EXISTENCE FOR NONLOCAL VARIATIONAL PROBLEMS IN PERIDYNAMICS

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Abstract. We present an existence theory based on minimization of the nonlocal energies appearing in peridynamics, which is a nonlocal continuum model in Solid Mechanics that avoids the use of deformation gradients. We employ the direct method of the calculus of variations in order to find minimizers of the energy of a deformation. Lower semicontinuity is proved under a weaker condition than convexity, whereas coercivity is proved via a nonlocal Poincaré inequality. We cover Dirichlet, Neumann and mixed boundary conditions. The existence theory is set in the Lebesgue L^p spaces and in the fractional Sobolev $W^{s,p}$ spaces, for $0 < s < 1$ and $1 < p < \infty$.

1. Introduction

Peridynamics is a nonlocal continuum model in Solid Mechanics introduced by Silling [39]. The main difference with the usual Cauchy–Green elasticity [15, 9] relies on the non-locality, which is reflected in the fact that points separated by a positive distance exert a force upon each other. Mathematically, deformations are not assumed to be weakly differentiable, in contrast with classical continuum mechanics, and in particular hyperelasticity, where they are required to be Sobolev. This makes peridynamics a suitable framework for problems where discontinuities appear naturally, such as fracture, dislocation, or, in general, multiscale materials. Later developments and variants of the original peridynamic theory are to be found in [40, 30, 41].

The peridynamic equation of motion [39] is typically a second-order hyperbolic equation whose corresponding elliptic operator is nonlocal, of the type of a p-Laplacian. This is in contrast to nonlocal diffusion problems (see, e.g., [8]), which lead to parabolic equations, again with a corresponding elliptic operator being nonlocal and often taken as the p-Laplacian.

This paper focuses on the variational formulation of equilibrium states in peridynamics. We thus ignore the time dependence, and make the fundamental object to be the energy (sometimes called *macroelastic*, see [39]) of the deformation $\mathbf{u} : \Omega \to \mathbb{R}^d$, to which external body and surface forces can be added. The macroelastic energy has typically the form

$$\int_\Omega \int_\Omega w(\mathbf{x} - \mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ represents the body, and $w : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ is the pairwise potential function. Here $n \in \mathbb{N}$ is the space dimension, and $d \in \mathbb{N}$ the dimension of the target space; in real applications, d should coincide with n (and with 3), but since we also want to treat the antiplane case $d = 1$, we prefer to carry out the proofs for a general d. Expression (1.1) reflects the two main features of the peridynamic theory: the non-locality (expressed as a double integral) and the absence of gradients, which are often replaced by weighted difference quotients.

As nearby particles interact with a stronger force than distant ones, it is natural to assume that the function $w(\cdot, \tilde{\mathbf{y}})$ blows up to infinity at $\mathbf{0}$, for each $\tilde{\mathbf{y}} \in \mathbb{R}^d$. It is also natural to assume that distant particles do not interact at all, so $w(\tilde{\mathbf{x}}, \cdot) \equiv 0$ if $|\tilde{\mathbf{x}}|$ is larger than a so-called horizon, where the function w has been normalized so that its minimum value is 0. As a matter of fact, the assumption that w vanishes when $|\tilde{\mathbf{x}}|$ is large adds a further difficulty in the mathematical analysis.

The function w has to satisfy additional mathematical properties in order to meet some physical requirements such as objectivity (see [41, Sect. 4]). In this paper we do not insist on those properties, but rather focus on the conditions on w that guarantee existence of minimizers for the total energy of the deformation. In truth, we work with functionals that are local perturbations of (1.1), corresponding to the addition of external forces. Our aim is to prove the existence of minimizers for those functionals under fairly general assumptions on the potential w; in particular, our existence theorems cover most of the existence results in peridynamics based on minimization, such as [2, 18, 26]. To that aim, we employ the direct method of the Calculus of Variations (e.g., [16]), so that semicontinuity and coercivity are the two main ingredients.

The first issue we find in our analysis is to determine the proper functional space to set the problem, and this depends on the growth conditions assumed on w. In this paper, they are quite general, and cover the cases of the papers mentioned above, in particular, when the singularity of $w(\cdot, \tilde{\mathbf{y}})$ at $\mathbf{0}$ obeys an inverse power law of the form

$$w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \sim \frac{|\tilde{\mathbf{y}}|^p}{|\tilde{\mathbf{x}}|^\alpha} \qquad \text{for } \tilde{\mathbf{x}} \sim \mathbf{0}, \tag{1.2}$$

for some $1 < p < \infty$ and $0 \leq \alpha < n + p$. For this special growth, we distinguish the weakly singular case $0 \leq \alpha < n$ and the strongly singular case $n < \alpha < n + p$.

When $0 \leq \alpha < n$, the analysis of the lower semicontinuity is reduced to the recent study carried out by Elbau [22] and lies in the functional framework of Lebesgue L^p spaces. The weak lower semicontinuity is proved in [22] to be equivalent to an interesting convexity property of the integrand w, of a different nature that those convexity properties equivalent to weak lower semicontinuity for local problems (see, e.g., [16, Ch. 8]); we will discuss this issue in Section 3 in our particular peridynamics framework. The coercivity for the Dirichlet problem was proved by Andreu *et al.* [7] in their study of nonlocal diffusion problems, and later used by [3, 26] in the context of peridynamics. The coercivity for the Neumann and mixed problem was proved by Aksoylu & Mengesha [2] using a Poincaré-type inequality proved by Ponce [36] in his study of nonlocal characterizations of Sobolev spaces (see also [13]). As a matter of fact, we shall need some adaptations of those results to our context. At this point, we ought to mention that *Dirichlet* and *mixed* boundary value problems have a slightly different meaning than for local problems, one the reasons being that L^p functions do not have traces of the boundary $\partial\Omega$. In contrast, Dirichlet conditions in the context of peridynamics prescribe the value of the deformation in a set of positive measure.

The lower semicontinuity in the case $n < \alpha < n+p$ is in fact trivial, since the functional framework is that of the fractional Sobolev spaces $W^{s,p}$ with $s = \frac{\alpha - n}{p}$, and weak convergence in $W^{s,p}$ implies (for a subsequence) convergence a.e. The coercivity, on the other hand, is a consequence of an improved Poincaré-type inequality in fractional Sobolev spaces recently proved in Hurri-Syrjänen & Vähäkangas [27]. It is worth mentioning that the need of improved Poincaré-type inequalities is a result of the assumption that $w(\tilde{\mathbf{x}}, \cdot)$ vanishes for $|\tilde{\mathbf{x}}|$ large.

The existence theory for the critical case $\alpha = n$ is also covered by reducing it to the case $0 \leq \alpha < n$ and to the functional framework of L^p spaces. In doing that, we do not provide a full characterization of the lower semicontinuity, so that our conditions on w may not be optimal.

Nonlocal variational problems, of which (1.1) is a particular case, have attracted a great attention in the mathematical community in the last two decades, coming from fields such as statistical mechanics [5], abstract results involving nonlocality of gradients [34, 33], ferromagnetism [38], nonlocal p-Laplacian [8], imaging [28, 24, 12], characterization of Sobolev spaces [13, 14, 36, 37, 32], as well as, of course, peridynamics [2, 18, 21, 26, 25, 4].

The outline of the paper is as follows. In Section 2 we present the mechanical model, make the general assumptions of the paper, and explain the notation used. In Section 3 we prove the lower semicontinuity of the nonlocal energy in the weak topology of L^p, by means of a nonlocal convexity property of the integrand w. In Section 4 we obtain the inequalities that allows us to prove the coercivity in L^p, for Dirichlet, mixed and Neumann boundary conditions. Section 5 uses the results of Sections 3 and 4 to show the existence of minimizers of the energy in the L^p context. Section 6 presents the key inequalities for the coercivity in $W^{s,p}$, again for the three types of boundary conditions. Section 7 proves the existence theorems for deformations in $W^{s,p}$, using the results of the previous two sections. In Section 8 we write down the Euler–Lagrange equations corresponding to the minimizers.

2. Model, notation and general assumptions

In this section we present the mechanical model. We refer to the papers [39, 40, 2, 41] for further motivation and physical interpretation. In particular, our model follows closely that of Hinds & Radu [26].

Let Ω be a non-empty open bounded subset of $\mathbb{R}^n$ representing the reference configuration of the body. The nonlocal theory requires the distinction within Ω of an interior part Ω_0 of the body and nonlocal boundary Ω_1, so that Ω is the disjoint union of Ω_0 and Ω_1. When needed, we will assume that $\Omega_0 + B(\mathbf{0}, \delta) \subset \Omega$ for some $\delta > 0$. This number $\delta > 0$ represents the horizon of the potential w, and measures the distance after which there is no interaction between particles, i.e., $w(\tilde{\mathbf{x}}, \cdot) = 0$ if $|\tilde{\mathbf{x}}| \geq \delta$. Thus, we are making the natural assumption that the inner part Ω_0 of the body does not interact with the exterior $\mathbb{R}^n \setminus \Omega$. In some studies (e.g., [2]), the stronger condition $\Omega_0 + B(\mathbf{0}, \delta) = \Omega$ is imposed, but this is not really needed in our model. Of course, $\Omega_0 + B(\mathbf{0}, \delta)$ denotes the set of points in $\mathbb{R}^n$ that can be expressed as a sum of an element of Ω_0 plus an element of $B(\mathbf{0}, \delta)$, and $B(\mathbf{0}, \delta)$ is the open ball of centre $\mathbf{0}$ and radius δ. The set Ω can thus be regarded as a nonlocal closure of Ω_0, and the deformation $\mathbf{u}$ is defined on the whole Ω. In fact, we will assume $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ for some $1 < p < \infty$.

The macroeslastic energy of a deformation $\mathbf{u}$ is calculated through (1.1). Here $w : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ is the pairwise potential function, and $\tilde{\Omega}$ is the set of $\mathbf{x} - \mathbf{x}'$ with $\mathbf{x}, \mathbf{x}' \in \Omega$. Clearly, $\tilde{\Omega}$ is open. Thanks to Fubini's theorem, without loss of generality we may assume that

$$w(-\tilde{\mathbf{x}}, -\tilde{\mathbf{y}}) = w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \qquad \text{for all } \tilde{\mathbf{x}} \in \tilde{\Omega} \text{ and } \tilde{\mathbf{y}} \in \mathbb{R}^d, \tag{2.1}$$

just by substituting $w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ with

$$\frac{1}{2} \left[w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) + w(-\tilde{\mathbf{x}}, -\tilde{\mathbf{y}}) \right].$$

Equality (2.1) will be assumed throughout the paper, even though not explicitly stated; in turn, it is the realization in this context of Newton's third law (see [39, Eqs. (6) and (27)]) The function w is required to be $\mathcal{L}^n(\tilde{\Omega}) \times \mathcal{B}(\mathbb{R}^d)$-measurable, i.e., Lebesgue measurable in the first n variables, and Borel measurable in the last d variables. This guarantees that the integrand in (1.1) is Lebesgue measurable. The expression *a.e.* for *almost everywhere* or *almost every* refers to the Lebesgue measure, which is denoted by $\mathcal{L}^n$ when the underlying space is $\mathbb{R}^n$.

External body and surfaces forces are added to the macroeslastic energy to conform the total energy. Those external forces have the form

$$-\int_{\Omega} F(\mathbf{x}, \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \tag{2.2}$$

for some potential function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ assumed to be $\mathcal{L}^n(\Omega) \times \mathcal{B}(\mathbb{R}^d)$-measurable. The part of the integral (2.2) in Ω_0 corresponds to the body force, while the part in Ω_1 corresponds to the surface force; we recall that, in the context of peridynamics, notions like *boundary* or *surface* have positive volume. The distinction between Ω_0 and Ω_1 is part of the mechanical model, but it hardly affects the mathematical analysis. In many practical cases, both body and surface forces are linear, so that

$$F(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{y}, \qquad \mathbf{x} \in \Omega, \quad \mathbf{y} \in \mathbb{R}^d \tag{2.3}$$

for a given measurable $\mathbf{f} : \Omega \to \mathbb{R}^d$. Here $\cdot$ denotes the scalar (inner) product in $\mathbb{R}^d$.

Thus, the total energy of a deformation is

$$\mathcal{I}(\mathbf{u}) := \int_{\Omega} \int_{\Omega} w(\mathbf{x} - \mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} - \int_{\Omega} F(\mathbf{x}, \mathbf{u}(\mathbf{x}))\, \mathrm{d}\mathbf{x}, \tag{2.4}$$

so equilibrium solutions in the static theory correspond to critical points of $\mathcal{I}$. In fact, this paper analyzes the existence of global minimizers of $\mathcal{I}$.

The nonlocal boundary conditions present some peculiarities (see [25, 19]). Dirichlet conditions prescribe $\mathbf{u} = \mathbf{b}$ a.e. in a measurable subset Ω_D of Ω_1, for a given $\mathbf{b} : \Omega_D \to \mathbb{R}^d$. The Neumann part of the nonlocal boundary is $\Omega_1 \setminus \Omega_D$. A pure Dirichlet problem corresponds to $\mathcal{L}^n(\Omega_D) = \mathcal{L}^n(\Omega_1)$, a pure Neumann problem corresponds to $\mathcal{L}^n(\Omega_D) = 0$, while a mixed problem corresponds to $0 < \mathcal{L}^n(\Omega_D) < \mathcal{L}^n(\Omega_1)$. In the particular but interesting case where F is of the form (2.3) with $\mathbf{f} \in L^1(\Omega, \mathbb{R}^d)$ satisfying $\int_\Omega \mathbf{f} = \mathbf{0}$, the energy $\mathcal{I}$ of (2.4) is invariant under translations. Hence, to avoid the trivial non-uniqueness of the pure Neumann problem given by translations, the normalization condition $\int_\Omega \mathbf{u} = \mathbf{0}$ is imposed.

Now we say some words about the notation. We write $\mathbf{x}$ for the coordinates in the reference configuration Ω, and $\mathbf{y}$ in the deformed configuration $\mathbb{R}^d$. We write $\tilde{\mathbf{x}}$ for coordinates in $\tilde{\Omega}$, while $\tilde{\mathbf{y}}$ is reserved for coordinates of functions whose argument is typically a difference between two points in the deformed configuration. Thus, the natural notation for the variables of w is $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$. Vector quantities are written in boldface.

For $1 \leq p \leq \infty$, the Lebesgue L^p spaces are defined in the usual way, and p' denotes the conjungate exponent of p. We will always indicate the domain and target sets, as in, for example, $L^p(\Omega, \mathbb{R}^d)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^p(\Omega)$. Given a measurable subset A of $\mathbb{R}^n$, the expression $⨍_A \mathbf{u}$ indicates $\frac{1}{\mathcal{L}^n(A)} \int_A \mathbf{u}$.

We will also use fractional Sobolev spaces: for $0 < s < 1$ and $1 \le p < \infty$, the space $W^{s,p}(A,\mathbb{R}^d)$ is the set of functions $\mathbf{u} \in L^p(A,\mathbb{R}^d)$ such that the fractional Sobolev seminorm

$$|\mathbf{u}|_{W^{s,p}(A,\mathbb{R}^d)} := \left(\int_A \int_A \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x} - \mathbf{x}'|^{n+sp}} \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}}$$

is finite. The corresponding norm is defined as

$$\|\mathbf{u}\|_{W^{s,p}(A,\mathbb{R}^d)} := \left(\|\mathbf{u}\|_{L^p(A,\mathbb{R}^d)}^p + |\mathbf{u}|_{W^{s,p}(A,\mathbb{R}^d)}^p \right)^{\frac{1}{p}}.$$

Weak convergence is indicated by $\rightharpoonup$, while strong or a.e. convergence is indicated by $\to$. We recall that $W^{s,p}$ is a reflexive Banach space when $1 < p < \infty$.

For any real-valued function w, its positive and negative parts are denoted, respectively, by $w^+ := \max\{w, 0\}$ and $w^- := -\min\{w, 0\}$. The characteristic function of a subset A of $\mathbb{R}^n$ is denoted by χ_A

For the convenience of the reader, we write down the fractional Sobolev immersions that will be used in the paper. The following result is well known; proofs can be found, e.g., in [1, Ch. 7], [31, Ch. 14] or [17, Sects. 6–8].

Proposition 2.1. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$. Let $0 < s < 1$ and $1 \le p < \infty$. Then the following assertions hold:*

i) If $sp < n$, define

$$p^* := \frac{np}{n - sp}.$$

Then $W^{s,p}(\Omega,\mathbb{R}^d)$ is continuously embedded in $L^q(\Omega,\mathbb{R}^d)$ for all $q \in [1, p^*]$, and compactly embedded for all $q \in [1, p^*)$.

ii) If $sp = n$, then $W^{s,p}(\Omega,\mathbb{R}^d)$ is compactly embedded in $L^q(\Omega,\mathbb{R}^d)$ for all $q \in [1,\infty)$.

iii) If $sp > n$, define

$$\alpha^* := \frac{sp - n}{p}$$

Then $W^{s,p}(\Omega,\mathbb{R}^d)$ is continuously embedded in $C^{0,\alpha}(\Omega,\mathbb{R}^d)$ for all $\alpha \in (0, \alpha^*]$, and compactly embedded for all $\alpha \in (0, \alpha^*)$.

In the statement above, the set $C^{0,\alpha}(\Omega,\mathbb{R}^d)$ denotes the Banach space of Hölder continuous functions (up to the boundary) of exponent α.

3. Weak lower semicontinuity in L^p

The first question that arises in an existence theory is the choice of the functional space. If the growth of w is of the form (1.2) with $0 \le \alpha < n$ then L^p is the natural space to set the problem, as shown in the following inequality.

Lemma 3.1. *Let Ω be a bounded measurable subset of $\mathbb{R}^n$. Let $1 \le p < \infty$ and $0 \le \alpha < n$. Then there exist $C_1, C_2 > 0$, depending only on n, p, α and Ω, such that for any $\mathbf{u} \in L^p(\Omega,\mathbb{R}^d)$,*

$$C_1 \int_\Omega \left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p \mathrm{d}\mathbf{x} \le \int_\Omega \int_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x} - \mathbf{x}'|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le C_2 \int_\Omega \left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p \mathrm{d}\mathbf{x}.$$

Proof. Using Jensen's inequality, we find that for all $\mathbf{x} \in \Omega$,

$$\left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p = \left| \fint_\Omega [\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')] \, \mathrm{d}\mathbf{x}' \right|^p \le \fint_\Omega |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \, \mathrm{d}\mathbf{x}' \le \left(\sup_{\mathbf{x}' \in \Omega} |\mathbf{x} - \mathbf{x}'|^\alpha \right) \fint_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x} - \mathbf{x}'|^\alpha} \, \mathrm{d}\mathbf{x}',$$

so, denoting by diam the diameter of a set, we have that

$$\int_\Omega \left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p \mathrm{d}\mathbf{x} \le (\operatorname{diam} \Omega)^\alpha \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega \int_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x} - \mathbf{x}'|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x},$$

which shows the first inequality. For the second, we have that

$$\int_\Omega \int_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x} - \mathbf{x}'|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le 2^{p-1} \int_\Omega \int_\Omega \frac{\left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p + \left| \mathbf{u}(\mathbf{x}') - \fint_\Omega \mathbf{u} \right|^p}{|\mathbf{x} - \mathbf{x}'|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = 2^p \int_\Omega \int_\Omega \frac{\left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p}{|\mathbf{x} - \mathbf{x}'|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}.$$

Now,

$$\int_\Omega \int_\Omega \frac{\left|\mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u}\right|^p}{|\mathbf{x}-\mathbf{x}'|^\alpha} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \le \left(\sup_{\mathbf{x}\in\Omega} \int_\Omega \frac{1}{|\mathbf{x}-\mathbf{x}'|^\alpha} \mathrm{d}\mathbf{x}' \right) \int_\Omega \left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p \mathrm{d}\mathbf{x}$$

and

$$\sup_{\mathbf{x}\in\Omega} \int_\Omega \frac{1}{|\mathbf{x}-\mathbf{x}'|^\alpha} \, \mathrm{d}\mathbf{x}' \le \int_{B(\mathbf{0}, \operatorname{diam}\Omega)} \frac{1}{|\mathbf{x}|^\alpha} \, \mathrm{d}\mathbf{x} = \sigma_n \frac{(\operatorname{diam}\Omega)^{n-\alpha}}{n-\alpha},$$

where σ_n denotes the area of the unit sphere in $\mathbb{R}^n$. This shows the second inequality. $\square$

Lemma 3.1 is not actually used in the paper, but it explains why L^p is the correct space under the growth condition (1.2) with $0 \le \alpha < n$.

The main results of this section show the lower semicontinuity of the macroeslastic energy (1.1) under weak convergence in L^p. A useful tool will be Young measures. Since they are only used in this section, we will not explain them in detail, but rather we assume the reader to have some familiarity with them. We just mention their fundamental property, while we refer for the proofs and more background to [35], [10, Sect. 4.3] and, in particular, [23, Ch. 8], whose notation is closely followed here. The above-mentioned fundamental property is that if $1 \le p < \infty$ and $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ is a sequence bounded in $L^p(\Omega, \mathbb{R}^d)$ then, for a subsequence (not relabelled) there exists a Young measure $(\nu_{\mathbf{x}})_{\mathbf{x}\in\Omega}$ such that $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ converges to $(\nu_{\mathbf{x}})_{\mathbf{x}\in\Omega}$ as $j \to \infty$ in the sense of Young measures. This means that $\nu_{\mathbf{x}}$ is a probability measure in $\mathbb{R}^d$ for a.e. $\mathbf{x} \in \Omega$ with the property that for any Borel set E of $\mathbb{R}^d$, the map $\mathbf{x} \mapsto \nu_{\mathbf{x}}(E)$ is measurable in Ω; moreover,

$$\int_\Omega \int_{\mathbb{R}^d} |\mathbf{y}|^p \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \, \mathrm{d}\mathbf{x} < \infty \tag{3.1}$$

and for every continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$ of compact support,

$$\lim_{j\to\infty} \int_\Omega \varphi(\mathbf{u}_j(\mathbf{x})) \, \mathrm{d}\mathbf{x} = \int_\Omega \int_{\mathbb{R}^d} \varphi(\mathbf{y}) \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \, \mathrm{d}\mathbf{x}.$$

We say that $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ generates $(\nu_{\mathbf{x}})_{\mathbf{x}\in\Omega}$.

We start with an auxiliary result calculating the Young measure of the difference $\mathbf{u}_j(\mathbf{x}) - \mathbf{u}_j(\mathbf{x}')$, for which the following notation is useful.

Definition 3.2. *Given two probability measures μ_1 and μ_2 in $\mathbb{R}^d$, we define its convolution difference $\mu_1 \ominus \mu_2$ as*

$$(\mu_1 \ominus \mu_2)(E) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_E(\mathbf{y} - \mathbf{y}') \, \mathrm{d}\mu_1(\mathbf{y}) \, \mathrm{d}\mu_2(\mathbf{y}')$$

for any Borel set E of $\mathbb{R}^d$.

Note that $\mu_1 \ominus \mu_2$ is a probability measure in $\mathbb{R}^d$, and that for any continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$ of compact support,

$$\int_{\mathbb{R}^d} \varphi(\tilde{\mathbf{y}}) \, \mathrm{d}(\mu_1 \ominus \mu_2)(\tilde{\mathbf{y}}) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(\mathbf{y} - \mathbf{y}') \, \mathrm{d}\mu_1(\mathbf{y}) \, \mathrm{d}\mu_2(\mathbf{y}'). \tag{3.2}$$

In fact, thanks to a standard approximation result, equality (3.2) holds true for continuous functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ such that the left-hand side of (3.2) is finite.

We employ the notation $\mu_1 \times \mu_2$ for the (sometimes called *tensorial* or *Cartesian*) product of the measures μ_1 and μ_2 (see, e.g., [6, Th. 1.74]).

Lemma 3.3. *Let $1 \le p < \infty$. Let $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ be a sequence bounded in $L^p(\Omega, \mathbb{R}^d)$ generating the Young measure $(\nu_{\mathbf{x}})_{\mathbf{x}\in\Omega}$. For each $j \in \mathbb{N}$, define $\bar{\mathbf{u}}_j \in L^p(\Omega \times \Omega, \mathbb{R}^d)$ as*

$$\bar{\mathbf{u}}_j(\mathbf{x}, \mathbf{x}') := \mathbf{u}_j(\mathbf{x}) - \mathbf{u}_j(\mathbf{x}'), \qquad (\mathbf{x}, \mathbf{x}') \in \Omega \times \Omega. \tag{3.3}$$

Then $\{\bar{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ generates the Young measure $(\nu_{\mathbf{x}} \ominus \nu_{\mathbf{x}'})_{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega}$ in $L^p(\Omega \times \Omega, \mathbb{R}^d)$.

Proof. For each $j \in \mathbb{N}$, define $\mathbf{v}_j \in L^p(\Omega \times \Omega, \mathbb{R}^d \times \mathbb{R}^d)$ as

$$\mathbf{v}_j(\mathbf{x}, \mathbf{x}') := (\mathbf{u}_j(\mathbf{x}), \mathbf{u}_j(\mathbf{x}')) , \qquad (\mathbf{x}, \mathbf{x}') \in \Omega \times \Omega.$$

It was essentially proved in Pedregal [34, Prop. 2.3] (see also the proof of [22, Th. 11]) that the sequence $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ generates the Young measure $(\nu_{\mathbf{x}} \times \nu_{\mathbf{x}'})_{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega}$. Note that

$$\|\bar{\mathbf{u}}_j\|_{L^p(\Omega\times\Omega,\mathbb{R}^d)} \leq 2\, \mathcal{L}^n(\Omega)^{\frac{1}{p}} \, \|\mathbf{u}_j\|_{L^p(\Omega,\mathbb{R}^d)} , \qquad j \in \mathbb{N},$$

and let $\left(\bar{\nu}_{(\mathbf{x},\mathbf{x}')}\right)_{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega}$ be the Young measure generated for a subsequence of $\{\bar{\mathbf{u}}_j\}_{j\in\mathbb{N}}$. We use the probabilistic representation formula for Young measures of Ball [11, p. 6], according to which for each $\mathbf{x}_0, \mathbf{x}_0' \in \Omega$,

$$\begin{aligned} \bar{\nu}_{(\mathbf{x}_0,\mathbf{x}_0')}(E) &= \lim_{R\to 0} \lim_{j\to\infty} \frac{\mathcal{L}^{2n}\left(\{(\mathbf{x},\mathbf{x}') \in B((\mathbf{x}_0,\mathbf{x}_0'),R) : \bar{\mathbf{u}}_j(\mathbf{x},\mathbf{x}') \in E\}\right)}{\mathcal{L}^{2n}\left(B((\mathbf{x}_0,\mathbf{x}_0'),R)\right)}, \\ \left(\nu_{\mathbf{x}_0} \times \nu_{\mathbf{x}_0'}\right)(E') &= \lim_{R\to 0} \lim_{j\to\infty} \frac{\mathcal{L}^{2n}\left(\{(\mathbf{x},\mathbf{x}') \in B((\mathbf{x}_0,\mathbf{x}_0'),R) : \mathbf{v}_j(\mathbf{x},\mathbf{x}') \in E'\}\right)}{\mathcal{L}^{2n}\left(B((\mathbf{x}_0,\mathbf{x}_0'),R)\right)}, \end{aligned} \tag{3.4}$$

for any Borel set E of $\mathbb{R}^d$, and any Borel set E' of $\mathbb{R}^{2d}$. Now, let E be an open set of $\mathbb{R}^d$, and define $E' := \left\{(\mathbf{y},\mathbf{y}') \in \mathbb{R}^d \times \mathbb{R}^d : \mathbf{y} - \mathbf{y}' \in E\right\}$, which is easily seen to be an open set of $\mathbb{R}^{2d}$. Then

$$\{(\mathbf{x},\mathbf{x}') \in B((\mathbf{x}_0,\mathbf{x}_0'),R) : \bar{\mathbf{u}}_j(\mathbf{x},\mathbf{x}') \in E\} = \{(\mathbf{x},\mathbf{x}') \in B((\mathbf{x}_0,\mathbf{x}_0'),R) : \mathbf{v}_j(\mathbf{x},\mathbf{x}') \in E'\}$$

for every $R > 0$, and $\chi_E(\mathbf{y} - \mathbf{y}') = \chi_{E'}(\mathbf{y},\mathbf{y}')$ for every $\mathbf{y},\mathbf{y}' \in \mathbb{R}^d$. Consequently, using also (3.4) and Definition 3.2, we find that

$$\begin{aligned} \bar{\nu}_{(\mathbf{x}_0,\mathbf{x}_0')}(E) = \left(\nu_{\mathbf{x}_0} \times \nu_{\mathbf{x}_0'}\right)(E') &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_{E'}(\mathbf{y},\mathbf{y}')\,\mathrm{d}\nu_{\mathbf{x}_0}(\mathbf{y})\,\mathrm{d}\nu_{\mathbf{x}_0'}(\mathbf{y}') \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \chi_E(\mathbf{y}-\mathbf{y}')\,\mathrm{d}\nu_{\mathbf{x}_0}(\mathbf{y})\,\mathrm{d}\nu_{\mathbf{x}_0'}(\mathbf{y}') = \left(\nu_{\mathbf{x}_0} \ominus \nu_{\mathbf{x}_0'}\right)(E). \end{aligned}$$

Thus, the probability measures $\bar{\nu}_{(\mathbf{x}_0,\mathbf{x}_0')}$ and $\nu_{\mathbf{x}_0} \ominus \nu_{\mathbf{x}_0'}$ coincide in all open sets of $\mathbb{R}^d$, and, hence (see, e.g., [6, Prop. 1.8]), $\bar{\nu}_{(\mathbf{x}_0,\mathbf{x}_0')} = \nu_{\mathbf{x}_0} \ominus \nu_{\mathbf{x}_0'}$. By uniqueness, we conclude that the whole sequence $\{\bar{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ generates the Young measure $(\nu_{\mathbf{x}} \ominus \nu_{\mathbf{x}'})_{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega}$. □

The following observation, which follows from an immediate application of Fubini's theorem, will be used throughout the paper: if a property $P(\tilde{\mathbf{x}})$ holds for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$ then the property $P(\mathbf{x} - \mathbf{x}')$ holds for a.e. $(\mathbf{x},\mathbf{x}') \in \Omega \times \Omega$.

A characterization in terms of w of the lower semicontinuity of nonlocal functionals more general than (1.1) with respect to the weak topology of L^p was given in a recent paper of Elbau [22]. As there is a gap in the proof of [22, Th. 11], we have decided, for the convenience of the reader, to write down the full proof of the lower semicontinuity result (the sufficient condition) for the situation at hand. We do not include here the necessity part, as we are only concerned with existence. We instead refer the interested reader to [22, Th. 11] for the proof. Nevertheless, it is worth emphasizing that the convexity property *d)* in the following Proposition 3.4 is equivalent to the sequential weak lower semicontinuity of the functional (1.1) in L^p.

Proposition 3.4. *Let Ω be a non-empty, bounded open subset of $\mathbb{R}^n$. Let $1 < p < \infty$. Let $w : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega}) \times \mathcal{B}(\mathbb{R}^d)$-measurable. Assume that:*

a) There exist $a_1 \in L^1(\tilde{\Omega})$ with

$$a_1 \geq 0, \qquad a_1(-\tilde{\mathbf{x}}) = a_1(\tilde{\mathbf{x}}), \quad \text{a.e. } \tilde{\mathbf{x}} \in \tilde{\Omega}$$

and $1 \leq q < p$ such that for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$ and all $\tilde{\mathbf{y}} \in \mathbb{R}^d$,

$$w^-(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \leq a_1(\tilde{\mathbf{x}})\,(1 + |\tilde{\mathbf{y}}|^q)\,.$$

b) There exist $a_2 \in L^1_{\mathrm{loc}}(\tilde{\Omega})$ with $a_2 \geq 0$ and $C > 0$ such that for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$ and all $\tilde{\mathbf{y}} \in \mathbb{R}^d$,

$$w^+(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) \leq a_2(\tilde{\mathbf{x}}) + C\,|\tilde{\mathbf{y}}|^p\,.$$

c) $w(\tilde{\mathbf{x}},\cdot)$ is continuous for a.e. $\tilde{\mathbf{x}} \in \Omega$.

d) For a.e. $\mathbf{x} \in \Omega$ *and all* $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$*, the function*

$$\mathbf{y} \mapsto \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{y}-\mathbf{u}(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'$$

is convex in $\mathbb{R}^d$*.*

For each $j \in \mathbb{N}$*, let* $\mathbf{u}_j, \mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ *be such that* $\mathbf{u}_j \rightharpoonup \mathbf{u}$ *in* $L^p(\Omega, \mathbb{R}^d)$ *as* $j \to \infty$*. Then*

$$\int_\Omega \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \le \liminf_{j\to\infty} \int_\Omega \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{u}_j(\mathbf{x})-\mathbf{u}_j(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}. \tag{3.5}$$

Proof. By passing to a subsequence, we can assume that the inferior limit in the right hand side of (3.5) is in fact a finite limit, and that the sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ generates a Young measure $\nu = (\nu_{\mathbf{x}})_{\mathbf{x}\in\Omega}$ in $L^p(\Omega, \mathbb{R}^d)$. Thanks to a standard result in the theory of Young measures (see, e.g., [23, Cor. 8.8]), there exists a sequence $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega, \mathbb{R}^d)$ generating the Young measure ν, and such that the sequence $\{|\mathbf{v}_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable.

By Lemma 3.3, the sequence $\{\bar{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ defined by (3.3) generates the Young measure $(\nu_{\mathbf{x}} \ominus \nu_{\mathbf{x}'})_{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega}$ in $L^p(\Omega\times\Omega, \mathbb{R}^d)$. We now show that the sequence of functions

$$\Omega\times\Omega \ni (\mathbf{x},\mathbf{x}') \mapsto w^-(\mathbf{x}-\mathbf{x}', \mathbf{u}_j(\mathbf{x})-\mathbf{u}_j(\mathbf{x}')) \tag{3.6}$$

is equiintegrable. Indeed, using *a)* we find that for a.e. $(\mathbf{x},\mathbf{x}') \in \Omega\times\Omega$ and all $j \in \mathbb{N}$,

$$0 \le w^-(\mathbf{x}-\mathbf{x}', \mathbf{u}_j(\mathbf{x})-\mathbf{u}_j(\mathbf{x}')) \le a_1(\mathbf{x}-\mathbf{x}')\left[1+|\mathbf{u}_j(\mathbf{x})-\mathbf{u}_j(\mathbf{x}')|^q\right] \le a_1(\mathbf{x}-\mathbf{x}')\left[1+2^{q-1}\left(|\mathbf{u}_j(\mathbf{x})|^q+|\mathbf{u}_j(\mathbf{x}')|^q\right)\right].$$

As the sum of equiintegrable sequences is equiintegrable, to show that (3.6) is equiintegrable, it suffices to show that each of the three sequences of functions

$$(\mathbf{x},\mathbf{x}') \mapsto a_1(\mathbf{x}-\mathbf{x}'), \qquad (\mathbf{x},\mathbf{x}') \mapsto a_1(\mathbf{x}-\mathbf{x}')\,|\mathbf{u}_j(\mathbf{x})|^q, \qquad (\mathbf{x},\mathbf{x}') \mapsto a_1(\mathbf{x}-\mathbf{x}')\,|\mathbf{u}_j(\mathbf{x}')|^q. \tag{3.7}$$

is equiintegrable in $\Omega\times\Omega$. As

$$\int_{\Omega\times\Omega} a_1(\mathbf{x}-\mathbf{x}')\, \mathrm{d}(\mathbf{x},\mathbf{x}') = \int_\Omega \int_{\mathbf{x}-\Omega} a_1(\mathbf{x}')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \le \mathcal{L}^n(\Omega)\, \|a_1\|_{L^1(\tilde{\Omega})},$$

the first one is equiintegrable. For the second, we observe that for each $t > 0$ and $j \in \mathbb{N}$,

$$\int_{\{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega: a_1(\mathbf{x}-\mathbf{x}')|\mathbf{u}_j(\mathbf{x})|^q>t\}} a_1(\mathbf{x}-\mathbf{x}')\,|\mathbf{u}_j(\mathbf{x})|^q\, \mathrm{d}(\mathbf{x},\mathbf{x}') = \int_\Omega |\mathbf{u}_j(\mathbf{x})|^q \int_{\{\mathbf{x}'\in\mathbf{x}-\Omega: a_1(\mathbf{x}')|\mathbf{u}_j(\mathbf{x})|^q>t\}} a_1(\mathbf{x}')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}$$
$$\le \int_\Omega |\mathbf{u}_j(\mathbf{x})|^q \left[\int_{\{\mathbf{x}'\in\mathbf{x}-\Omega: a_1(\mathbf{x}')>t^{1/2}\}} a_1(\mathbf{x}')\, \mathrm{d}\mathbf{x}' + \int_{\{\mathbf{x}'\in\mathbf{x}-\Omega: |\mathbf{u}_j(\mathbf{x})|^q>t^{1/2}\}} a_1(\mathbf{x}')\, \mathrm{d}\mathbf{x}'\right] \mathrm{d}\mathbf{x};$$

moreover,

$$\sup_{j\in\mathbb{N}} \int_\Omega |\mathbf{u}_j(\mathbf{x})|^q \int_{\{\mathbf{x}'\in\mathbf{x}-\Omega: a_1(\mathbf{x}')>t^{1/2}\}} a_1(\mathbf{x}')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \le \sup_{j\in\mathbb{N}} \|\mathbf{u}_j\|^q_{L^q(\Omega,\mathbb{R}^d)} \int_{\{\mathbf{x}'\in\tilde{\Omega}: a_1(\mathbf{x}')>t^{1/2}\}} a_1(\mathbf{x}')\, \mathrm{d}\mathbf{x}'$$

and

$$\sup_{j\in\mathbb{N}} \int_\Omega |\mathbf{u}_j(\mathbf{x})|^q \int_{\{\mathbf{x}'\in\mathbf{x}-\Omega: |\mathbf{u}_j(\mathbf{x})|^q>t^{1/2}\}} a_1(\mathbf{x}')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} = \sup_{j\in\mathbb{N}} \int_{\{\mathbf{x}\in\Omega: |\mathbf{u}_j(\mathbf{x})|^q>t^{1/2}\}} |\mathbf{u}_j(\mathbf{x})|^q \int_\Omega a_1(\mathbf{x}-\mathbf{x}')\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}$$
$$\le \|a_1\|_{L^1(\tilde{\Omega})} \sup_{j\in\mathbb{N}} \int_{\{\mathbf{x}\in\Omega: |\mathbf{u}_j(\mathbf{x})|^q>t^{1/2}\}} |\mathbf{u}_j(\mathbf{x})|^q\, \mathrm{d}\mathbf{x},$$

which proves the equiintegrability, since $a_1 \in L^1(\tilde{\Omega})$ and the sequence $\{|\mathbf{u}_j|^q\}_{j\in\mathbb{N}}$ is equiintegrable because $q < p$. For the third sequence in (3.7), we notice that, thanks to Fubini's theorem and the symmetry of a_1, for each $j \in \mathbb{N}$,

$$\int_{\{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega: a_1(\mathbf{x}-\mathbf{x}')|\mathbf{u}_j(\mathbf{x}')|^q>t\}} a_1(\mathbf{x}-\mathbf{x}')\,|\mathbf{u}_j(\mathbf{x}')|^q\, \mathrm{d}(\mathbf{x},\mathbf{x}') = \int_{\{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega: a_1(\mathbf{x}-\mathbf{x}')|\mathbf{u}_j(\mathbf{x})|^q>t\}} a_1(\mathbf{x}-\mathbf{x}')\,|\mathbf{u}_j(\mathbf{x})|^q\, \mathrm{d}(\mathbf{x},\mathbf{x}').$$

Thus, as the second sequence of (3.7) is equiintegrable, so is the third. Hence the sequence (3.6) is equiintegrable, so we can apply the main lower semicontinuity theorem for Young measures (e.g., [23, Th. 8.6]) to obtain that

$$\int_\Omega \int_\Omega \int_{\mathbb{R}^d} w(\mathbf{x}-\mathbf{x}', \tilde{\mathbf{y}})\, \mathrm{d}\left(\nu_{\mathbf{x}} \ominus \nu_{\mathbf{x}'}\right)(\tilde{\mathbf{y}})\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \le \liminf_{j\to\infty} \int_\Omega \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{u}_j(\mathbf{x})-\mathbf{u}_j(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x}. \tag{3.8}$$

Recall from (3.2) that

$$\int_\Omega \int_\Omega \int_{\mathbb{R}^d} w(\mathbf{x}-\mathbf{x}', \tilde{\mathbf{y}}) \, \mathrm{d}\left(\nu_{\mathbf{x}} \ominus \nu_{\mathbf{x}'}\right)(\tilde{\mathbf{y}}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = \int_\Omega \int_\Omega \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} w(\mathbf{x}-\mathbf{x}', \mathbf{y}-\mathbf{y}') \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \, \mathrm{d}\nu_{\mathbf{x}'}(\mathbf{y}') \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \tag{3.9}$$

Now, for each $\mathbf{x} \in \Omega$ and each Young measure $\mu = (\mu_{\mathbf{x}'})_{\mathbf{x}' \in \Omega}$, define $\Phi_{\mathbf{x},\mu} : \mathbb{R}^n \to \mathbb{R}$ as

$$\Phi_{\mathbf{x},\mu}(\mathbf{y}) := \int_\Omega \int_{\mathbb{R}^d} w(\mathbf{x}-\mathbf{x}', \mathbf{y}-\mathbf{y}') \, \mathrm{d}\mu_{\mathbf{x}'}(\mathbf{y}') \, \mathrm{d}\mathbf{x}'. \tag{3.10}$$

Note that $\Phi_{\mathbf{x},\mu}$ takes finite values thanks to the growth conditions *a–b)* and to the integrability property (3.1). The definition is made so that

$$\int_\Omega \int_\Omega \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} w(\mathbf{x}-\mathbf{x}', \mathbf{y}-\mathbf{y}') \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \, \mathrm{d}\nu_{\mathbf{x}'}(\mathbf{y}') \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = \int_\Omega \int_{\mathbb{R}^d} \Phi_{\mathbf{x},\nu}(\mathbf{y}) \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \, \mathrm{d}\mathbf{x}. \tag{3.11}$$

Note that, via the usual identification of a function $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ with the Young measure $\left(\delta_{\mathbf{u}(\mathbf{x})}\right)_{\mathbf{x} \in \Omega}$ (where δ denotes Dirac's delta), we have that

$$\Phi_{\mathbf{x},\mathbf{u}}(\mathbf{y}) = \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{y}-\mathbf{u}(\mathbf{x}')) \, \mathrm{d}\mathbf{x}'.$$

Thus,

$$\int_\Omega \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}')) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} = \int_\Omega \Phi_{\mathbf{x},\mathbf{u}}(\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x}. \tag{3.12}$$

Also note that the symmetry (2.1) of w yields

$$\int_\Omega \Phi_{\mathbf{x},\nu}(\mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} = \int_\Omega \int_{\mathbb{R}^d} \Phi_{\mathbf{x},\mathbf{u}}(\mathbf{y}) \, \mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}) \, \mathrm{d}\mathbf{x}. \tag{3.13}$$

Now we show that for a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{y} \in \mathbb{R}^n$, the sequence of functions $\{f_{j,\mathbf{x},\mathbf{y}}\}_{j\in\mathbb{N}}$ in Ω defined by

$$f_{j,\mathbf{x},\mathbf{y}}(\mathbf{x}') := w(\mathbf{x}-\mathbf{x}', \mathbf{y}-\mathbf{v}_j(\mathbf{x}')), \qquad \mathbf{x}' \in \Omega$$

is equiintegrable. The sequence $\{f^-_{j,\mathbf{x},\mathbf{y}}\}_{j\in\mathbb{N}}$ is equiintegrable because of the same argument that showed that (3.6) was equiintegrable. Let us show that $\{f^+_{j,\mathbf{x},\mathbf{y}}\}_{j\in\mathbb{N}}$ is equiintegrable. We have

$$0 \le f^+_{j,\mathbf{x},\mathbf{y}}(\mathbf{x}') \le a_2(\mathbf{x}-\mathbf{x}') + C\,|\mathbf{y}-\mathbf{v}_j(\mathbf{x}')|^p \le a_2(\mathbf{x}-\mathbf{x}') + 2^{p-1}C\left(|\mathbf{y}|^p + |\mathbf{v}_j(\mathbf{x}')|^p\right).$$

As $a_2(\mathbf{x}-\cdot) \in L^1(\Omega)$ and the sequence $\{|\mathbf{v}_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable, then the sequence $\{f^+_{j,\mathbf{x},\mathbf{y}}\}_{j\in\mathbb{N}}$ is equiintegrable. Therefore, when we define $f_{\mathbf{x},\mathbf{y}} : \Omega \to \mathbb{R}$ as

$$f_{\mathbf{x},\mathbf{y}}(\mathbf{x}') := \int_{\mathbb{R}^d} w(\mathbf{x}-\mathbf{x}', \mathbf{y}-\mathbf{y}') \, \mathrm{d}\nu_{\mathbf{x}'}(\mathbf{y}'), \qquad \mathbf{x}' \in \Omega,$$

we have, by the fundamental theorem for Young measures (see, e.g., [23, Th. 8.6]),

$$\lim_{j\to\infty} \int_\Omega f_{j,\mathbf{x},\mathbf{y}}(\mathbf{x}') \, \mathrm{d}\mathbf{x}' = \int_\Omega f_{\mathbf{x},\mathbf{y}}(\mathbf{x}') \, \mathrm{d}\mathbf{x}'.$$

Now for a.e. $\mathbf{x} \in \Omega$, each $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^d$, $0 \le \lambda \le 1$ and $j \in \mathbb{N}$, assumption *d)* shows that

$$\Phi_{\mathbf{x},\mathbf{v}_j}(\lambda\mathbf{y}_1 + (1-\lambda)\mathbf{y}_2) \le \lambda\,\Phi_{\mathbf{x},\mathbf{v}_j}(\mathbf{y}_1) + (1-\lambda)\,\Phi_{\mathbf{x},\mathbf{v}_j}(\mathbf{y}_2),$$

that is to say,

$$\int_\Omega f_{j,\mathbf{x},\lambda\mathbf{y}_1+(1-\lambda)\mathbf{y}_2}(\mathbf{x}') \, \mathrm{d}\mathbf{x}' \le \lambda \int_\Omega f_{j,\mathbf{x},\mathbf{y}_1}(\mathbf{x}') \, \mathrm{d}\mathbf{x}' + (1-\lambda) \int_\Omega f_{j,\mathbf{x},\mathbf{y}_2}(\mathbf{x}') \, \mathrm{d}\mathbf{x}'.$$

Taking limits as $j \to \infty$, we find that

$$\int_\Omega f_{\mathbf{x},\lambda\mathbf{y}_1+(1-\lambda)\mathbf{y}_2}(\mathbf{x}') \, \mathrm{d}\mathbf{x}' \le \lambda \int_\Omega f_{\mathbf{x},\mathbf{y}_1}(\mathbf{x}') \, \mathrm{d}\mathbf{x}' + (1-\lambda) \int_\Omega f_{\mathbf{x},\mathbf{y}_2}(\mathbf{x}') \, \mathrm{d}\mathbf{x}',$$

so

$$\Phi_{\mathbf{x},\nu}(\lambda\mathbf{y}_1 + (1-\lambda)\mathbf{y}_2) \le \lambda\,\Phi_{\mathbf{x},\nu}(\mathbf{y}_1) + (1-\lambda)\,\Phi_{\mathbf{x},\nu}(\mathbf{y}_2).$$

Thus, $\Phi_{\mathbf{x},\nu}$ is convex.

The weak convergence of $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ and its convergence in the sense of Young measures show that

$$\mathbf{u}(\mathbf{x}) = \int_{\mathbb{R}^d} \mathbf{y}\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}), \qquad \text{a.e. } \mathbf{x}\in\Omega.$$

This and Jensen's inequality imply that

$$\Phi_{\mathbf{x},\nu}(\mathbf{u}(\mathbf{x})) \le \int_{\mathbb{R}^d} \Phi_{\mathbf{x},\nu}(\mathbf{y})\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y}), \qquad \text{a.e. } \mathbf{x}\in\Omega$$

so

$$\int_\Omega \Phi_{\mathbf{x},\nu}(\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \le \int_\Omega\int_{\mathbb{R}^d} \Phi_{\mathbf{x},\nu}(\mathbf{y})\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\,\mathrm{d}\mathbf{x}. \tag{3.14}$$

Analogously,

$$\int_\Omega \Phi_{\mathbf{x},\mathbf{u}}(\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \le \int_\Omega\int_{\mathbb{R}^d} \Phi_{\mathbf{x},\mathbf{u}}(\mathbf{y})\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\,\mathrm{d}\mathbf{x}. \tag{3.15}$$

Putting together the relations (3.8), (3.9), (3.11), (3.12), (3.13), (3.14) and (3.15) we obtain

$$\begin{aligned}\int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} &= \int_\Omega \Phi_{\mathbf{x},\mathbf{u}}(\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \le \int_\Omega\int_{\mathbb{R}^d} \Phi_{\mathbf{x},\mathbf{u}}(\mathbf{y})\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega \Phi_{\mathbf{x},\nu}(\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \le \int_\Omega\int_{\mathbb{R}^d} \Phi_{\mathbf{x},\nu}(\mathbf{y})\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega\int_\Omega\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} w(\mathbf{x}-\mathbf{x}',\mathbf{y}-\mathbf{y}')\,\mathrm{d}\nu_{\mathbf{x}}(\mathbf{y})\,\mathrm{d}\nu_{\mathbf{x}'}(\mathbf{y}')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\\
&= \int_\Omega\int_\Omega\int_{\mathbb{R}^d} w(\mathbf{x}-\mathbf{x}',\tilde{\mathbf{y}})\,\mathrm{d}\left(\nu_{\mathbf{x}}\ominus\nu_{\mathbf{x}'}\right)(\tilde{\mathbf{y}})\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\\
&\le \liminf_{j\to\infty}\int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{u}_j(\mathbf{x})-\mathbf{u}_j(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},\end{aligned}$$

as desired. □

Note that the assumptions of Proposition 3.4 are slightly more general than necessary for applications in peridynamics, since the energy function w is usually assumed to be non-negative. Thus, assumption *a)* and, hence, the first part of the proof showing the equiintegrability of (3.6) can be dispensed with. In fact, continuity of $w(\tilde{\mathbf{x}},\cdot)$ can be relaxed to lower semicontinuity, as the following result shows.

Proposition 3.5. *Let Ω be a non-empty, bounded open subset of $\mathbb{R}^n$. Let $1\le p<\infty$. Let $w:\tilde{\Omega}\times\mathbb{R}^d\to\mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega})\times\mathcal{B}(\mathbb{R}^d)$-measurable. Assume that:*

a) There exist $a\in L^1_{\mathrm{loc}}(\tilde{\Omega})$ with $a\ge 0$ and $C>0$ such that for a.e. $\tilde{\mathbf{x}}\in\tilde{\Omega}$ and all $\tilde{\mathbf{y}}\in\mathbb{R}^d$,

$$0\le w(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\le a(\tilde{\mathbf{x}})+C\,|\tilde{\mathbf{y}}|^p.$$

b) $w(\tilde{\mathbf{x}},\cdot)$ is lower semicontinuous for a.e. $\tilde{\mathbf{x}}\in\Omega$.

c) For a.e. $\mathbf{x}\in\Omega$ and all $\mathbf{u}\in L^p(\Omega,\mathbb{R}^d)$, the function

$$\mathbf{y}\mapsto\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{y}-\mathbf{u}(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'$$

is convex in $\mathbb{R}^d$.

For each $j\in\mathbb{N}$, let $\mathbf{u}_j,\mathbf{u}\in L^p(\Omega,\mathbb{R}^d)$ be such that $\mathbf{u}_j\rightharpoonup\mathbf{u}$ in $L^p(\Omega,\mathbb{R}^d)$ as $j\to\infty$. Then (3.5) holds.

Proof. We apply a standard approximation procedure for lower semicontinuous functions. For each $k\in\mathbb{N}$, define $w_k:\tilde{\Omega}\times\mathbb{R}^d\to[0,\infty)$ as

$$w_k(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) := \inf\left\{w(\tilde{\mathbf{x}},\tilde{\mathbf{z}})+k|\tilde{\mathbf{z}}-\tilde{\mathbf{y}}| : \tilde{\mathbf{z}}\in\mathbb{R}^d\right\}, \qquad (\tilde{\mathbf{x}},\tilde{\mathbf{y}})\in\tilde{\Omega}\times\mathbb{R}^d.$$

Then (see, e.g., [6, Lemma 1.61]) w_k is $\mathcal{L}^n(\tilde{\Omega})\times\mathcal{B}(\mathbb{R}^d)$-measurable and $w_k(\tilde{\mathbf{x}},\cdot)$ is continuous for a.e. $\tilde{\mathbf{x}}\in\tilde{\Omega}$. Moreover, $w_k(\tilde{\mathbf{x}},\cdot)\nearrow w(\tilde{\mathbf{x}},\cdot)$ as $k\to\infty$ pointwise in $\mathbb{R}^d$ for all $\tilde{\mathbf{x}}\in\tilde{\Omega}$. Note that we can apply Proposition 3.4, even when $p=1$, since the assumption $p > 1$ was only used there in order to prove the equiintegrability of (3.6). Therefore, for each $k \in \mathbb{N}$,

$$\begin{aligned}\int_\Omega \int_\Omega w_k(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} &\leq \liminf_{j\to\infty} \int_\Omega \int_\Omega w_k(\mathbf{x}-\mathbf{x}', \mathbf{u}_j(\mathbf{x}) - \mathbf{u}_j(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} \\ &\leq \liminf_{j\to\infty} \int_\Omega \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{u}_j(\mathbf{x}) - \mathbf{u}_j(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x},\end{aligned}$$

whereas by the monotone convergence theorem we have that

$$\lim_{k\to\infty} \int_\Omega \int_\Omega w_k(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} = \int_\Omega \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x},$$

which concludes inequality (3.5). □

If $w(\tilde{\mathbf{x}}, \cdot)$ is convex, then the growth conditions of w can be relaxed as follows.

Proposition 3.6. *Let Ω be a non-empty, bounded open subset of $\mathbb{R}^n$. Let $1 < p < \infty$. Let $w : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega}) \times \mathcal{B}(\mathbb{R}^d)$-measurable. Assume that:*

a) There exist $a \in L^1(\tilde{\Omega})$ with

$$a \geq 0, \qquad a(-\tilde{\mathbf{x}}) = a(\tilde{\mathbf{x}}), \quad a.e.\ \tilde{\mathbf{x}} \in \tilde{\Omega}$$

and $1 \leq q < p$ such that for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$ and all $\tilde{\mathbf{y}} \in \mathbb{R}^d$,

$$w^-(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \leq a(\tilde{\mathbf{x}})\,(1 + |\tilde{\mathbf{y}}|^q)\,.$$

b) For a.e. $\tilde{\mathbf{x}} \in \Omega$, the function $w(\tilde{\mathbf{x}}, \cdot)$ is convex and lower semicontinuous in $\mathbb{R}^d$.

For each $j \in \mathbb{N}$, let $\mathbf{u}_j, \mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ be such that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ as $j \to \infty$. Then (3.5) holds.

Proof. As in Proposition 3.4, we can assume that the inferior limit in the right hand side of (3.5) is a finite limit, and that the sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ generates a Young measure $\nu = (\nu_\mathbf{x})_{\mathbf{x}\in\Omega}$ in $L^p(\Omega, \mathbb{R}^d)$. Moreover, there exists a sequence $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ of functions in $L^p(\Omega, \mathbb{R}^d)$ generating the same Young measure ν, and such that the sequence $\{|\mathbf{v}_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable.

By Lemma 3.3, the sequence $\{\bar{\mathbf{u}}_j\}_{j\in\mathbb{N}}$ defined by (3.3) generates the Young measure $(\nu_\mathbf{x} \ominus \nu_{\mathbf{x}'})_{(\mathbf{x},\mathbf{x}')\in\Omega\times\Omega}$ in $L^p(\Omega\times\Omega, \mathbb{R}^d)$. The argument of Proposition 3.4 shows that the sequence of functions (3.6) is equiintegrable. Therefore, (3.8) holds, and so does (3.9).

For each $\mathbf{x} \in \Omega$ and each Young measure $\mu = (\mu_{\mathbf{x}'})_{\mathbf{x}'\in\Omega}$, define $\Phi_{\mathbf{x},\mu} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ as in (3.10). Then (3.11), (3.12) and (3.13) hold. Moreover, the convexity of $w(\tilde{\mathbf{x}}, \cdot)$ for a.e. $\tilde{\mathbf{x}} \in \Omega$ implies at once that $\Phi_{\mathbf{x},\mu}$ is convex. Hence, the same argument of Proposition 3.4 shows that (3.14) and (3.15) hold, and the proof is concluded. □

We finish this section with some remarks about condition *d)* of Proposition 3.4, referring to Elbau [22] for more insight, but, at the same time, admitting that a better understanding is still pending. If w does not depend on $\tilde{\mathbf{x}}$ (which is a non-realistic assumption for peridynamics), condition *d)* is easily seen to be equivalent to the usual convexity of w, whereas if w does depend on $\tilde{\mathbf{x}}$, condition *d)* is weaker, as will be shown in the following paragraph. This is in contrast with local variational problems, in which the convexity property for the integrand that is equivalent to weak lower semicontinuity of the functional only concerns the variable $\tilde{\mathbf{y}}$ (see, e.g., [10, 23, 16]). We also refer to Pedregal [34] for a characterization of the weak lower semicontinuity through a convexity property in a nonlocal but slightly different situation.

We now present an example showing that condition *d)* of Proposition 3.4 is weaker than the requirement of $w(\tilde{\mathbf{x}}, \cdot)$ to be convex, even in dimension 1. Let $n = 1$, $d = 1$ and $\Omega = (0, 1)$, hence $\tilde{\Omega} = (-1, 1)$. Additionally, let $h : [-1, 1) \to \mathbb{R}$ be any smooth function such that $h(-1) < 0$ and

$$\int_{-1+t}^{t} h \geq 0, \quad h(t) = h(-t) \qquad \text{for all } t \in (0, 1).$$

Define $w : (-1,1) \times \mathbb{R} \to \mathbb{R}$ as $w(\tilde{x}, \tilde{y}) := h(\tilde{x}) \tilde{y}^2$. Then $w(\tilde{x}, \cdot)$ is not convex in $\mathbb{R}$ for $\tilde{x}$ in a neigbourhood of -1, but for all $x \in \Omega$ and all $u \in L^p(\Omega)$, the function

$$y \mapsto \int_\Omega w(x - x', y - u(x')) \, \mathrm{d}x'$$

is convex in $\mathbb{R}$, since its second derivative satisfies for all $y \in \mathbb{R}$,

$$\int_\Omega \frac{\partial^2 w}{\partial \tilde{y}^2}(x - x', y - u(x')) \, \mathrm{d}x' = 2 \int_\Omega h(x - x') \, \mathrm{d}x' = 2 \int_{x-1}^{x} h(t) \, \mathrm{d}t \geq 0.$$

4. Coercivity in L^p

The inequality needed for the coercivity for the Dirichlet problem was proved by Andreu *et al.* [7, Prop. 2.5]. In the context of peridynamics it was used in Aksoylu & Parks [3, Prop. 4.1] and Hinds & Radu [26, Lemma 3.5]. In any of those papers, the following result is proved.

Proposition 4.1. *Let Ω be a bounded domain of $\mathbb{R}^n$. Let Ω_0 be a non-empty open subset of Ω for which there is a $\delta > 0$ satisfying $\Omega_0 + B(\mathbf{0}, \delta) \subset \Omega$. Let $1 \leq p < \infty$. Then there exists $\lambda > 0$ such that for all $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$,*

$$\int_\Omega |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \leq \lambda \int_\Omega \int_{\Omega \cap B(\mathbf{x}, \delta)} |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} + \lambda \int_{\Omega \setminus \Omega_0} |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x}. \tag{4.1}$$

Regarding the coercivity inequality for the Neumann problem, it was observed by Aksoylu & Mengesha [2] that it can be easily obtained by invoking the Poincaré-type inequalities obtained by Ponce [36, 37] and, earlier, by Bourgain, Brezis & Mironescu [13, 14] in their study of nonlocal characterizations of Sobolev spaces. The proof of [2, Cor. 3.4] is adapted as follows.

Proposition 4.2. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $1 \leq p < \infty$. Then there exists $\lambda > 0$ such that for all $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$,*

$$\int_\Omega \left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p \mathrm{d}\mathbf{x} \leq \lambda \int_\Omega \int_{\Omega \cap B(\mathbf{x}, \delta)} |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \tag{4.2}$$

Proof. For each $j \in \mathbb{N}$ define the function $\rho_j : [0, \infty) \to [0, \infty)$ as

$$\rho_j(t) := \frac{(n+p) \, j^{n+p}}{\sigma_n} \, t^p \, \chi_{[0,1]}(jt), \qquad t \geq 0,$$

where σ_n denotes the surface area of the unit sphere of $\mathbb{R}^n$. It is immediate to check that

$$\int_{\mathbb{R}^n} \rho_j(|\mathbf{x}|) \, \mathrm{d}\mathbf{x} = 1 \qquad \text{for all } j \in \mathbb{N}$$

and

$$\lim_{j \to \infty} \int_{\mathbb{R}^n \setminus B(\mathbf{0}, \eta)} \rho_j(|\mathbf{x}|) \, \mathrm{d}\mathbf{x} = 0 \qquad \text{for all } \eta > 0.$$

Moreover, when $n = 1$ we have that for all $0 < \theta_0 < 1$,

$$\int_{\mathbb{R}} \inf_{\theta \in [\theta_0, 1]} \rho_j(|\theta t|) \, \mathrm{d}t \geq \theta_0^p \int_{\mathbb{R}} \rho_j(|t|) \, \mathrm{d}t = \theta_0^p.$$

Hence, by [36, Thms. 1.1 and 1.3], there exist $c > 0$ and $j \in \mathbb{N}$ such that $\frac{1}{j} \leq \delta$ and for any $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$,

$$\int_\Omega \left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p \mathrm{d}\mathbf{x} \leq c \int_\Omega \int_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x} - \mathbf{x}'|^p} \rho_j \left(|\mathbf{x} - \mathbf{x}'| \right) \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}.$$

The definition of ρ_j and the inequality $\frac{1}{j} \leq \delta$ show that

$$\int_\Omega \int_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x} - \mathbf{x}'|^p} \rho_j \left(|\mathbf{x} - \mathbf{x}'| \right) \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq \frac{(n+p) \, j^{n+p}}{\sigma_n} \int_\Omega \int_{\Omega \cap B(\mathbf{x}, \delta)} |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x},$$

which concludes the proof. ∎

It was again observed by Aksoylu & Mengesha [2] that the same techniques can also be used to prove the coercivity inequality for the mixed problem. The proof of the following result is, thus, a straightforward adaptation of [2, Cor. 3.4], who proved it for the case $p = 2$, with the variant given in Proposition 4.2 to cover the case of any $1 \leq p < \infty$.

Proposition 4.3. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $1 \leq p < \infty$. Let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(\mathbf{0}, \delta) \subset \Omega$. Let Ω_D be a measurable subset of $\Omega \setminus \Omega_0$ with positive measure. Then there exists $\lambda > 0$ such that for all $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ with $\mathbf{u} = \mathbf{0}$ a.e. in Ω_D, we have*

$$\int_\Omega |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \leq \lambda \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}.$$

We will in fact use Proposition 4.3 in the following form.

Corollary 4.4. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $1 \leq p < \infty$. Let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(\mathbf{0}, \delta) \subset \Omega$. Let Ω_D be a measurable subset of $\Omega \setminus \Omega_0$ with positive measure. Then there exists $\lambda > 0$ such that for all $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$,*

$$\int_\Omega |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \leq \lambda \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} + \lambda \int_{\Omega_D} |\mathbf{u}(\mathbf{x})|^p \, \mathrm{d}\mathbf{x}.$$

5. Existence of minimizers in L^p

With the lower semicontinuity and coercivity results at hand, we pass now to show the existence of minimizers of the total energy in the functional setting of L^p spaces. We start with the Dirichlet and mixed boundary conditions.

Theorem 5.1. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $1 < p < \infty$. Let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(\mathbf{0}, \delta) \subset \Omega$. Let Ω_D be a measurable subset of $\Omega \setminus \Omega_0$ with positive measure. Let $w : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega}) \times \mathcal{B}(\mathbb{R}^d)$-measurable. Assume that:*

a) There exists $c_0 > 0$ such that

$$w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \geq c_0 \chi_{B(\mathbf{0},\delta)}(\tilde{\mathbf{x}}) \, |\tilde{\mathbf{y}}|^p \,, \qquad \text{for a.e. } \tilde{\mathbf{x}} \in \tilde{\Omega} \text{ and all } \tilde{\mathbf{y}} \in \mathbb{R}^d.$$

b) There exist $a_1 \in L^1(\tilde{\Omega})$ and $C > 0$ such that

$$w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \leq a_1(\tilde{\mathbf{x}}) + C \, |\tilde{\mathbf{y}}|^p \,, \qquad \text{for a.e. } \tilde{\mathbf{x}} \in \tilde{\Omega} \text{ and all } \tilde{\mathbf{y}} \in \mathbb{R}^d.$$

c) $w(\tilde{\mathbf{x}}, \cdot)$ is lower semicontinuous for a.e. $\tilde{\mathbf{x}} \in \Omega$.

d) For a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$, the function

$$\mathbf{y} \mapsto \int_\Omega w(\mathbf{x} - \mathbf{x}', \mathbf{y} - \mathbf{u}(\mathbf{x}')) \, \mathrm{d}\mathbf{x}'$$

is convex in $\mathbb{R}^d$.

Let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\Omega) \times \mathcal{B}(\mathbb{R}^d)$-measurable and satisfy that for a.e. $\mathbf{x} \in \Omega$, the function $F(\mathbf{x}, \cdot)$ is concave, upper semicontinuous and

$$F^+(\mathbf{x}, \mathbf{y}) \leq a_2(\mathbf{x}) + c_1 |\mathbf{y}|^q, \quad F^+(\mathbf{x}, \mathbf{y}) \leq a_2(\mathbf{x}) + \mathbf{a}_3(\mathbf{x}) \cdot \mathbf{y}, \qquad \text{for a.e. } \mathbf{x} \in \Omega \text{ and all } \mathbf{y} \in \mathbb{R}^d. \tag{5.1}$$

for some $1 \leq q < p$, some $c_1 > 0$, some $a_2 \in L^1(\Omega)$ and some $\mathbf{a}_3 \in L^{p'}(\Omega, \mathbb{R}^d)$. Let $\mathbf{b} \in L^p(\Omega_D, \mathbb{R}^d)$ satisfy that

$$\int_{\Omega \setminus \Omega_D} F^-(\mathbf{x}, \mathbf{0}) \, \mathrm{d}\mathbf{x} + \int_{\Omega_D} F^-(\mathbf{x}, \mathbf{b}(\mathbf{x})) \, \mathrm{d}\mathbf{x} < \infty. \tag{5.2}$$

Let $\mathcal{A}$ be the set of $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ such that $\mathbf{u} = \mathbf{b}$ a.e. on Ω_D. Let $\mathcal{I}$ be as in (2.4). Then there exists a minimizer of $\mathcal{I}$ in $\mathcal{A}$.

Proof. Let $\mathbf{u} \in \mathcal{A}$. Assumption *a)* and Corollary 4.4 yield the estimate

$$\frac{c_0}{\lambda} \, \|\mathbf{u}\|^p_{L^p(\Omega, \mathbb{R}^d)} - c_0 \, \|\mathbf{b}\|^p_{L^p(\Omega_D, \mathbb{R}^d)} \leq \int_\Omega \int_\Omega w(\mathbf{x} - \mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}, \tag{5.3}$$

where λ is the constant of Corollary 4.4, while the first bound of (5.1) and Hölder's inequality show that

$$\int_\Omega F(\mathbf{x},\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \le \|a_2\|_{L^1(\Omega)} + c_2 \|\mathbf{u}\|^q_{L^p(\Omega,\mathbb{R}^d)} \tag{5.4}$$

for some $c_2 > 0$ independent of $\mathbf{u}$. Using Young's inequality, we find that

$$c_2 \|\mathbf{u}\|^q_{L^p(\Omega,\mathbb{R}^d)} \le c_3 + \frac{c_0}{2\lambda} \|\mathbf{u}\|^p_{L^p(\Omega,\mathbb{R}^d)}, \tag{5.5}$$

for some $c_3 > 0$ independent of $\mathbf{u}$. From (5.3), (5.4) and (5.5) we conclude that there exists $c_4 > 0$, such that for all $\mathbf{u} \in \mathcal{A}$,

$$\mathcal{I}(\mathbf{u}) \ge \frac{c_0}{2\lambda} \|\mathbf{u}\|^p_{L^p(\Omega,\mathbb{R}^d)} - c_4. \tag{5.6}$$

Let $\bar{\mathbf{b}} \in L^p(\Omega,\mathbb{R}^d)$ be the extension of $\mathbf{b}$ to Ω by zero. Assumption *b)* shows that

$$\int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\bar{\mathbf{b}}(\mathbf{x})-\bar{\mathbf{b}}(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \le \mathcal{L}^n(\Omega)\|a_1\|_{L^1(\tilde\Omega)} + 2^p C \mathcal{L}^n(\Omega)\|\mathbf{b}\|_{L^p(\Omega_D,\mathbb{R}^d)},$$

while assumption (5.2) yields

$$\int_\Omega F^-(\mathbf{x},\bar{\mathbf{b}}(\mathbf{x}))\,\mathrm{d}\mathbf{x} < \infty.$$

hence $\mathcal{I}(\bar{\mathbf{b}}) < \infty$.

Clearly, $\mathcal{A} \ne \varnothing$, since $\bar{\mathbf{b}} \in \mathcal{A}$. Estimate (5.6) shows that $\mathcal{I}$ is bounded below in $\mathcal{A}$, and, as just seen, $\mathcal{I}$ is not identically infinity. So let $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ be a minimizing sequence of $\mathcal{I}$ in $\mathcal{A}$. By (5.6), for a subsequence, there exists $\mathbf{u} \in L^p(\Omega,\mathbb{R}^d)$ such that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $L^p(\Omega,\mathbb{R}^d)$ as $j \to \infty$. Proposition 3.5 shows that

$$\int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} \le \liminf_{j\to\infty} \int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{u}_j(\mathbf{x})-\mathbf{u}_j(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}, \tag{5.7}$$

while standard arguments on lower semicontinuity for convex functions (e.g., [23, Th. 6.54]) show that

$$\int_\Omega F(\mathbf{x},\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x} \ge \limsup_{j\to\infty} \int_\Omega F(\mathbf{x},\mathbf{u}_j(\mathbf{x}))\,\mathrm{d}\mathbf{x}.$$

In total, $\mathcal{I}(\mathbf{u}) \le \liminf_{j\to\infty} \mathcal{I}(\mathbf{u}_j)$. Clearly, $\mathbf{u}_j|_{\Omega_D} \rightharpoonup \mathbf{u}|_{\Omega_D}$ in $L^p(\Omega_D,\mathbb{R}^d)$ as $j \to \infty$, so $\mathbf{u} = \mathbf{b}$ a.e. in Ω_D. Hence $\mathbf{u} \in \mathcal{A}$ and, thus, $\mathbf{u}$ is a minimizer of $\mathcal{I}$ in $\mathcal{A}$. □

Note that the assumptions on F in Theorem 5.1 are satisfied when $F(\mathbf{x},\mathbf{y}) = \mathbf{f}(\mathbf{x})\cdot\mathbf{y}$ for a given $\mathbf{f} \in L^r(\Omega,\mathbb{R}^d)$ with $r > p'$. This is typically the case for external forces, and even more so when dealing with Neumann boundary conditions, as presented in the following result, which remains true for nonlinear forces F satisfying the assumptions of Theorem 5.1.

Theorem 5.2. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $1 < p < \infty$. Let $w : \tilde\Omega \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\tilde\Omega)\times\mathcal{B}(\mathbb{R}^d)$-measurable. Let assumptions* a)–d) *of Theorem 5.1 hold. Let $\mathbf{f} \in L^{p'}(\Omega,\mathbb{R}^d)$. Let $\mathcal{A}$ be the set of $\mathbf{u} \in L^p(\Omega,\mathbb{R}^d)$ such that $\int_\Omega \mathbf{u} = \mathbf{0}$. Define $\mathcal{I}$ as*

$$\mathcal{I}(\mathbf{u}) := \int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad \mathbf{u} \in \mathcal{A}. \tag{5.8}$$

Then there exists a minimizer of $\mathcal{I}$ in $\mathcal{A}$.

Proof. Assumption *a)* and Proposition 4.2 yield the estimate

$$\frac{c_0}{\lambda}\|\mathbf{u}\|^p_{L^p(\Omega,\mathbb{R}^d)} \le \int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}, \tag{5.9}$$

where λ is the constant of Proposition 4.2, while Hölder's and Young's inequalities show that

$$\int_\Omega \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x} \le \|\mathbf{f}\|_{L^{p'}(\Omega,\mathbb{R}^d)}\|\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^d)} \le c_1 + \frac{c_0}{2\lambda}\|\mathbf{u}\|^p_{L^p(\Omega,\mathbb{R}^d)} \tag{5.10}$$

for some $c_1 > 0$ independent of $\mathbf{u}$. From (5.9) and (5.10) we conclude that for all $\mathbf{u} \in \mathcal{A}$,

$$\mathcal{I}(\mathbf{u}) \ge \frac{c_0}{2\lambda}\|\mathbf{u}\|^p_{L^p(\Omega,\mathbb{R}^d)} - c_1. \tag{5.11}$$

Assumption *b)* shows that

$$\int_\Omega \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{0}) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq \mathcal{L}^n(\Omega) \, \|a_2\|_{L^1(\tilde{\Omega})} \, ,$$

so $\mathcal{I}(\mathbf{0}) < \infty$. Clearly, $\mathcal{A} \neq \varnothing$, since $\mathbf{0} \in \mathcal{A}$. Estimate (5.11) shows that $\mathcal{I}$ is bounded below in $\mathcal{A}$, and, as just seen, $\mathcal{I}$ is not identically infinity. So let $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ be a minimizing sequence of $\mathcal{I}$ in $\mathcal{A}$. By (5.11), for a subsequence, there exists $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ such that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ as $j \to \infty$.

Proposition 3.5 shows that inequality (5.7) holds, while weak convergence yields $\int_\Omega \mathbf{f} \cdot \mathbf{u}_j \to \int_\Omega \mathbf{f} \cdot \mathbf{u}$ as $j \to \infty$. Therefore, $\mathcal{I}(\mathbf{u}) \leq \liminf_{j\to\infty} \mathcal{I}(\mathbf{u}_j)$. Clearly, $\int_\Omega \mathbf{u}_j \to \int_\Omega \mathbf{u}$ as $j \to \infty$, so $\int_\Omega \mathbf{u} = \mathbf{0}$. Hence $\mathbf{u} \in \mathcal{A}$ and $\mathbf{u}$ is a minimizer of $\mathcal{I}$ in $\mathcal{A}$. □

6. Coercivity in $W^{s,p}$

When the growth of w is of the form (1.2) with $n < \alpha < n + p$, the natural functional spaces to set the problem are the fractional Sobolev spaces. We will need the following two coercivity inequalities, essentially proved in Hurri-Syrjänen & Vähäkangas [27].

Proposition 6.1. *Let $0 < s < 1$, $1 \leq p < \infty$ and $\delta > 0$. Let Ω be a Lipschitz domain. Then there exists $\lambda > 0$ such that for all $\mathbf{u} \in W^{s,p}(\Omega, \mathbb{R}^d)$,*

$$\int_\Omega \left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p \mathrm{d}\mathbf{x} \leq \lambda \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x}-\mathbf{x}'|^{n+sp}} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \tag{6.1}$$

and

$$\int_\Omega \int_\Omega \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x}-\mathbf{x}'|^{n+sp}} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq \lambda \int_\Omega \int_{\Omega \cap B(\mathbf{x},\delta)} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x}-\mathbf{x}'|^{n+sp}} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}. \tag{6.2}$$

Proof. Inequality (6.1) is a particular case of [27, Cor. 4.6]. As for (6.2), we notice that

$$\int_\Omega \int_{\Omega \setminus B(\mathbf{x},\delta)} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p}{|\mathbf{x}-\mathbf{x}'|^{n+sp}} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq \frac{1}{\delta^{n+sp}} \int_\Omega \int_\Omega |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x}, \tag{6.3}$$

while the inequality

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \leq 2^{p-1} \left[\left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p + \left| \mathbf{u}(\mathbf{x}') - \fint_\Omega \mathbf{u} \right|^p \right], \qquad \mathbf{x}, \mathbf{x}' \in \Omega$$

provides, by integration,

$$\int_\Omega \int_\Omega |\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')|^p \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} \leq 2^p \mathcal{L}^n(\Omega) \int_\Omega \left| \mathbf{u}(\mathbf{x}) - \fint_\Omega \mathbf{u} \right|^p \mathrm{d}\mathbf{x}. \tag{6.4}$$

Putting together (6.3), (6.4) and (6.1), we obtain inequality (6.2), changing the value of the constant λ. □

We will need another Poincaré-type inequality for the situation at hand; a similar result can be found in [20, Lemma 4.3].

Lemma 6.2. *Let Ω be a Lipschitz domain and let Ω_D be a measurable set of Ω of positive measure. Let $0 < s < 1$ and $1 \leq p < \infty$. Then there exists $\lambda > 0$ such that for all $\mathbf{u} \in W^{s,p}(\Omega, \mathbb{R}^d)$ with $\mathbf{u} = \mathbf{0}$ a.e. on Ω_D, we have*

$$\|\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^d)} \leq \lambda \, |\mathbf{u}|_{W^{s,p}(\Omega,\mathbb{R}^d)} \, .$$

Proof. Assume, for a contradiction, that there exists a sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ in $W^{s,p}(\Omega, \mathbb{R}^d)$ such that

$$\|\mathbf{u}_j\|_{L^p(\Omega,\mathbb{R}^d)} = 1, \qquad \mathbf{u}_j = \mathbf{0} \text{ a.e. on } \Omega_D, \qquad |\mathbf{u}_j|_{W^{s,p}(\Omega,\mathbb{R}^d)} \leq \frac{1}{j}$$

for all $j \in \mathbb{N}$. Then there exists $\mathbf{u} \in W^{s,p}(\Omega, \mathbb{R}^d)$ such that, for a subsequence, $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{s,p}(\Omega, \mathbb{R}^d)$ and, due to Proposition 2.1, $\mathbf{u}_j \to \mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$, as $j \to \infty$. This $\mathbf{u}$ satisfies $\mathbf{u} = \mathbf{0}$ a.e. on Ω_D,

$$\|\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^d)} = \lim_{j\to\infty} \|\mathbf{u}_j\|_{L^p(\Omega,\mathbb{R}^d)} = 1 \quad \text{and} \quad |\mathbf{u}|_{W^{s,p}(\Omega,\mathbb{R}^d)} \leq \liminf_{j\to\infty} |\mathbf{u}_j|_{W^{s,p}(\Omega,\mathbb{R}^d)} = 0.$$

As Ω is connected and $|\mathbf{u}|_{W^{s,p}(\Omega,\mathbb{R}^d)} = 0$, then $\mathbf{u}$ is constant, necessarily $\mathbf{0}$, which contradicts that $\|\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^d)} = 1$. □

We will use Lemma 6.2 in the following form.

Corollary 6.3. *Let Ω be a Lipschitz domain and let Ω_D be a measurable set of Ω of positive measure. Let $0 < s < 1$ and $1 \leq p < \infty$. Let $\mathbf{b} \in W^{s,p}(\Omega, \mathbb{R}^d)$. Then there exists $\lambda > 0$ such that for all $\mathbf{u} \in W^{s,p}(\Omega, \mathbb{R}^d)$ with $\mathbf{u} = \mathbf{b}$ a.e. on Ω_D, we have*

$$\|\mathbf{u}\|_{W^{s,p}(\Omega,\mathbb{R}^d)} \leq \lambda \left(|\mathbf{u}|_{W^{s,p}(\Omega,\mathbb{R}^d)} + \|\mathbf{b}\|_{W^{s,p}(\Omega,\mathbb{R}^d)} \right).$$

7. Existence of minimizers in $W^{s,p}$

With the coercivity results at hand, we present the existence theorems in $W^{s,p}$. In this case, no convexity conditions are needed on w or $-F$, since, as a consequence of Proposition 2.1, weak convergence in $W^{s,p}$ implies (for a subsequence) convergence a.e. We start with Dirichlet and mixed boundary conditions.

Theorem 7.1. *Let $0 < s < 1$ and $1 < p < \infty$. Let Ω be a Lipschitz domain of $\mathbb{R}^n$ and fix $\delta > 0$. Let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(\mathbf{0}, \delta) \subset \Omega$. Let Ω_D be a measurable subset of $\Omega \setminus \Omega_0$ with positive measure. Let $w : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega}) \times \mathcal{B}(\mathbb{R}^d)$-measurable and satisfy the following conditions:*

a) There exist $c_0 > 0$ such that

$$w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \geq c_0 \frac{|\tilde{\mathbf{y}}|^p}{|\tilde{\mathbf{x}}|^{n+sp}} \chi_{B(\mathbf{0},\delta)}(\tilde{\mathbf{x}}), \qquad \text{for a.e. } \tilde{\mathbf{x}} \in \tilde{\Omega} \text{ and all } \tilde{\mathbf{y}} \in \mathbb{R}^d.$$

b) For a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$, the function $w(\tilde{\mathbf{x}}, \cdot)$ is lower semicontinuous.

Let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\Omega) \times \mathcal{B}(\mathbb{R}^d)$-measurable and satisfy that for a.e. $\mathbf{x} \in \Omega$, the function $F(\mathbf{x}, \cdot)$ is upper semicontinuous,

$$F^+(\mathbf{x}, \mathbf{y}) \leq a_1(\mathbf{x}) + a_2(\mathbf{x}) |\mathbf{y}|^q, \qquad \text{for a.e. } \mathbf{x} \in \Omega \text{ and all } \mathbf{y} \in \mathbb{R}^d, \tag{7.1}$$

for some $1 \leq q < p$, some $a_1 \in L^1(\Omega)$ and some $a_2 \in L^r(\Omega)$ with

$$\begin{aligned} r > \frac{p^*}{p^* - q}, \quad p^* := \frac{np}{n - sp} \qquad & \text{if } sp < n, \\ r > 1 \qquad & \text{if } sp = n, \\ r = 1 \qquad & \text{if } sp > n. \end{aligned} \tag{7.2}$$

Let $\mathbf{b} \in W^{s,p}(\Omega, \mathbb{R}^d)$. Let $\mathcal{A}$ be the set of $\mathbf{u} \in W^{s,p}(\Omega, \mathbb{R}^d)$ such that $\mathbf{u} = \mathbf{b}$ a.e. on Ω_D. Let $\mathcal{I}$ be as in (2.4), and assume $\mathcal{I}(\mathbf{b}) < \infty$. Then there exists a minimizer of $\mathcal{I}$ in $\mathcal{A}$.

Proof. Note that $\mathcal{A} \neq \varnothing$ and that $\mathcal{I}$ is not identically ∞ in $\mathcal{A}$. Assumption *a)*, Proposition 6.1 and Corollary 6.3 show that there exist $c_1, c_2 > 0$ such that for all $\mathbf{u} \in \mathcal{A}$,

$$c_1 \|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)} \leq \int_\Omega \int_\Omega w(\mathbf{x} - \mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')) \, \mathrm{d}\mathbf{x}' \, \mathrm{d}\mathbf{x} + c_2. \tag{7.3}$$

From this point, the proof is divided according to the cases $sp < n$, $sp = n$ and $sp > n$.

Case $sp < n$. Using estimate (7.1) and Hölder's inequality, we find that for all $\mathbf{u} \in \mathcal{A}$,

$$\int_\Omega F(\mathbf{x}, \mathbf{u}(\mathbf{x})) \, \mathrm{d}\mathbf{x} \leq \|a_1\|_{L^1(\Omega)} + \mathcal{L}^n(\Omega)^{\frac{1}{r'} - \frac{q}{p^*}} \|a_2\|_{L^r(\Omega)} \|\mathbf{u}\|^q_{L^{p^*}(\Omega,\mathbb{R}^d)}. \tag{7.4}$$

Now, Young's inequality, the fractional Sobolev immersion $W^{s,p}(\Omega, \mathbb{R}^d) \subset L^{p^*}(\Omega, \mathbb{R}^d)$ (see Proposition 2.1) and Corollary 6.3 show that there exists $c_3 > 0$ such that for all $\mathbf{u} \in \mathcal{A}$,

$$\mathcal{L}^n(\Omega)^{\frac{1}{r'} - \frac{q}{p^*}} \|a_2\|_{L^r(\Omega)} \|\mathbf{u}\|^q_{L^{p^*}(\Omega,\mathbb{R}^d)} \leq c_3 + \frac{c_1}{2} \|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)}. \tag{7.5}$$

Inequalities (7.3), (7.4) and (7.5) conclude that there exists $c_4 > 0$ such that for all $\mathbf{u} \in \mathcal{A}$,

$$\mathcal{I}(\mathbf{u}) \geq \frac{c_1}{2} \|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)} - c_4. \tag{7.6}$$

Hence $\mathcal{I}$ is bounded below in $\mathcal{A}$. Take a minimizing sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ of $\mathcal{I}$ in $\mathcal{A}$. Due to bound (7.6), for a subsequence, there exists $\mathbf{u} \in W^{s,p}(\Omega, \mathbb{R}^d)$ such that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{s,p}(\Omega, \mathbb{R}^d)$ and $\mathbf{u}_j \to \mathbf{u}$ a.e. as $j \to \infty$, thanks to the compact immersion $W^{s,p}(\Omega,\mathbb{R}^d)\subset L^p(\Omega,\mathbb{R}^d)$ (see Proposition 2.1). In particular, $\mathbf{u}=\mathbf{b}$ a.e. in Ω_D, and, hence, $\mathbf{u}\in\mathcal{A}$. By Fatou's lemma,

$$\int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x}\le\liminf_{j\to\infty}\int_\Omega\int_\Omega w(\mathbf{x}-\mathbf{x}',\mathbf{u}_j(\mathbf{x})-\mathbf{u}_j(\mathbf{x}'))\,\mathrm{d}\mathbf{x}'\,\mathrm{d}\mathbf{x},\tag{7.7}$$

and

$$\int_\Omega F^-(\mathbf{x},\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}\le\liminf_{j\to\infty}\int_\Omega F^-(\mathbf{x},\mathbf{u}_j(\mathbf{x}))\,\mathrm{d}\mathbf{x}.\tag{7.8}$$

On the other hand, the sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ is bounded in $L^{p^*}(\Omega,\mathbb{R}^d)$, again by Proposition 2.1, hence $\{a_2|\mathbf{u}_j|^q\}_{j\in\mathbb{N}}$ is bounded in $L^s(\Omega,\mathbb{R}^d)$ with

$$\frac{1}{r}+\frac{q}{p^*}=\frac{1}{s},$$

so $s>1$. Therefore, the sequence $\{a_2|\mathbf{u}_j|^q\}_{j\in\mathbb{N}}$ is equiintegrable. Bound (7.1) and Vitali's convergence theorem show that

$$\int_\Omega F^+(\mathbf{x},\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}\ge\limsup_{j\to\infty}F^+(\mathbf{x},\mathbf{u}_j(\mathbf{x}))\,\mathrm{d}\mathbf{x}.\tag{7.9}$$

Inequalities (7.7), (7.8) and (7.9) show that

$$\mathcal{I}(\mathbf{u})\le\liminf_{j\to\infty}\mathcal{I}(\mathbf{u}_j).\tag{7.10}$$

Hence $\mathbf{u}$ is a minimizer of $\mathcal{I}$ in $\mathcal{A}$.

Case $sp=n$. Choose $1<p^*<\infty$ big enough so that

$$r>\frac{p^*}{p^*-q}.$$

As the compact immersion $W^{s,p}(\Omega,\mathbb{R}^d)\subset L^{p^*}(\Omega,\mathbb{R}^d)$ holds (see Proposition 2.1), one can repeat the proof of the previous case.

Case $sp>n$. Using estimate (7.1), we find that

$$\int_\Omega F^+(\mathbf{x},\mathbf{u}(\mathbf{x}))\,\mathrm{d}\mathbf{x}\le\|a_1\|_{L^1(\Omega)}+\|a_2\|_{L^1(\Omega)}\|\mathbf{u}\|^q_{L^\infty(\Omega,\mathbb{R}^d)}.$$

Now, Young's inequality, the fractional Sobolev immersion $W^{s,p}(\Omega,\mathbb{R}^d)\subset L^\infty(\Omega,\mathbb{R}^d)$ (see Proposition 2.1) and Corollary 6.3 show that there exists $c_3>0$ such that for all $\mathbf{u}\in\mathcal{A}$,

$$\|a_2\|_{L^1(\Omega)}\|\mathbf{u}\|^q_{L^\infty(\Omega,\mathbb{R}^d)}\le c_3+\frac{c_1}{2}\|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)}.$$

As before, we conclude that there exists $c_4>0$ such that bound (7.6) holds for all $\mathbf{u}\in\mathcal{A}$, hence $\mathcal{I}$ is bounded below in $\mathcal{A}$. Take a minimizing sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ of $\mathcal{I}$ in $\mathcal{A}$. Then, for a subsequence, there exists $\mathbf{u}\in W^{s,p}(\Omega,\mathbb{R}^d)$ such that $\mathbf{u}_j\rightharpoonup\mathbf{u}$ in $W^{s,p}(\Omega,\mathbb{R}^d)$ and $\mathbf{u}_j\to\mathbf{u}$ uniformly as $j\to\infty$ (see Proposition 2.1). In particular, $\mathbf{u}=\mathbf{b}$ a.e. in Ω_D, and, hence, $\mathbf{u}\in\mathcal{A}$. By Fatou's lemma, inequalities (7.7) and (7.8) hold. It is easy to check that the sequence $\{a_2|\mathbf{u}_j|^q\}_{j\in\mathbb{N}}$ is equiintegrable. As before, inequalities (7.9) and (7.10) hold. Hence $\mathbf{u}$ is a minimizer of $\mathcal{I}$ in $\mathcal{A}$. □

For Neumann boundary conditions, the result is as follows. As in Theorem 5.2, we present it with linear forces, but the result is also true for nonlinear forces F satisfying the assumptions of Theorem 7.1.

Theorem 7.2. *Let $0<s<1$ and $1<p<\infty$. Let Ω be a Lipschitz domain and fix $\delta>0$. Let $w:\tilde{\Omega}\times\mathbb{R}^d\to\mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega})\times\mathcal{B}(\mathbb{R}^d)$-measurable and satisfy conditions* a–b) *of Theorem 7.1. Let $\mathbf{f}\in L^r(\Omega,\mathbb{R}^d)$ be with*

$$\begin{aligned} r&=(p^*)',\quad p^*:=\frac{np}{n-sp} && \text{if } sp<n,\\ r&>1 && \text{if } sp=n,\\ r&=1 && \text{if } sp>n.\end{aligned}$$

Let $\mathcal{A}$ be the set of $\mathbf{u}\in W^{s,p}(\Omega,\mathbb{R}^d)$ such that $\int_\Omega\mathbf{u}=\mathbf{0}$. Let $\mathcal{I}$ be as in (5.8), and assume that $\mathcal{I}$ is not identically infinity in $\mathcal{A}$. Then there exists a minimizer of $\mathcal{I}$ in $\mathcal{A}$.

Proof. Note that $\mathcal{A} \neq \varnothing$ since $\mathbf{0} \in \mathcal{A}$. In addition, we are assuming that $\mathcal{I}$ is not identically ∞ in $\mathcal{A}$. Assumption *a)* of Theorem 7.1 and Proposition 6.1 show that there exists $c_1 > 0$ such that for all $\mathbf{u} \in \mathcal{A}$,

$$c_1 \|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)} \leq \int_\Omega \int_\Omega w(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')) \,\mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x}. \tag{7.11}$$

From this point, the proof is divided according to the cases $sp < n$, $sp = n$ and $sp > n$.

Case $sp < n$. Using Hölder's and Young's inequalities, as well as Proposition 2.1, we obtain that there exists $c_2 > 0$ such that for all $\mathbf{u} \in \mathcal{A}$,

$$\int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \,\mathrm{d}\mathbf{x} \leq \|\mathbf{f}\|_{L^{(p^*)'}(\Omega,\mathbb{R}^d)} \|\mathbf{u}\|_{L^{p^*}(\Omega,\mathbb{R}^d)} \leq c_2 + \frac{c_1}{2} \|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)}, \tag{7.12}$$

which, together with (7.11) yields

$$I(\mathbf{u}) \geq \frac{c_1}{2} \|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)} - c_2.$$

Hence $\mathcal{I}$ is bounded below in $\mathcal{A}$. Take a minimizing sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}}$ of $\mathcal{I}$ in $\mathcal{A}$. Then, for a subsequence, there exists $\mathbf{u} \in W^{s,p}(\Omega,\mathbb{R}^d)$ such that $\mathbf{u}_j \rightharpoonup \mathbf{u}$ in $W^{s,p}(\Omega,\mathbb{R}^d)$ and $\mathbf{u}_j \to \mathbf{u}$ a.e. as $j \to \infty$. In particular, $\int_\Omega \mathbf{u} = \mathbf{0}$, and, hence, $\mathbf{u} \in \mathcal{A}$. By Fatou's lemma, inequality (7.7) holds, while by weak convergence, $\int_\Omega \mathbf{f} \cdot \mathbf{u}_j \to \int_\Omega \mathbf{f} \cdot \mathbf{u}$ as $j \to \infty$. Therefore, inequality (7.10) holds and $\mathbf{u}$ is a minimizer of $\mathcal{I}$ in $\mathcal{A}$.

Case $sp = n$. Instead of (7.12), we have

$$\int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \,\mathrm{d}\mathbf{x} \leq \|\mathbf{f}\|_{L^r(\Omega,\mathbb{R}^d)} \|\mathbf{u}\|_{L^{r'}(\Omega,\mathbb{R}^d)} \leq c_2 + \frac{c_1}{2} \|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)},$$

and we can repeat the argument of the previous case.

Case $sp > n$. Instead of (7.12), we have

$$\int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \,\mathrm{d}\mathbf{x} \leq \|\mathbf{f}\|_{L^1(\Omega,\mathbb{R}^d)} \|\mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^d)} \leq c_2 + \frac{c_1}{2} \|\mathbf{u}\|^p_{W^{s,p}(\Omega,\mathbb{R}^d)},$$

and we can repeat the argument of the previous cases. □

8. Euler–Lagrange equations

In this section we write out the Euler–Lagrange equations satisfied by the minimizers of $\mathcal{I}$. The results are divided according to the spaces and boundary conditions used.

The following result, which is an immediate consequence of the differentiation under the integral sign (see, e.g., [29, Ch. 13, §2, Lemma 2.2]), shows the derivative of the energy under growth conditions compatible with L^p.

Lemma 8.1. *Let Ω be a bounded open set of $\mathbb{R}^n$. Let $1 \leq p < \infty$ and $\mathbf{u}, \mathbf{v} \in L^p(\Omega,\mathbb{R}^d)$. Let $a_1 \in L^1(\tilde{\Omega})$, $a_2 \in L^{p'}(\tilde{\Omega})$, $b_1 \in L^1(\Omega)$ and $b_2 \in L^{p'}(\Omega)$. Let $w : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega}) \times \mathcal{B}(\mathbb{R}^d)$-measurable, and $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\Omega) \times \mathcal{B}(\mathbb{R}^d)$-measurable. Suppose $w(\tilde{\mathbf{x}}, \cdot)$ is differentiable in $\mathbb{R}^d$ for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$ with derivative $D_2 w(\tilde{\mathbf{x}}, \cdot)$, and $F(\mathbf{x}, \cdot)$ is differentiable in $\mathbb{R}^d$ for a.e. $\mathbf{x} \in \Omega$ with derivative $D_2 F(\mathbf{x}, \cdot)$. Assume that there exists $C > 0$ for which*

$$\begin{aligned} &|w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})| \leq a_1(\tilde{\mathbf{x}}) + C |\tilde{\mathbf{y}}|^p, \quad |D_2 w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})| \leq a_2(\tilde{\mathbf{x}}) + C |\tilde{\mathbf{y}}|^{p-1}, \qquad \text{for a.e. } \tilde{\mathbf{x}} \in \tilde{\Omega} \text{ and all } \tilde{\mathbf{y}} \in \mathbb{R}^d, \\ &|F(\mathbf{x}, \mathbf{y})| \leq b_1(\mathbf{x}) + C |\mathbf{y}|^p, \quad |D_2 F(\mathbf{x}, \mathbf{y})| \leq b_2(\mathbf{x}) + C |\mathbf{y}|^{p-1}, \qquad \text{for a.e. } \mathbf{x} \in \Omega \text{ and all } \mathbf{y} \in \mathbb{R}^d. \end{aligned} \tag{8.1}$$

Let $\mathcal{I}$ be as in (2.4). Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=0} \mathcal{I}(\mathbf{u} + \tau\mathbf{v}) = \int_\Omega \int_\Omega D_2 w(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')) \cdot (\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x}')) \,\mathrm{d}\mathbf{x}' \,\mathrm{d}\mathbf{x} - \int_\Omega D_2 F(\mathbf{x}, \mathbf{u}(\mathbf{x})) \cdot \mathbf{v}(\mathbf{x}) \,\mathrm{d}\mathbf{x}. \tag{8.2}$$

For growth conditions compatible with $W^{s,p}$, the corresponding result is the following, where the embeddings of Proposition 2.1 are used.

Lemma 8.2. *Let Ω be a Lipschitz domain in $\mathbb{R}^d$. Let $0 < s < 1$ and $1 \leq p < \infty$. Let $\mathbf{u}, \mathbf{v} \in W^{s,p}(\Omega,\mathbb{R}^d)$. Let $w : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega}) \times \mathcal{B}(\mathbb{R}^d)$-measurable, and $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\Omega) \times \mathcal{B}(\mathbb{R}^d)$-measurable. Suppose $w(\tilde{\mathbf{x}}, \cdot)$ is differentiable in $\mathbb{R}^d$ for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$ with derivative $D_2 w(\tilde{\mathbf{x}}, \cdot)$, and $F(\mathbf{x}, \cdot)$ is differentiable in $\mathbb{R}^d$ for a.e. $\mathbf{x} \in \Omega$ with derivative $D_2 F(\mathbf{x}, \cdot)$.*

i) If $sp < n$, assume that there exist $C > 0$, $a_1 \in L^1(\tilde{\Omega})$, $a_2 \in L^{(p^*)'}(\tilde{\Omega})$, $b_1 \in L^1(\Omega)$ and $b_2 \in L^{(p^*)'}(\Omega)$, with $p^* := \frac{np}{n-sp}$, such that for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$, all $\tilde{\mathbf{y}} \in \mathbb{R}^d$, a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{y} \in \mathbb{R}^d$,

$$|w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})| \leq a_1(\tilde{\mathbf{x}}) + C\left(|\tilde{\mathbf{y}}|^{p^*} + \frac{|\tilde{\mathbf{y}}|^p}{|\tilde{\mathbf{x}}|^{n+sp}}\right), \qquad |D_2 w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})| \leq a_2(\tilde{\mathbf{x}}) + C\left(|\tilde{\mathbf{y}}|^{p^*-1} + \frac{|\tilde{\mathbf{y}}|^{p-1}}{|\tilde{\mathbf{x}}|^{n+sp}}\right),$$

$$|F(\mathbf{x}, \mathbf{y})| \leq b_1(\mathbf{x}) + C\left(|\mathbf{y}|^{p^*} + \frac{|\mathbf{y}|^p}{|\mathbf{x}|^{n+sp}}\right), \qquad |D_2 F(\mathbf{x}, \mathbf{y})| \leq b_2(\mathbf{x}) + C\left(|\mathbf{y}|^{p^*-1} + \frac{|\mathbf{y}|^{p-1}}{|\mathbf{x}|^{n+sp}}\right).$$

ii) If $sp = n$, assume that there exist $C > 0$, $a_1 \in L^1(\tilde{\Omega})$, $a_2 \in L^{r'}(\tilde{\Omega})$, $b_1 \in L^1(\Omega)$ and $b_2 \in L^{r'}(\Omega)$, with $1 < r < \infty$, such that for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$, all $\tilde{\mathbf{y}} \in \mathbb{R}^d$, a.e. $\mathbf{x} \in \Omega$ and all $\mathbf{y} \in \mathbb{R}^d$,

$$|w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})| \leq a_1(\tilde{\mathbf{x}}) + C\left(|\tilde{\mathbf{y}}|^{r} + \frac{|\tilde{\mathbf{y}}|^p}{|\tilde{\mathbf{x}}|^{n+sp}}\right), \qquad |D_2 w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})| \leq a_2(\tilde{\mathbf{x}}) + C\left(|\tilde{\mathbf{y}}|^{r-1} + \frac{|\tilde{\mathbf{y}}|^{p-1}}{|\tilde{\mathbf{x}}|^{n+sp}}\right),$$

$$|F(\mathbf{x}, \mathbf{y})| \leq b_1(\mathbf{x}) + C\left(|\mathbf{y}|^{r} + \frac{|\mathbf{y}|^p}{|\mathbf{x}|^{n+sp}}\right), \qquad |D_2 F(\mathbf{x}, \mathbf{y})| \leq b_2(\mathbf{x}) + C\left(|\mathbf{y}|^{r-1} + \frac{|\mathbf{y}|^{p-1}}{|\mathbf{x}|^{n+sp}}\right).$$

iii) If $sp > n$, assume that for each compact $K \subset \mathbb{R}^d$,

$$\int_\Omega \int_\Omega \sup_{\mathbf{y} \in K} |w(\mathbf{x} - \mathbf{x}', \mathbf{y})| \, d\mathbf{x}' \, d\mathbf{x} < \infty, \qquad \int_\Omega \int_\Omega \sup_{\mathbf{y} \in K} |D_2 w(\mathbf{x} - \mathbf{x}', \mathbf{y})| \, d\mathbf{x}' \, d\mathbf{x} < \infty,$$

$$\int_\Omega \sup_{\mathbf{y} \in K} |F(\mathbf{x}, \mathbf{y})| \, d\mathbf{x} < \infty, \qquad \int_\Omega \sup_{\mathbf{y} \in K} |D_2 F(\mathbf{x}, \mathbf{y})| \, d\mathbf{x} < \infty.$$

Then equality (8.2) *holds, where $\mathcal{I}$ is as in* (2.4).

The Euler–Lagrange equations are as follows.

Theorem 8.3. *Let Ω be a bounded open set of $\mathbb{R}^n$, and let $1 \leq p < \infty$. Let $w : \tilde{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\tilde{\Omega}) \times \mathcal{B}(\mathbb{R}^d)$-measurable, and $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{L}^n(\Omega) \times \mathcal{B}(\mathbb{R}^d)$-measurable. Suppose $w(\tilde{\mathbf{x}}, \cdot)$ is differentiable in $\mathbb{R}^d$ for a.e. $\tilde{\mathbf{x}} \in \tilde{\Omega}$ with derivative $D_2 w(\tilde{\mathbf{x}}, \cdot)$, and $F(\mathbf{x}, \cdot)$ is differentiable in $\mathbb{R}^d$ for a.e. $\mathbf{x} \in \Omega$ with derivative $D_2 F(\mathbf{x}, \cdot)$. The following assertions hold:*

1) Let Ω_D be a measurable subset of Ω of positive measure. Assume either of the following:
  a) Let $\mathbf{b} \in L^p(\Omega_D, \mathbb{R}^d)$, and let $\mathcal{A}$ be the set of $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ such that $\mathbf{u} = \mathbf{b}$ a.e. on Ω_D. Assume the bounds (8.1) *for some $a_1 \in L^1(\tilde{\Omega})$, $a_2 \in L^{p'}(\tilde{\Omega})$, $b_1 \in L^1(\Omega)$ and $b_2 \in L^{p'}(\Omega)$.*
  b) Let $0 < s < 1$ and suppose that Ω is a Lipschitz domain. Let $\mathbf{b} \in W^{s,p}(\Omega, \mathbb{R}^d)$, and let $\mathcal{A}$ be the set of $\mathbf{u} \in W^{s,p}(\Omega, \mathbb{R}^d)$ such that $\mathbf{u} = \mathbf{b}$ a.e. on Ω_D. Suppose, further, that $\Omega \setminus \Omega_D$ coincides a.e. with an open set. Assume any of conditions i)–iii) *of Lemma 8.2.*

Let $\mathcal{I}$ be as in (2.4), *and let $\mathbf{u}$ be a minimizer of $\mathcal{I}$ in $\mathcal{A}$. Then*

$$2 \int_\Omega D_2 w(\mathbf{x} - \mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')) \, d\mathbf{x}' = D_2 F(\mathbf{x}, \mathbf{u}(\mathbf{x})), \qquad \text{a.e. } \mathbf{x} \in \Omega \setminus \Omega_D. \tag{8.3}$$

2) Assume either of the following:
  a) Let $\mathcal{A}$ be the set of $\mathbf{u} \in L^p(\Omega, \mathbb{R}^d)$ such that $\int_\Omega \mathbf{u} = \mathbf{0}$. Assume the bounds (8.1) *for some $a_1 \in L^1(\tilde{\Omega})$, $a_2 \in L^{p'}(\tilde{\Omega})$, $b_1 \in L^1(\Omega)$ and $b_2 \in L^{p'}(\Omega)$.*
  b) Let $0 < s < 1$ and suppose that Ω is a Lipschitz domain. Let $\mathcal{A}$ be the set of $\mathbf{u} \in W^{s,p}(\Omega, \mathbb{R}^d)$ such that $\int_\Omega \mathbf{u} = \mathbf{0}$. Assume any of conditions i)–iii) *of Lemma 8.2.*

Let $\mathcal{I}$ be as in (2.4), *and let $\mathbf{u}$ be a minimizer of $\mathcal{I}$ in $\mathcal{A}$. Then there exists $\mathbf{a} \in \mathbb{R}^d$ such that*

$$2 \int_\Omega D_2 w(\mathbf{x} - \mathbf{x}', \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')) \, d\mathbf{x}' = D_2 F(\mathbf{x}, \mathbf{u}(\mathbf{x})) + \mathbf{a}, \qquad \text{a.e. } \mathbf{x} \in \Omega. \tag{8.4}$$

Proof. Assume first case *1)*. Let $\mathbf{v}$ belong to $L^p(\Omega, \mathbb{R}^d)$ or to $W^{s,p}(\Omega, \mathbb{R}^d)$, according to whether option *1a)* or *1b)* holds. In addition, assume that $\mathbf{v} = \mathbf{0}$ a.e. on Ω_D. Then, $\mathbf{u} + \tau \mathbf{v} \in \mathcal{A}$ for all $\tau \in \mathbb{R}$. As $\mathbf{u}$ is a minimizer, we apply

Lemmas 8.1 or 8.2 (according to whether option *1a)* or *1b)* holds), and obtain that

$$\int_\Omega \int_\Omega D_2 w(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}')) \cdot (\mathbf{v}(\mathbf{x})-\mathbf{v}(\mathbf{x}'))\, \mathrm{d}\mathbf{x}'\, \mathrm{d}\mathbf{x} - \int_\Omega D_2 F(\mathbf{x}, \mathbf{u}(\mathbf{x})) \cdot \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x} = 0.$$

Changing the order of integration, using the symmetry

$$D_2 w(-\tilde{\mathbf{x}}, -\tilde{\mathbf{y}}) = -D_2 w(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}), \qquad \text{for a.e. } \tilde{\mathbf{x}} \in \tilde{\Omega} \text{ and all } \tilde{\mathbf{y}} \in \mathbb{R}^d$$

coming from (2.1), and applying the boundary condition $\mathbf{v} = \mathbf{0}$ a.e. on Ω_D, we arrive at

$$\int_{\Omega \setminus \Omega_D} \left[2 \int_\Omega D_2 w(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}'))\mathrm{d}\mathbf{x}' - D_2 F(\mathbf{x}, \mathbf{u}(\mathbf{x})) \right] \cdot \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x} = 0.$$

In case *1a)*, this implies at once that (8.3) holds. In case *1b)*, this also implies equality (8.3), thanks to a classic approximation result (see, e.g., [16, Th. 3.40]), using that an a.e. representant of $\Omega \setminus \Omega_D$ is open.

Now we assume case *2)*. Let $\mathbf{v}$ belong to $L^p(\Omega, \mathbb{R}^d)$ or to $W^{s,p}(\Omega, \mathbb{R}^d)$, according to whether option *2a)* or *2b)* holds. In addition, assume that $\int_\Omega \mathbf{v} = \mathbf{0}$. Then, $\mathbf{u} + \tau \mathbf{v} \in \mathcal{A}$ for all $\tau \in \mathbb{R}$. As before, we arrive at the equality

$$\int_\Omega \left[2 \int_\Omega D_2 w(\mathbf{x}-\mathbf{x}', \mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}'))\mathrm{d}\mathbf{x}' - D_2 F(\mathbf{x}, \mathbf{u}(\mathbf{x})) \right] \cdot \mathbf{v}(\mathbf{x})\, \mathrm{d}\mathbf{x} = 0. \tag{8.5}$$

Hence the function inside the square brackets of (8.5) is orthogonal to the closed hyperplane of $L^p(\Omega, \mathbb{R}^d)$ or of $W^{s,p}(\Omega, \mathbb{R}^d)$ formed by the functions with zero integral in Ω. Thus, there exists $\mathbf{a} \in \mathbb{R}^d$ such that (8.4) holds. ☐

We finish this paper with a brief mention that the Euler–Lagrange equation (8.3) can be given an interpretation in terms of Neumann boundary conditions. Indeed, let Ω_0, Ω_1 be nonempty measurable disjoint subsets with union Ω, such that $\Omega_D \subset \Omega_1$. Define $\Omega_N := \Omega_1 \setminus \Omega_D$, and, for definiteness, assume $\mathcal{L}^n(\Omega_N) > 0$. Equality (8.3) is split into two: for a.e. $\mathbf{x} \in \Omega_0$ and for a.e. $\mathbf{x} \in \Omega_N$. The equality for a.e. $\mathbf{x} \in \Omega_0$ corresponds to the equation satisfied in the inner part of the body. The equality for a.e. $\mathbf{x} \in \Omega_N$, on the other hand, corresponds to the equation satisfied on the Neumann part of the nonlocal boundary of the body, and can be given an interpretation of a nonlocal flux through the boundary of Ω_N, thus mimicking what happens for the local equations. This nonlocal calculus is developed in [25, 18, 19], to which we refer for further explanation.

Acknowledgements. We thank Pablo Pedregal for his suggestion to study this problem, and Javier Cárcamo for explaining to us a probabilistic interpretation of Lemma 3.3. C.M.-C. has been supported by Project MTM2011-28198 of the Spanish Ministry of Economy and Competitivity, the ERC Starting Grant n. 307179, the "Ramón y Cajal" program and the European Social Fund.

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