{v}(t), S(t)) \leq \bar{\mathcal{I}}(t)(\mathbf{v}, S)$, so $(\mathbf{v}(t), S(t))$ is a minimizer of $\bar{\mathcal{I}}(t)$ in $\mathfrak{B}_{S(t)}$. Repeating the argument with $(\mathbf{v}, S) = (\mathbf{v}(t), S(t))$, we obtain that the inequality in (11.5) becomes in fact the first equality of (11.4). ∎

12. Energy balance. In this section we show that the function $t \mapsto \bar{I}(t) := \bar{\mathcal{I}}(t)(\mathbf{v}(t), S(t))$ is absolutely continuous on $[0,1]$ with derivative $(\bar{\mathcal{I}}^c)'(t)(\mathbf{v}(t))$. This is the energy balance property of the system (see, e.g., [47, 25] for a physical interpretation and discussion), and shows that $\bar{\mathcal{I}}^c$ is indeed the conservative part of the energy, while $\mathcal{S}_0$ is the dissipative part. The proof follows the lines of [18, Sect. 5].

Proposition 12.1. *The function $\bar{I}$ is absolutely continuous on $[0,1]$, and*

$$\bar{I}'(t) = (\bar{\mathcal{I}}^c)'(t)(\mathbf{v}(t)) \qquad a.e.\ t \in [0,1].$$

Proof. Thanks to Proposition 9.1, we can apply Fatou's lemma to the function $\theta_\infty := \limsup_{k\to\infty} \theta_k$ and thus obtain that $\theta_\infty \in L^\infty([0,1])$ and

$$\limsup_{k\to\infty} \int_0^{\tau_k(t)} \theta_k(s)\,\mathrm{d}s \leq \int_0^t \theta_\infty(s)\,\mathrm{d}s \tag{12.1}$$

for all $t \in [0,1]$. It was proved in Proposition 11.2 that for all $s \in [0,1]$,

$$\lim_{k\to\infty} \bar{\mathcal{I}}^c(s)(\mathbf{v}_k(s)) = \bar{\mathcal{I}}^c(s)(\mathbf{v}(s)). \tag{12.2}$$

As in [18, Lemma 5.1], we apply [25, Prop. 3.3] to conclude that

$$\lim_{k\to\infty} (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}_k(s)) = (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s)), \qquad s \in [0,1] \tag{12.3}$$

Indeed, let us check that the assumptions of [25, Prop. 3.3] are satisfied. The functional $\bar{\mathcal{I}}^c : [0,1] \times \mathfrak{A} \to \mathbb{R}$ is of class C^1 in its first variable (by Corollary 6.4) and lower semicontinuous in its second variable with respect to the convergence of Definition 7.2 (by Proposition 7.5). The existence of the modulus of continuity stated in Corollary 6.4 shows then that the assumptions of [25, Prop. 3.3] are satisfied and, hence, (12.3) holds true. Thus, $\theta_\infty(s) = (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s))$ for a.e. $s \in [0,1]$. Therefore, by (11.4), (11.5), Proposition 9.1 and (12.1),

$$\bar{\mathcal{I}}(t)(\mathbf{v}(t), S(t)) = \lim_{k\to\infty} \bar{\mathcal{I}}(\tau_k(t))(\mathbf{v}_k(t), S_k(t)) \leq \bar{\mathcal{I}}(0)(\mathbf{v}^0, S^0) + \int_0^t (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s))\,\mathrm{d}s,$$

which is one inequality of the energy balance. In particular,

$$\sup_{t\in[0,1]} \bar{\mathcal{I}}^c(t)(\mathbf{v}(t)) < \infty.$$

Thanks to ($\bar{\mathrm{W}}$6) and Proposition 6.3, we infer that

$$\sup_{t\in[0,1]} \bar{\mathcal{I}}^c(0)(\mathbf{v}(t)) < \infty. \tag{12.4}$$

For the reverse inequality of the energy balance, we employ the technique of [16, Lemmas 4.12 and 5.7], ultimately based on the approximation of a Lebesgue integral by Riemann sums. In our context, we use the formulation of [18, Sect. 5.2], according to which for each $t \in [0,1]$ there exists a sequence $\left\{\{s_k^i\}_{0\leq i\leq i_k}\right\}_{k\in\mathbb{N}}$ of subdivisions of $[0,t]$ such that

$$0 = s_0^0 < s_k^1 < \cdots < s_k^{i_k} = t, \qquad \lim_{k\to\infty} \max_{1\leq i\leq i_k} \left(s_k^i - s_k^{i-1}\right) = 0$$

and

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \left(\bar{\mathcal{I}}^c\right)' (s_k^i)(\mathbf{v}(s_k^i)) - \left(\bar{\mathcal{I}}^c\right)' (s)(\mathbf{v}(s)) \right| \mathrm{d}s = 0. \tag{12.5}$$

Now, the existence of the modulus of continuity of Corollary 6.4, as well as bound (12.4) show that

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| \left(\bar{\mathcal{I}}^c\right)' (s_k^i)(\mathbf{v}(s_k^i)) - \left(\bar{\mathcal{I}}^c\right)' (s)(\mathbf{v}(s_k^i)) \right| \mathrm{d}s = 0. \tag{12.6}$$

Combining (12.5) and (12.6), we obtain

$$\lim_{k\to\infty} \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} \left| (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s_k^i)) - (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s)) \right| \mathrm{d}s = 0. \tag{12.7}$$

By Corollary 6.4 and Proposition 11.2,

$$\begin{aligned}
\bar{\mathcal{I}}(s_k^i)(\mathbf{v}(s_k^i), S(s_k^i)) &= \bar{\mathcal{I}}(s_k^{i-1})(\mathbf{v}(s_k^i), S(s_k^i)) + \int_{s_k^{i-1}}^{s_k^i} (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s_k^i))\,\mathrm{d}s \\
&\geq \bar{\mathcal{I}}(s_k^{i-1})(\mathbf{v}(s_k^{i-1}), S(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i} (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s_k^i))\,\mathrm{d}s,
\end{aligned}$$

for each $k \in \mathbb{N}$ and $1 \leq i \leq i_k$. Iterating this inequality we obtain

$$\bar{\mathcal{I}}(t)(\mathbf{v}(t), S(t)) \geq \bar{\mathcal{I}}(0)(\mathbf{v}^0, S^0) + \sum_{i=1}^{i_k} \int_{s_k^{i-1}}^{s_k^i} (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s_k^i)) \,\mathrm{d}s,$$

and passing to the limit as $k \to \infty$ and using (12.7) we arrive at

$$\bar{\mathcal{I}}(t)(\mathbf{v}(t), S(t)) \geq \bar{\mathcal{I}}(0)(\mathbf{v}^0, S^0) + \int_0^t (\bar{\mathcal{I}}^c)'(s)(\mathbf{v}(s)) \,\mathrm{d}s,$$

which is the other inequality of the energy balance, and concludes the proof. ∎

13. Proof of Theorem 5.2. Let $(\mathbf{u}^0, S^0) \in \mathfrak{B}$ be a minimizer of $\mathcal{I}$ in $\mathfrak{B}_{S^0}(0)$, and define $\mathbf{v}^0 := \boldsymbol{\phi}(0) \circ \mathbf{u}^0$. By Proposition 6.1 and definition (6.1), $(\mathbf{v}^0, S^0)$ is a minimizer of $\bar{\mathcal{I}}(0)$ in $\mathfrak{B}_{S^0}$. The procedure of Sections 9–12, and especially Propositions 10.3, 11.2 and 12.1, show that for each $t \in [0,1]$ there exists $\mathbf{v}(t) \in \mathfrak{A}$ such that, when one defines $S(t) := S^0 \cup \bigcup_{s \in [0,t]} C(\mathbf{v}(s))$, the following conditions hold:

(a) $(\mathbf{v}(0), S(0)) = (\mathbf{v}^0, S^0)$.
(b) For every $t \in [0,1]$, the pair $(\mathbf{v}(t), S(t))$ is a minimizer of $\bar{\mathcal{I}}(t)$ in $\mathfrak{B}_{S(t)}$.
(c) The function $t \mapsto \bar{I}(t) := \bar{\mathcal{I}}(t)(\mathbf{v}(t), S(t))$ is absolutely continuous on $[0,1]$, and $\bar{I}'(t) = (\bar{\mathcal{I}}^c)'(t)(\mathbf{v}(t))$ for a.e. $t \in [0,1]$.

For each $t \in [0,1]$, call $\mathbf{u}(t) := \boldsymbol{\psi}(t) \circ \mathbf{v}(t)$. Again Proposition 6.1 and definition (6.1) assure that the family $\{(\mathbf{u}(t), S(t))\}_{t \in [0,1]}$ satisfies the conditions of Theorem 5.2, and that the function I defined therein coincides with the function $\bar{I}$ of Proposition 12.1. The expression for $(\bar{\mathcal{I}}^c)'$ given in Corollary 6.4, as well as Proposition 6.1, conclude the proof. ∎

We finish the paper with an interpretation of the term $I'(t)$ given in Theorem 5.2. It was shown in [18, Eq. (2.27)] that if W, Ω, K and $\mathbf{u}(t)$ are regular enough then

$$\begin{aligned} &\int_\Omega D_2 W(\mathbf{x}, D\mathbf{u}(t)) \cdot D\left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t) \circ \mathbf{u}(t)\right) \mathrm{d}\mathbf{x} \\ &= \int_{\Gamma_D} \left(D_2 W\left(\mathbf{x}, D\mathbf{u}(t)\right) \boldsymbol{\nu}\right) \cdot \boldsymbol{\psi}'(t) \,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

where $\boldsymbol{\nu} : \Gamma_D \to \mathbb{S}^{n-1}$ is the normal vector. Moreover, assume, additionally, that for each $t \in [0,1]$ and $\mathbf{a} \in C(\mathbf{u}(t))$, the set $\operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})$ is open and of class C^2. For example, if $\mathbf{u}(t)$ is in $W^{1,p}$ with $p > n-1$, then $\operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})$ is open (see [51]), but there are no available results as for its regularity. Then $\partial^* \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})$ coincides with $\partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})$, and is a C^2 manifold of dimension $n-1$ without boundary. Thus, we can apply the divergence theorem on manifolds (see, e.g., [56, Eq. 7.6]) to conclude that

$$\begin{aligned} &\int_{\partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})} \operatorname{div}^{\partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})} \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t)\right) \mathrm{d}\mathcal{H}^{n-1} \\ &= - \int_{\partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})} \left(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t)\right) \cdot \mathbf{H}(t, \mathbf{a}) \,\mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

where $\mathbf{H}(t, \mathbf{a}) : \partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a}) \to \mathbb{R}^n$ is the mean curvature vector,

$$\mathbf{H}(t, \mathbf{a}) := -\left(\operatorname{div}^{\partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})} \boldsymbol{\nu}_{t,\mathbf{a}}\right) \boldsymbol{\nu}_{t,\mathbf{a}},$$

and $\boldsymbol{\nu}_{t,\mathbf{a}} : \partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a}) \to \mathbb{S}^{n-1}$ is the normal vector of $\partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})$. Hence, under this extra regularity assumptions, we have that

$$\begin{aligned} I'(t) = & \int_{\Gamma_D} \left(D_2 W\left(\mathbf{x}, D\mathbf{u}(t)\right) \boldsymbol{\nu}\right) \cdot \boldsymbol{\psi}'(t) \, \mathrm{d}\mathcal{H}^{n-1} \\ & - \sum_{\mathbf{a} \in C(\mathbf{u}(t))} \kappa_1(\mathbf{a}) \int_{\partial \operatorname{im}_{\mathrm{T}}(\mathbf{u}(t), \mathbf{a})} \big(\boldsymbol{\psi}'(t) \circ \boldsymbol{\phi}(t)\big) \cdot \mathbf{H}(t, \mathbf{a}) \, \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

Thus, from this expression we may rephrase the energy balance property as follows: the increment in the energy equals the work of the external forces given by the boundary conditions, and acting on Γ_D and also on the new surface created from cavitation by the deformation, the latter through the mean curvature.

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REFERENCES

[1] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford University Press, New York, 2000.

[2] L. Badea and M. Predeleanu, *On the dynamic cavitation in solids*, in Advanced Methods in Materials Processing Defects, M. Predeleanu and P. Gilormini, eds., vol. 45 of Studies in Applied Mechanics, Elsevier, 1997, pp. 3–12.

[3] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Ration. Mech. Anal., 63 (1977), pp. 337–403.

[4] ———, *Discontinuous equilibrium solutions and cavitation in nonlinear elasticity*, Philos. Trans. R. Soc. Lond. Ser. A, 306 (1982), pp. 557–611.

[5] ———, *Some open problems in elasticity*, in Geometry, mechanics, and dynamics, Springer, New York, 2002, pp. 3–59.

[6] J. M. Ball, J. C. Currie, and P. J. Olver, *Null Lagrangians, weak continuity, and variational problems of arbitrary order*, J. Funct. Anal., 41 (1981), pp. 135–174.

[7] J. M. Ball and F. Murat, *$W^{1,p}$-quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal., 58 (1984), pp. 225–253.

[8] S. Biwa, *Cavitation in finite elasticity with surface energy effects*, Int. J. Non-Linear Mech., 41 (2006), pp. 1084–1094.

[9] E. Bouchbinder, T.-S. Lo, and I. Procaccia, *Dynamic failure in amorphous solids via a cavitation instability*, Phys. Rev. E, 77 (2008), p. 025101.

[10] H. Brezis and L. Nirenberg, *Degree theory and BMO. I. Compact manifolds without boundaries*, Selecta Math. (N.S.), 1 (1995), pp. 197–263.

[11] C. B. Bucknall, D. S. Ayre, and D. J. Dijkstra, *Detection of rubber particle cavitation in toughened plastics using thermal contraction tests*, Polymer, 41 (2000), pp. 5937–5947.

[12] M.-S. Chou-Wang and C. O. Horgan, *Cavitation in nonlinear elastodynamics for neo-Hookean materials*, Int. J. Eng. Sci., 27 (1989), pp. 967–973.

[13] T.-J. Chuang, K. I. Kagawa, J. R. Rice, and L. B. Sills, *Non-equilibrium models for diffusive cavitation of grain interfaces*, Acta Metall., 27 (1979), pp. 265–284.

[14] S. Conti and C. De Lellis, *Some remarks on the theory of elasticity for compressible Neohookean materials*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2 (2003), pp. 521–549.

[15] A. Cristiano, A. Marcellan, R. Long, C.-Y. Hui, J. Stolk, and C. Creton, *An experimental investigation of fracture by cavitation of model elastomeric networks*, J. Polym. Sci. B Polym. Phys., 48 (2010), pp. 1409–1422.

[16] G. Dal Maso, G. A. Francfort, and R. Toader, *Quasistatic crack growth in nonlinear elasticity*, Arch. Ration. Mech. Anal., 176 (2005), pp. 165–225.

[17] G. Dal Maso, A. Giacomini, and M. Ponsiglione, *A variational model for quasistatic crack growth in nonlinear elasticity: some qualitative properties of the solutions*, Boll. Unione Mat. Ital. (9), 2 (2009), pp. 371–390.

[18] G. Dal Maso and G. Lazzaroni, *Quasistatic crack growth in finite elasticity with non-interpenetration*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), pp. 257–290.

[19] G. Dal Maso and R. Toader, *A model for the quasi-static growth of brittle fractures: existence and approximation results*, Arch. Ration. Mech. Anal., 162 (2002), pp. 101–135.

[20] U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab, *Minimal surfaces. I*, vol. 295 of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, 1992.

[21] A. Dorfmann, *Stress-softening of elastomers in hydrostatic tension*, Acta Mech., 165 (2003), pp. 117–137.

[22] H. Federer, *Geometric measure theory*, Springer, New York, 1969.

[23] C. Fond, *Cavitation criterion for rubber materials: A review of void-growth models*, J. Polym. Sci. B Polym. Phys., 39 (2001), pp. 2081–2096.

[24] I. Fonseca and W. Gangbo, *Degree theory in analysis and applications*, Oxford University Press, New York, 1995.

[25] G. Francfort and A. Mielke, *Existence results for a class of rate-independent material models with nonconvex elastic energies*, J. Reine Angew. Math., 595 (2006), pp. 55–91.

[26] G. A. Francfort and C. J. Larsen, *Existence and convergence for quasi-static evolution in brittle fracture*, Comm. Pure Appl. Math., 56 (2003), pp. 1465–1500.

[27] A. Gent, *Cavitation in rubber: a cautionary tale*, Rubber Chem. Tech., 63 (1990), pp. G49–G53.

[28] A. Gent and C. Wang, *Fracture mechanics and cavitation in rubber-like solids*, J. Mater. Sci., 26 (1991), pp. 3392–3395.

[29] A. N. Gent and P. B. Lindley, *Internal rupture of bonded rubber cylinders in tension*, Proc. Roy. Soc. London Ser. A, 249 (1959), pp. 195–205.

[30] A. N. Gent and D. A. Tompkins, *Surface energy effects for small holes or particles in elastomers*, J. Polym. Sci. A-2 Polym. Phys., 7 (1969), pp. 1483–1487.

[31] S. H. Goods and L. M. Brown, *The nucleation of cavities by plastic deformation*, Acta Metall., 27 (1979), pp. 1–15.

[32] A. Goriely, D. E. Moulton, and R. Vandiver, *Elastic cavitation, tube hollowing, and differential growth in plants and biological tissues*, EPL, 91 (2010), p. 18001.

[33] A. A. Griffith, *The phenomena of rupture and flow in solids*, Philos. Trans. Roy. Soc. London Ser. A, 221 (1921), pp. 163–198.

[34] A. L. Gurson, *Continuum theory of ductile rupture by void nucleation and growth: Part I—Yield criteria and flow rules for porous ductile media*, J. Eng. Mater. Technol., 99 (1977), pp. 2–15.

[35] D. Henao, *Variational Modelling of Cavitation and Fracture in Nonlinear Elasticity*, PhD thesis, University of Oxford, 2009.

[36] D. Henao and C. Mora-Corral, *Invertibility and weak continuity of the determinant for the modelling of cavitation and fracture in nonlinear elasticity*, Arch. Ration. Mech. Anal, 197 (2010), pp. 619–655.

[37] ———, *Fracture surfaces and the regularity of inverses for BV deformations*, Arch. Ration. Mech. Anal., 201 (2011), pp. 575–629.

[38] ———, *Lusin's condition and the distributional determinant for deformations with finite energy*, Adv. Calc. Var., 5 (2011), pp. 355–409.

[39] D. Henao and S. Serfaty, *Energy estimates and cavity interaction for a critical-exponent cavitation model*, Comm. Pure Appl. Math., (2012). Doi: 10.1002/cpa.21396.

[40] C. Horgan and D. Polignone, *Cavitation in nonlinearly elastic solids: A review*, Appl. Mech. Rev., 48 (1995), pp. 471–485.

[41] D. H. Kaelble, *Cavitation in viscoelastic media*, Trans. Soc. Rheol., 15 (1971), pp. 275–296.

[42] G. Lazzaroni, *Quasistatic crack growth in finite elasticity with Lipschitz data*, Ann. Mat. Pura Appl. (4), 190 (2011), pp. 165–194.

[43] O. Lopez-Pamies, M. I. Idiart, and T. Nakamura, *Cavitation in elastomeric solids: I—A defect-growth theory*, J. Mech. Phys. Solids, 59 (2011), pp. 1464–1487.

[44] A. Mainik and A. Mielke, *Existence results for energetic models for rate-independent systems*, Calc. Var. Partial Differ. Equ., 22 (2005), pp. 73–99.

[45] A. Mielke, *Energetic formulation of multiplicative elasto-plasticity using dissipation distances*, Contin. Mech. Thermodyn., 15 (2003), pp. 351–382.

[46] ———, *Existence of minimizers in incremental elasto-plasticity with finite strains*, SIAM J. Math. Anal., 36 (2004), pp. 384–404.

[47] ———, *Evolution of rate-independent systems*, in Evolutionary equations. Vol. II, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2005, pp. 461–559.

[48] A. Mielke and F. Theil, *On rate-independent hysteresis models*, NoDEA-Nonlinear Differ. Equ. Appl., 11 (2004), pp. 151–189.

[49] A. Mielke, F. Theil, and V. I. Levitas, *A variational formulation of rate-independent phase transformations using an extremum principle*, Arch. Ration. Mech. Anal., 162 (2002), pp. 137–177.
[50] S. Müller, Det = det. *A remark on the distributional determinant*, C. R. Acad. Sci. Paris Sér. I Math., 311 (1990), pp. 13–17.
[51] S. Müller and S. J. Spector, *An existence theory for nonlinear elasticity that allows for cavitation*, Arch. Ration. Mech. Anal., 131 (1995), pp. 1–66.
[52] S. Müller, Q. Tang, and B. S. Yan, *On a new class of elastic deformations not allowing for cavitation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 11 (1994), pp. 217–243.
[53] K. A. Pericak-Spector and S. J. Spector, *Nonuniqueness for a hyperbolic system: cavitation in nonlinear elastodynamics*, Arch. Ration. Mech. Anal., 101 (1988), pp. 293–317.
[54] N. Petrinic, J. L. Curiel Sosa, C. R. Siviour, and B. C. F. Elliott, *Improved predictive modelling of strain localisation and ductile fracture in a Ti-6Al-4V alloy subjected to impact loading*, Journal de Physique IV, 134 (2006), pp. 147–155.
[55] J. R. Rice and D. M. Tracey, *On the ductile enlargement of voids in triaxial stress fields*, J. Mech. Phys. Solids, 17 (1969), pp. 201–217.
[56] L. Simon, *Lectures on geometric measure theory*, vol. 3 of Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra, 1983.
[57] J. Sivaloganathan and S. J. Spector, *On the existence of minimizers with prescribed singular points in nonlinear elasticity*, J. Elast., 59 (2000), pp. 83–113.
[58] M. Spivak, *A comprehensive introduction to differential geometry. Vol. IV*, Publish or Perish, Boston, 1975.
[59] V. Šverák, *Regularity properties of deformations with finite energy*, Arch. Ration. Mech. Anal., 100 (1988), pp. 105–127.
[60] V. Tvergaard, *Material failure by void growth and coalescence*, in Advances in Applied Mechanics, Academic Press, San Diego, 1990, pp. 83–151.
[61] M. Williams and R. Schapery, *Spherical flaw instability in hydrostatic tension*, Int. J. Fract. Mech., 1 (1965), pp. 64–71.
[62] X. Xu and D. Henao, *An efficient numerical method for cavitation in nonlinear elasticity*, Math. Models Methods Appl. Sci., 21 (2011), pp. 1733–1760.
[63] F. Zaïri, M. Naït-Abdelaziz, and K. Woznica, *On the cavitation problem and the influence of the surface energy in a viscoplastic medium*, C. R. Mecanique, 333 (2005), pp. 431–437.
[64] W. P. Ziemer, *Weakly differentiable functions*, Springer, New York, 1989.