DIFFEOMORPHIC APPROXIMATION OF CONTINUOUS ALMOST EVERYWHERE INJECTIVE SOBOLEV DEFORMATIONS IN THE PLANE

STANISLAV HENCL AND CARLOS MORA-CORRAL

Abstract. In this note we prove that given a continuous Sobolev $W^{1,p}$ deformation f, with $1 < p < \infty$, from a planar domain to $\mathbb{R}^2$ which is injective almost everywhere, we can find a sequence f_k of diffeomorphisms with $f_k - f \in W_0^{1,p}$ such that $f_k \to f$ uniformly and in the Sobolev norm.

1. Introduction

The possibility of approximating a Sobolev homeomorphism by diffeomorphism has attracted much attention in the last years [20, 4, 15, 16, 8, 21, 14]. This problem arises naturally in nonlinear elasticity [3] and geometric function theory. In this paper we follow many ideas from Iwaniec, Kovalev and Onninen [15], where they prove that a Sobolev $W^{1,p}$ homeomorphism in the plane can be approximated by diffeomorphisms uniformly and in the Sobolev norm, for $1 < p < \infty$.

The main aim of this note is to remove the assumption of homeomorphism and replace it by almost everywhere injectivity. This is of interest in the theory since many natural models of deformations fail to be injective by a set of measure zero, and still conserve the property of non-interpenetration of matter. The study of mappings f in the spirit of Theorem 1.1 below was initiated by Šverák [25] and Müller and Spector [22], and later developed in [23, 24, 7, 10, 11, 12]. They were studying properties of elastic deformations allowing or forbidding cavitation.

Theorem 1.1. *Let $p > 1$, $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $f \in W^{1,p}(\Omega, \mathbb{R}^2) \cap C(\overline{\Omega}, \mathbb{R}^2)$ satisfy $\det Df > 0$ a.e., condition INV holds, the distributional Jacobian satisfies $\operatorname{Det} Df = \det Df$, and $f(\Omega) \cap f(\partial\Omega) = \emptyset$. Then there are C^∞ smooth diffeomorphisms $f_k \in W^{1,p}(\Omega, \mathbb{R}^2)$ onto $f(\Omega)$, with $f_k - f \in W_0^{1,p}(\Omega, \mathbb{R}^2)$, such that $f_k \to f$ uniformly and in $W^{1,p}(\Omega, \mathbb{R}^2)$.*

Let us point out that assumption $f(\Omega) \cap f(\partial\Omega) = \emptyset$ is needed for the openness of the set $f(\Omega)$, which is necessary for the existence of a sequence of homeomorphisms $f_k : \Omega \to f(\Omega)$ approximating f.

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To illustrate that our result applies to some wild mappings let us describe one example. It is easy to see that the mapping $h : [-1,1]^2 \to \mathbb{R}^2$ defined as

$$h(x_1, x_2) = [x_1, |x_1|x_2]$$

is Lipschitz and the preimage of $[0,0]$ is the line segment $[-1,1] \times \{0\}$. Moreover, this mapping can be extended to a Lipschitz mapping $h : [-2,2]^2 \to \mathbb{R}^2$ so that $h|_{(-2,2)^2 \setminus [-1,1]^2}$ is a diffeomorphism (see [2] for details). Recall now that there is a quasiconformal mapping $g : [-2,2]^2 \to [-2,2]^2$ that maps a part of the von Koch snowflake to the segment $[-1,1] \times \{0\}$ (see, e.g., [19, Example 10.9]). The composition $f := h \circ g$ belongs to $W^{1,2}$ (as a composition of a $W^{1,2}$ mapping and a quasiconformal mapping) and satisfies all assumptions of Theorem 1.1 and, hence, it can be approximated by diffeomorphisms. On the other hand, it collapses a part of the von Koch snowflake to one point.

It would be interesting to ascertain whether the assumption of continuity in Theorem 1.1 can be removed, and, in particular, whether it still holds when we allow for cavities (in which case, the assumption $\operatorname{Det} Df = \det Df$ is dropped). Of course, the uniform convergence will be lost. Unfortunately, it seems that the existing techniques are not strong enough to tackle this problem. Our result is in the direction of [17, Question 1.2], where they asked for a characterization of limits of Sobolev homeomorphisms. A natural candidate, as they suggested, is the class of monotone maps. Another possible candidate is some class of functions satisfying condition INV.

The sketch of the proof is as follows. The first step consists in showing that the set

$$S := \{y \in f(\Omega) : \operatorname{diam} f^{-1}(\{y\}) > 0\}$$

satisfies $\mathcal{H}^1(S) = 0$. This will be essential for showing that the restriction of our mapping to some properly chosen grid in $f(\Omega)$ is one-to-one. For the proof of $\mathcal{H}^1(S) = 0$, we follow the ideas of [25]. Then we show that $f(\Omega)$ is open. For the construction of the approximation we use the techniques of [15], where we use the p-harmonic replacement on each cell of the induced grid in Ω. With this, we construct a homeomorphism that approximates f, which is smooth except on the grid. From this step, it would be possible to smooth it out along the edges, as in the remaining steps of [15], or we can just refer directly to their statement.

2. Preliminaries

By $|E|$ we denote the Lebesgue measure of the set $E \subset \mathbb{R}^2$, and by $\mathcal{H}^1(E)$ its 1-dimensional Hausdorff measure. The preimage of an $F \subset \mathbb{R}^2$ is denoted by $f^{-1}(F)$, which is well defined since f is continuous.

We now explain the assumptions of Theorem 1.1 and their consequences. Recall that $f \in C(\overline{\Omega}, \mathbb{R}^2)$. For any open $U \subset \Omega$ and $y \in \mathbb{R}^2 \setminus f(\partial U)$, we denote by $\deg(f, U, y)$ the degree of f on U at y. For an introduction to degree theory, see, e.g., [9]. We define $\operatorname{im}_{\mathrm{T}}(f, U)$ as the set of $y \in \mathbb{R}^2 \setminus f(\partial U)$ such that $\deg(f, U, y) \neq 0$. By the continuity of the degree, $\operatorname{im}_{\mathrm{T}}(f, U)$ is open.

We say that f satisfies condition INV (a concept introduced in [22]) when for all $x_0 \in \Omega$ and a.e. $r \in (0, \operatorname{dist}(x_0, \partial\Omega))$,

(1) $f(x) \in \operatorname{im}_{\mathrm{T}}(f, B(x_0, r))$ for a.e. $x \in B(x_0, r)$,
(2) $f(x) \notin \operatorname{im}_{\mathrm{T}}(f, B(x_0, r))$ for a.e. $x \in \Omega \setminus B(x_0, r)$.

Condition INV is a stronger concept of a.e. invertibility, yet weaker than homeomorphism. It was shown in [22, Lemma 3.4] that if f satisfies INV and $\det Df > 0$ a.e., then there exists a set $\Omega_0 \subset \Omega$ of full measure such that $f|_{\Omega_0}$ is one-to-one. It is proved in [**?**, Theorem 3.3] that its inverse belongs to $W^{1,1}(\mathrm{im_T}(f,\Omega), \mathbb{R}^2)$.

By [22, Lemma 3.5], for all $x_0 \in \Omega$ and a.e. $r \in (0, \mathrm{dist}(x_0, \partial\Omega))$,

$$\deg(f, B(x_0,r), y) \in \{0,1\} \quad \text{for every } y \in \mathbb{R}^2 \setminus f(\partial B(x_0,r)).$$

In fact, thanks to the analysis of [22, Section 9] we have that if U is a C^2 open set such that $\overline{U} \subset \Omega$ and we call, for $r > 0$,

$$U_r := \{x \in \mathbb{R}^2 : \mathrm{dist}(x,U) < r\}, \qquad U_{-r} := \{x \in U : \mathrm{dist}(x, \partial U) > r\},$$

we have that there exists $\delta > 0$ such that for a.e. $r \in (-\delta, \delta)$,

$$\deg(f, U_r, y) \in \{0,1\} \quad \text{for every } y \in \mathbb{R}^2 \setminus f(\partial U_r). \tag{2.1}$$

Now let $U \subset \Omega$ be open. One can easily construct an increasing sequence U_j of C^2 open sets as in (2.1) such that $\bigcup_{j\in\mathbb{N}} U_j = U$. For a given $y \in \mathbb{R}^2 \setminus f(\partial U)$, one can see that $y \notin f(U \setminus U_j)$ for j large enough. Therefore, thanks to the excision property of the degree, $\deg(f,U,y) = \deg(f,U_j,y)$, so we conclude from (2.1) that

$$\deg(f, U, y) \in \{0,1\} \quad \text{for every } y \in \mathbb{R}^2 \setminus f(\partial U). \tag{2.2}$$

For the definition of the distributional Jacobian $\mathrm{Det}\, Df$, see [22, Section 8]. The equality $\mathrm{Det}\, Df = \det Df$ means that the distribution $\mathrm{Det}\, Df$ can be identified with the function $\det Df$. The geometrical meaning of that equality is that the function f does not create cavities (see [22]).

Under the assumptions of Theorem 1.1, the map f satisfies Lusin's condition (N) (see [22, Theorem 10.1]), i.e., for every $E \subset \Omega$ with $|E| = 0$ we have $|f(E)| = 0$. As $\det Df > 0$ a.e., it also satisfies (N^{-1}), i.e., for every $F \subset \mathbb{R}^2$ with $|F| = 0$ we have $|f^{-1}(F)| = 0$.

2.1. **p-harmonic replacement.** Analogously to [15, 17], we need to extend our mapping from the boundary of some cell U in the plane to $\overline{U}$ so that the extension is a diffeomorphism in U. For this, we will use the p-harmonic extension of both coordinate functions of the boundary mapping. Recall that the function $u : U \to \mathbb{R}$ is p-harmonic if it solves the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$ in the weak sense.

Proposition 2.1. *Let U be a bounded Jordan domain and $u_0 \in W^{1,p}(U) \cap C(\overline{U})$. There exists a unique, p-harmonic function $u \in W^{1,p}(U) \cap C(\overline{U})$ such that $u - u_0 \in W_0^{1,p}(U)$. Moreover,*

$$\int_U |\nabla u|^p \,\mathrm{d}x = \inf\Big\{ \int_U |\nabla w|^p \,\mathrm{d}x : \ w \in u_0 + W_0^{1,p}(U) \Big\}.$$

We use the generalization of the Radó-Kneser-Choquet theorem for harmonic mappings whose idea goes back to Kneser [18] and Choquet [6]. This generalization to p-harmonic mappings is by Alessandrini and Sigalotti [1]. We say that a mapping is p-harmonic when both coordinate functions are p-harmonic, i.e., the system of equations is uncoupled. The important property that we seek is that the extended mapping is a homeomorphism (in fact, a diffeomorphism).

Theorem 2.2. *Suppose that $U \subset \mathbb{R}^2$ is a simply connected Jordan domain, G a bounded convex domain and $h \in C(\overline{U}, \mathbb{R}^2) \cap W^{1,p}_{\mathrm{loc}}(U, \mathbb{R}^2)$ a p-harmonic mapping that sends ∂U homeomorphically onto ∂G. Then $h : U \overset{onto}{\to} G$ is a C^∞-diffeomorphism.*

3. Smallness of S

Recall that $S = \{y \in f(\Omega) : \operatorname{diam} f^{-1}(\{y\}) > 0\}$. In this section we follow the ideas of Šverák [25] to prove that $\mathcal{H}^1(S) = 0$. As our assumptions are different, we include the proof for the convenience of the reader. First, it is clear that $S \subset f(\Omega \setminus \Omega_0)$, since f is one-to-one in Ω_0 (see Preliminaries). Now, $|f(\Omega \setminus \Omega_0)| = 0$ because $|\Omega \setminus \Omega_0| = 0$ and f satisfies (N) condition. Thus, $|S| = 0$. By the (N^{-1}) condition,

$$|f^{-1}(S)| = 0. \tag{3.1}$$

Moreover, by definition of S, the inverse function $f^{-1} : f(\Omega) \setminus S \to \Omega$ is well defined and one-to-one.

We now adapt the proof of [2, Theorem 1 (iv)] to show that $f^{-1}(\{y\})$ is connected for every $y \in f(\Omega)$. If it were not connected, there would exist two disjoint, nonempty compact sets K_1, K_2 in Ω such that $f^{-1}(\{y\}) = K_1 \cup K_2$, as $f^{-1}(\{y\})$ is compact and $f(\Omega) \cap f(\partial\Omega) = \emptyset$. Choosing disjoint open sets $G_i \subset \Omega$ such that $K_i \subset G_i$, $i \in \{1, 2\}$, we would have that, using (2.2) and the additivity of the degree,

$$\deg(f, G_1, y) + \deg(f, G_2, y) = \deg(f, G_1 \cup G_2, y) \leq 1.$$

But, thanks to [22, Corollary 7.5], condition INV shows that $f(G_i) \subset \operatorname{im}_{\mathrm{T}}(f, G_i) \cup f(\partial G_i)$, so

$$\deg(f, G_1, y) = \deg(f, G_2, y) = 1,$$

which is a contradiction.

As, for every $y \in S$, the set $f^{-1}(\{y\})$ is connected and contains more than one point, we have that $\mathcal{H}^1\big(\pi(f^{-1}(\{y\}))\big) > 0$, where $\pi : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto some coordinate. Suppose, looking for a contradiction, that $\mathcal{H}^1(S) > 0$. By taking intervals with rational endpoints, we can find a non-degenerate interval $J \subset \mathbb{R}$ and a set $K \subset S$ with $\mathcal{H}^1(K) > 0$ such that

$$J \subset \pi_1\big(f^{-1}(\{y\})\big), \qquad \text{for every } y \in K, \tag{3.2}$$

where, without loss of generality, $\pi_1(x_1, x_2) := x_1$. We can find $t \in J$ such that $f(t, \cdot) \in W^{1,p}(\Omega^t, \mathbb{R}^2)$, where

$$\Omega^t := \{x_2 \in \mathbb{R} : (t, x_2) \in \Omega\}$$

and

$$\begin{aligned} f(t, \cdot) : \Omega^t &\to \mathbb{R}^2, \\ x_2 &\mapsto f(t, x_2). \end{aligned}$$

Moreover, we can also require that $\mathcal{H}^1\big((f^{-1}(K))^t\big) = 0$, thanks to (3.1). Since $f(t, \cdot)$ is absolutely continuous, we obtain that

$$\mathcal{H}^1\big(f(t, \cdot)((f^{-1}(K))^t)\big) = 0.$$

By (3.2) and, since $t \in J$, we can check that

$$K \subset f(t, \cdot)\big((f^{-1}(K))^t\big),$$

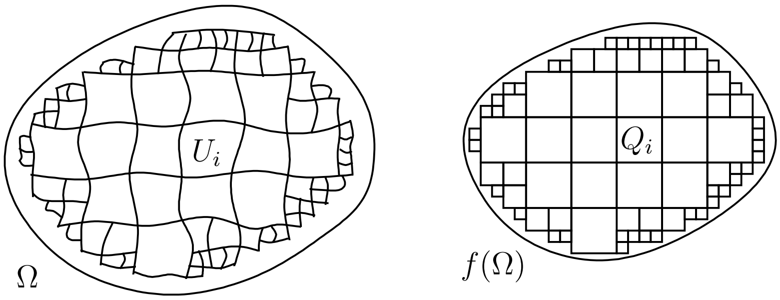

FIGURE 1. Sketch of the grids in Ω and $f(\Omega)$.

and, hence, $\mathcal{H}^1(K) = 0$, which is a contradiction.

4. Diffeomorphic approximation

By the existence property of the degree we have that

$$\mathrm{im}_{\mathrm{T}}(f,\Omega) \subset f(\Omega).$$

Thus,

$$f^{-1}(\mathrm{im}_{\mathrm{T}}(f,\Omega)) \subset f^{-1}(f(\Omega)) \subset \overline{\Omega}.$$

The assumption $f(\Omega) \cap f(\partial\Omega) = \emptyset$ yields $f^{-1}(f(\Omega)) \subset \Omega$. Condition INV implies, due to [22, Corollary 7.5], that $f(\Omega) \subset \mathrm{im}_{\mathrm{T}}(f,\Omega) \cup f(\partial\Omega)$. All in all, we have that

$$f(\Omega) = \mathrm{im}_{\mathrm{T}}(f,\Omega) \quad \text{and} \quad f^{-1}(f(\Omega)) = \Omega.$$

In particular, $f(\Omega)$ is open.

For a fixed $\delta > 0$ we can find a family $\{Q_i\}_{i\in\mathbb{N}}$ of closed rectangles with pairwise disjoint interiors such that

$$f(\Omega) = \bigcup_{i=1}^{\infty} Q_i,$$

$\operatorname{diam} Q_i \leq \delta$ for all $i \in \mathbb{N}$ and

$$\operatorname{diam} Q \to 0 \text{ as } \operatorname{dist}(Q, \partial f(\Omega)) \to 0. \tag{4.1}$$

As $\mathcal{H}^1(S) = 0$ we can additionally require that $S \cap \partial Q_i = \emptyset$ for all $i \in \mathbb{N}$. Indeed, we start with a shifted $\frac{\delta}{2}$-grid on the plane that does not intersect S. We do sucessive refinements of the rectangles on the grid close to the boundary such that none of the rectangle boundaries intersect S. See Figure 1.

Let us fix $i \in \mathbb{N}$. As $S \cap \partial Q_i = \emptyset$ and f^{-1} is well defined in $f(\Omega) \setminus S$, we infer that f^{-1} is one-to-one on ∂Q_i. As $f^{-1}(\partial Q_i)$ is compact and f is continuous, we obtain that f is a homeomorphism from $f^{-1}(\partial Q_i)$ to ∂Q_i. Thus, $f^{-1}(\partial Q_i)$ is a Jordan curve, so it divides $\mathbb{R}^2$ into two components; we denote by U_i the bounded one, which is a Jordan domain. This allows us to use there the p-harmonic replacement described in Proposition 2.1 and Theorem 2.2, so we obtain a diffeomorphism $h_i : U_i \overset{\text{onto}}{\to} \operatorname{int} Q_i$ such that $h_i = f$ on ∂U_i.

Let us consider the mapping defined as

$$f_\delta(x) = h_i(x) \text{ for } x \in \overline{U_i}.$$

The mapping f_δ is well defined because $h_i = f$ on ∂U_i for all $i \in \mathbb{N}$. Moreover, it is continuous since the family $\{\overline{U}_i\}_{i\in\mathbb{N}}$ is locally finite. It is injective as the interiors of the Q_i's are disjoint. It is, in turn, a homeomorphism from Ω onto $f(\Omega)$ because each h_i is a homeomorphism from $\overline{U}_i$ onto Q_i. The construction shows that

$$\|f_\delta - f\|_{L^\infty(U_i,\mathbb{R}^2)} \leq \operatorname{diam} Q_i. \tag{4.2}$$

In particular, as $\operatorname{diam} Q_i \leq \delta$ for each i it follows that

$$\|f_\delta - f\|_{L^\infty(\Omega,\mathbb{R}^2)} \leq \delta. \tag{4.3}$$

Furthermore, properties (4.1) and (4.2) imply that $f_\delta - f \in W_0^{1,p}(\Omega, \mathbb{R}^2)$ (see [26, Theorem 2.2]).

Let us denote the coordinate functions of f_δ as $(f_\delta)_1$ and $(f_\delta)_2$. By Proposition 2.1 we know that

$$\|Df_\delta\|^p_{L^p} := \int_\Omega \big(|D(f_\delta)_1|^p + |D(f_\delta)_2|^p\big) \leq \|Df\|^p_{L^p}. \tag{4.4}$$

It follows that $f_{\frac{1}{i}}$ form a bounded sequence in $W^{1,p}$ and we can select a subsequence (denoted as f_i) that converges weakly. By (4.3) we know that $f_i \to f$ in L^∞ and hence $f_i \overset{W^{1,p}}{\rightharpoonup} f$. By the lower semicontinuity of the norm we obtain

$$\|Df\|_{L^p} \leq \liminf_{i\to\infty} \|Df_i\|_{L^p}$$

and in view of (4.4) we see that in fact $\lim_{i\to\infty} \|Df_i\|_{L^p} = \|Df\|_{L^p}$. As L^p is uniformly convex (since $1 < p < \infty$) and $f_i \overset{W^{1,p}}{\rightharpoonup} f$ we obtain that $f_i \overset{W^{1,p}}{\to} f$ (see, e.g., [5, Proposition III.30]).

Given $\varepsilon > 0$ we can thus find a Sobolev homeomorphism $f_i : \Omega \overset{\text{onto}}{\to} f(\Omega)$ such that $\|f_i - f\|_{L^\infty} < \varepsilon$, $\|f_i - f\|_{W^{1,p}} < \varepsilon$ and $f_i - f \in W_0^{1,p}(\Omega, f(\Omega))$. Using the result of [15] we can now find a diffeomorphism $g : \Omega \overset{\text{onto}}{\to} f(\Omega)$ such that $g \in W^{1,p}(\Omega, f(\Omega))$, $\|f_i - g\|_{L^\infty} < \varepsilon$, $\|f_i - g\|_{W^{1,p}} < \varepsilon$ and $f_i - g \in W_0^{1,p}(\Omega, f(\Omega))$, which concludes our proof.

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Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic

E-mail address: hencl@karlin.mff.cuni.cz

University Autónoma of Madrid, Faculty of Sciences, Department of Mathematics, E-28049 Madrid, Spain

E-mail address: carlos.mora@uam.es