L^2-Betti numbers and their analogues in positive characteristic

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Abstract

In this article, we give a survey of results on L^2-Betti numbers and their analogues in positive characteristic. The main emphasis is made on the Lück approximation conjecture and the strong Atiyah conjecture.

Contents

1 Introduction

Let G be a group and let K be a field. For every matrix $A \in \mathrm{Mat}_{n\times m}(K[G])$ and every normal subgroup N of G of finite index let us define

$$\begin{array}{rccc} \phi^A_{G/N}: & K[G/N]^n & \to & K[G/N]^m \\ & (x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_n)A \end{array}.$$

This is a K-linear map between two finite-dimensional K-vector spaces. Thus, we can define

$$\mathrm{rk}_{G/N}(A) = \frac{\dim_K \mathrm{Im}\, \phi^A_{G/N}}{|G:N|} = n - \frac{\dim_K \ker \phi^A_{G/N}}{|G:N|}. \tag{1}$$

Now, let $G > G_1 > G_2 > \dots$ be a descending chain of subgroups such that G_i is normal in G, the index $|G:G_i|$ is finite and $\cap_{i\ge 1} G_i = \{1\}$. For a given matrix A over $K[G]$, we want to study the sequence $\{\mathrm{rk}_{G/G_i}(A)\}_{i\ge 1}$. Concretely, we would like to answer the following questions.

Question 1.1 Let us assume the previous notation.

1. Does the sequence $\{\mathrm{rk}_{G/G_i}(A)\}_{i\ge 1}$ converge?
2. Assume that the limit $\lim\limits_{i\to\infty} \mathrm{rk}_{G/G_i}(A)$ exists. Does it depend on the chain $G > G_1 > G_2 > \dots$?
3. Assume that the limit $\lim\limits_{i\to\infty} \mathrm{rk}_{G/G_i}(A)$ exists. What are the possible values of the limit $\lim\limits_{i\to\infty} \mathrm{rk}_{G/G_i}(A)$?

These questions arise in very different situations. We will present several examples in Section 13. Let us formulate a conjecture which answers all these three questions.

Conjecture 1.2 Let us assume the previous notation. Then the following holds.

(1) The sequence $\{\mathrm{rk}_{G/G_i}(A)\}_{i\ge 1}$ converges.

(2) The limit $\lim\limits_{i\to\infty} \mathrm{rk}_{G/G_i}(A)$ does not depend on the chain $G > G_1 > G_2 > \dots$.

(3) Assume that there exists an upper bound for the orders of finite subgroups of G and let $\mathrm{lcm}(G)$ be the least common multiple of these orders. Then

$$\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}.$$

Informally, the first and second part of the conjecture is called the Lück approximation conjecture and the third part is called the strong Atiyah conjecture. In Section 2, we will introduce the original Lück approximation and strong Atiyah conjectures. They are formulated only for fields K which are subfields of the field $\mathbb{C}$ of complex numbers. The numbers $\mathrm{rk}_G(A)$ which will appear in these conjectures are generalizations of the L^2-Betti numbers invented by M. Atiyah. If K is of characteristic $p > 0$, then $\lim\limits_{i\to\infty} \mathrm{rk}_{G/G_i}(A)$ is what we call an analogue of an L^2-Betti number in positive characteristic.

If K is of characteristic 0, the parts (1) and (2) of Conjecture 1.2 are known to be true and the part (3) holds for many families of groups which include the groups from the class $\mathcal{D}$, Artin's braid groups, virtually special groups and torsion-free p-adic compact groups. If K is of positive characteristic, the parts (1) and (2) are only known when G is amenable and the part (3) when G is elementary amenable.

If the reader sees Conjecture 1.2 for the first time he or she might wonder what makes the cases of characteristic 0 and positive characteristic so different. A quick answer is that in characteristic 0 we can use different techniques from the theory of operator algebras, but we do not have any analogue of them in positive characteristic. Nevertheless, in this survey we will try to give a uniform treatment of both cases using the notion of Sylvester matrix rank function. This is the main difference of our exposition of this subject from the previous ones.

Our first motivation is to explain the main ideas behind the proofs of positive results concerning Conjecture 1.2 and the related conjectures. We will present the complete proofs of several results. Some of them are not new but they are formulated in the literature differently, so we think it will be convenient to include their proofs. In most cases we will give only a sketch of the proofs, providing the references where the complete proofs can be found. Another motivation is to collect together the main open problems in the area. We hope that this will stimulate further research in this subject.

The article is organized as follows. In Section 2 we introduce L^2-Betti numbers of groups and formulate the strong Atiyah conjecture and different variations of the Lück approximation conjecture. In Section 3 we recall basic facts about von Neumann regular and $*$-regular rings. In Section 4 we explain the notion or epic homomorphism and present the Cohn theory of epic division R-algebras. Section 5 is devoted to the theory of Sylvester matrix rank and Sylvester module rank functions. These concepts unify the notion of L^2-Betti numbers with their analogues in positive characteristic. Until now this subject has been presented in the literature only partially. Therefore, we try to describe a complete picture. We formulate several exciting questions about Sylvester rank functions. Some of them are not related to L^2-Betti numbers, but we still believe that they are of big interest. In Section 6 we give an algebraic reformulation of the conjectures described in Section 2. This algebraic point of view allows to use the techniques introduced in Sections 3, 4 and 5 in order to attack the conjectures formulated in Section 1 and 2. In Section 7 we prove the parts (1) and (2) and in Section 9 the part (3) of Conjecture 2.4 (this is a strong version of Conjecture 1.2) over an arbitrary field for amenable groups. In Section 8 we discuss the notions of natural extensions of Sylvester rank functions. They play an important role in the proofs of many results of this survey. In Section 10 we explain the proof of the general Lück approximation conjecture over the field of complex numbers for sofic groups. Section 11 is devoted to the Lück approximation and strong Atiyah conjecture for completed group algebras of virtually pro-p groups. We formulate questions similar to the ones from Section 1. Section 12 describes the known positive results on the strong Atiyah conjecture. Finally, in Section 13 we present several applications of Sylvester matrix rank functions and, in particular, L^2-Betti numbers in other parts of mathematics. The list of applications is far from being complete, and represents mathematical interests of the author of this survey.

There are many good sources to learn about L^2-invariants and their approximations, mostly due to W. Lück. First, of course, one should mention his book [78]. We also highly recommend a recent Lück's survey [81]. Other useful sources are the Ph.D. thesis of H. Reich [98], expository papers by P. Pansu [94] and B. Eckmann [29], another survey by W. Lück [79] and two recent lecture notes, one by H. Kammeyer [59] and another by S. Kionke [62].

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General conventions and notations

In this paper all rings and homomorphisms are unital. The letter K is reserved for a field and by an algebra we will mean always a K-algebra.

If R is a ring, an R-module will usually mean left R-module. The category of R-modules is denoted by R-Mod. $R[x]$ is the ring of polynomials over R and $R[x^{\pm 1}]$ is the ring of Laurent polynomials.

A $*$-ring is a ring R with a map $* : R \to R$ that is an **involution** (i. e. $(x^*)^* = x,\ (x+y)^* = x^* + y^*,\ (xy)^* = y^*x^*\ (x, y \in R)$). If K is a $*$-ring, then a $*$-algebra is an algebra with an involution $*$ satisfying $(\lambda x)^* = \lambda^* x^*\ (\lambda \in K,\ x \in R)$.

An element of a $*$-ring e is called a **projection** if e is an **idempotent** ($e^2 = e$) and e is **self-adjoint** ($e^* = e$).

If $n \geq 1$ we denote by I_n the n by n identity matrix. For matrices A and B, $A \oplus B$ denotes the direct sum of A and B:

$$A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

For a group G, $d(G)$ denotes the minimal number of generators of G. We denote by $\mathcal{F}(G)$ the set of finite subgroups of G. If there is an upper bound on the orders of finite subgroups of G, we denote by $\mathrm{lcm}(G)$ the least common multiple of these orders. We will write $\mathrm{lcm}(G) = \infty$ if there is no such bound.

For a countable set X, $l^2(X)$ will denote the Hilbert space with Hilbert basis the elements of X; thus $l^2(X)$ consists of all square summable formal sums $\sum_{x \in X} a_x x$

with $a_x \in \mathbb{C}$ and the inner product is

$$\langle \sum_{x\in X} a_x x, \sum_{x\in X} b_x x\rangle = \sum_{x\in X} a_x \overline{b_x}.$$

2 L^2-Betti numbers and generalizations of Conjecture 1.2

A countable group G acts by left and right multiplication on $l^2(G)$. The right action of G on $l^2(G)$ extends to an action of $\mathbb{C}[G]$ on $l^2(G)$ and so we obtain that the group algebra $\mathbb{C}[G]$ acts faithfully as bounded linear operators on $l^2(G)$. In what follows we will simply consider $\mathbb{C}[G]$ as a subalgebra of $\mathcal{B}(l^2(G))$, the algebra of bounded linear operators on $l^2(G)$.

A finitely generated **Hilbert** G-module is a closed subspace $V \le (l^2(G))^n$, invariant by the left action of G. A **morphism** between two finitely generated Hilbert G-modules U and V is a bounded G-equivariant map $\alpha : U \to V$.

Let $V \le (l^2(G))^n$ be a f.g. Hilbert G-module and $\mathrm{proj}_V : (l^2(G))^n \to (l^2(G))^n$ the orthogonal projection onto V. We put

$$\dim_G V := \mathrm{Tr}_G(\mathrm{proj}_V) := \sum_{i=1}^n \langle \mathrm{proj}_V \mathbf{1}_i, \mathbf{1}_i\rangle_{(l^2(G))^n},$$

where $\mathbf{1}_i$ is the element of $(l^2(G))^n$ having 1 in the ith entry and 0 in the rest of the entries. The number $\dim_G V$ is the **von Neumann dimension** of V. It does not depend on the embedding of V into $l^2(G)^n$. The reader can consult [78] where other properties of $\dim_G V$ are described.

Let $A \in \mathrm{Mat}_{n\times m}(\mathbb{C}[G])$ be a matrix over $\mathbb{C}[G]$. The action of A by right multiplication on $l^2(G)^n$ induces a bounded linear operator $\phi_G^A : (l^2(G))^n \to (l^2(G))^m$. Let us define

$$\mathrm{rk}_G(A) = \dim_G \overline{\mathrm{Im}\,\phi_G^A} = n - \dim_G \ker \phi_G^A.$$

Observe that this notation is compatible with the formula (1), because if G is finite, then $\mathrm{rk}_G = \frac{\mathrm{rk}_{\mathbb{C}}}{|G|}$.

If G is a quotient of a group F and $A \in \mathrm{Mat}_{n\times m}(\mathbb{C}[F])$ is a matrix over $\mathbb{C}[F]$, we denote by $\bar{A}$ the image of A in $\mathrm{Mat}_{n\times m}(\mathbb{C}[G])$. Abusing the notation, we will write ϕ_G^A for $\phi_G^{\bar{A}}$ and $\mathrm{rk}_G(A)$ for $\mathrm{rk}_G(\bar{A})$.

If G is not a countable group then rk_G is also well defined. Take a matrix A over $\mathbb{C}[G]$. Then the group elements that appear in A are contained in a finitely generated subgroup H of G. We will put $\mathrm{rk}_G(A) = \mathrm{rk}_H(A)$. One easily checks that the value $\mathrm{rk}_H(A)$ does not depend on the subgroup H.

In [9] M. F. Atiyah introduced for a closed Riemannian manifold (M, g) with universal covering $\tilde{M}$ the analytic L^2-Betti numbers $b_p^{(2)}(M, g)$ which measure the size of the space of harmonic square-integrable p-forms on $\tilde{M}$. J. Dodziuk [24] extended the notion of L^2-Betti numbers to the more general context of free cocompact actions of discrete groups on CW-complexes. In particular, he also showed that the analytic L^2-Betti numbers do not depend on the metric.

For a given subfield K of $\mathbb{C}$ we denote by $\mathcal{C}_K(G)$ the set of possible values $\mathrm{rk}_G(A)$ where A is a matrix over $K[G]$ and by $\mathcal{A}_K(G)$ the additive group generated by $\mathcal{C}_K(G)$. Over time it has been realized (see [29, Proposition 3.10.1]) that L^2-Betti numbers, arising from a given group G acting freely and cocompactly on CW-complexes, form a set that can be defined purely in terms of G, without mentioning CW-complexes. In our notation it is the set $\mathcal{C}_{\mathbb{Q}}(G)$. In this survey we will consider not only $\mathcal{C}_{\mathbb{Q}}(G)$ but also the sets $\mathcal{C}_K(G)$ where K is an arbitrary subfield of $\mathbb{C}$.

2.1 Atiyah's question and the general Atiyah problem

In [9, page 72] M. F. Atiyah asked whether L^2-Betti numbers of a closed manifold can be irrational. We reformulate this question as the following problem and we refer to it as **the general Atiyah problem** for G.

Problem 2.1 For a given group G and a given subfield K of $\mathbb{C}$ determine the group $\mathcal{A}_K(G)$.

Before the work of R. Grigorchuk and A. Zuk [50], it had been conjectured that

$$\mathcal{A}_{\mathbb{Q}}(G) = \langle \frac{1}{|H|} : H \leq G \rangle.$$

However, in [50] the authors showed that if $G = C_2 \wr \mathbb{Z}$ is the lamplighter group, then $1/3 \in \mathcal{A}_{\mathbb{Q}}(G)$. Observe that the finite subgroups of the lamplighter group have orders which are powers of 2. This result was used in [48] to produce a closed Riemannian manifold (M, g) of dimension 7 with $\pi_1(M)$ having only finite subgroups of order a power of 2 and such that $b_3^{(2)}(M, g) = \frac{1}{3}$.

Shortly afterwards W. Dicks and T. Schick described in [22] an element T from the group ring of $\mathbb{Z}[G]$ where $G = (C_2 \wr \mathbb{Z}) \times (C_2 \wr \mathbb{Z})$ such that $\mathrm{rk}_G(T)$ looked like an irrational number. The question of irrationality of that specific number remains open. This was the first evidence that the question of Atiyah has an affirmative answer. It was T. Austin [10] who first proved the existence of a group G with an irrational element in $\mathcal{C}_{\mathbb{Q}}(G)$. His construction was not explicit. Concrete examples appear in [42, 67, 97, 44]. These examples also leaded to constructions of closed Riemannian manifolds with irrational L^2-Betti numbers confirming the prediction of M. Atiyah. Moreover, in [42] Ł. Grabowski showed that any non-negative real number belongs to $\mathcal{C}_{\mathbb{Q}}(G)$ for some elementary amenable group G and the set of L^2-Betti numbers arising from finitely presented groups contains the set of all numbers with computable binary expansions.

All the previous examples involve groups having finite subgroups of unbounded order. This suggests that we have to consider the general Atiyah problem for groups with bounded orders of finite subgroups and, in particular, for torsion-free groups.

2.2 The strong Atiyah conjecture

Now let us state a conjecture that got the name of the strong Atiyah conjecture [78].

Conjecture 2.2 (**The strong Atiyah conjecture over K for a group G**) Let G be a group and let K be a subfield of $\mathbb{C}$. Assume that $\operatorname{lcm}(G) < \infty$. Then

$$\mathcal{A}_K(G) = \frac{1}{\operatorname{lcm}(G)}\mathbb{Z} = \langle \frac{1}{|H|} : H \leq G \rangle.$$

There is a considerable body of work to establish the strong Atiyah conjecture for suitable classes of groups and fields. We will present these results in Section 12. At this moment the conjecture is known over $\mathbb{C}$ for many families of groups which include the groups from the class $\mathcal{D}$, Artin's braid groups, virtually special groups and torsion-free p-adic compact groups.

2.3 The Lück approximation conjecture

Now we introduce the Lück approximation conjecture. It arised from a question of M. Gromov (which was solved by W. Lück in [76]) of whether L^2-Betti numbers of a compact Riemannian manifold can be approximated by ordinary normalized Betti numbers of finite covers of the manifold.

Conjecture 2.3 (**The Lück approximation conjecture over K for a group G**) Let K be a subfield of $\mathbb{C}$, F a finitely generated free group and $F > N_1 > N_2 > \ldots$ a chain of normal subgroups of F with intersection $N = \cap N_i$. Put $G_i = F/N_i$ and $G = F/N$. Then for every $A \in \operatorname{Mat}_{n\times m}(K[F])$,

$$\lim_{k\to\infty} \operatorname{rk}_{G_k}(A) = \operatorname{rk}_G(A).$$

This conjecture was formulated by W. Lück. When K is of characteristic 0, Conjecture 2.3 implies the first and the second part of Conjecture 1.2 and Conjecture 2.2 and Conjecture 2.3 together imply the third part of Conjecture 1.2.

2.4 The sofic Lück approximation conjecture

Let F be a free finitely generated group and assume that it is freely generated by a set S. Recall that an element w of F has **length** n if w can be expressed as a product of n elements from $S \cup S^{-1}$ and n is the smallest number with this property. $B_k(1_F)$ will denote the set of elements of F of length at most k.

Let N be a normal subgroup of F. We put $G = F/N$. We say that G is **sofic** if there is a family $\{X_k\}_{k\in\mathbb{N}}$ of finite F-sets (F acts on the right) such that if we put

$$T_{k,s} = \{x \in X_k : \ x = x\cdot w \text{ if } w \in B_s(1_F)\cap N, \text{ and } \ x \neq x\cdot w \text{ if } w \in B_s(1_F)\setminus N\},$$

then for every s,

$$\lim_{k\to\infty} \frac{|T_{k,s}|}{|X_k|} = 1.$$

The family of F-sets $\{X_k\}$ is called a **sofic approximation** of G.

This is one of many equivalent definitions of soficity for a finitely generated group; we have borrowed this one from [109, Proposition 1.4].

This definition has the following geometric meaning. The action of F on X_k converts X_k in an $S^{\pm 1}$-labeled graph. Let $T'_{k,s}$ be the set of vertices x of X_k such that the s-ball $B_s(x)$ in X_k and the s-ball $B_s(1_G)$ in G are isomorphic as $S^{\pm 1}$-labeled graphs. It is clear that

$$T'_{k,s} \subseteq T_{k,s} \subseteq T'_{k,2s}.$$

Thus, the soficity condition says that for every s most of the vertices of X_k are in $T'_{k,s}$ when k tends to infinity.

For an arbitrary group G we say that G is **sofic** if every finitely generated subgroup of G is sofic. Amenable groups and residually finite groups are sofic. It is important to note that no nonsofic group is known at this moment. On the other hand, all the results presented in this survey are about sofic groups.

Now, let us generalize slightly the notation introduced in Section 1. Let F be a group acting (on the right) on a finite set X and let K be a field. For every matrix $A \in \mathrm{Mat}_{n\times m}(K[F])$ let us define

$$\begin{array}{rccc} \phi_X^A: & K[X]^n & \to & K[X]^m \\ & (x_1,\dots,x_n) & \mapsto & (x_1,\dots,x_n)A \end{array}.$$

This is a K-linear map between two finite-dimensional K-vector spaces, and so, we can define

$$\mathrm{rk}_X(A) = \frac{\dim_K \mathrm{Im}\,\phi_X^A}{|X|} = n - \frac{\dim_K \ker \phi_X^A}{|X|}. \tag{2}$$

Conjecture 2.4 (**The sofic Lück approximation conjecture over K for a group G**) Let $\{X_k\}$ be a sofic approximation of $G = F/N$. Then

(1) for every $A \in \mathrm{Mat}_{n\times m}(K[F])$, there exists the limit $\lim\limits_{k\to\infty} \mathrm{rk}_{X_k}(A)$;

(2) the limit does not depend on the sofic approximation $\{X_i\}$;

(3) If K is a subfield of $\mathbb{C}$, then $\lim\limits_{k\to\infty} \mathrm{rk}_{X_k}(A) = \mathrm{rk}_G(A)$.

This conjecture generalizes the parts (1) and (2) of Conjecture 1.2. Conjecture 2.4 holds when K is of characteristic 0. When K is of positive characteristic, the first and second parts of Conjecture 2.4 hold when G is amenable. These results will be explained in Sections 7 and 10.

2.5 The Lück approximation in the space of marked groups.

Let F be a free group freely generated by a finite set S. The **space of marked groups** $\mathrm{MG}(F,S)$ can be identified with the set of normal subgroups of F with the metric $d(N_1,N_2) = e^{-n}$ where n is the largest integer such that the balls of radius n in the Cayley graphs of F/N_1 and F/N_2 with respect to the generators S are simplicially isomorphic (with respect to an isomorphism respecting the labelings). In this setting the approximation conjecture is stated in the following way.

Conjecture 2.5 (**The Lück approximation conjecture in the space of marked groups over K for a group G**) Let K be a subfield of $\mathbb{C}$. Let $\{N_k \in \mathrm{MG}(F,S)\}$

converge to $N \in \mathrm{MG}(F,S)$. Put $G = F/N$ and $G_k = F/N_k$. Then for every $A \in \mathrm{Mat}_{n\times m}(K[F])$,

$$\lim_{k\to\infty} \mathrm{rk}_{G_k}(A) = \mathrm{rk}_G(A).$$

Clearly Conjecture 2.5 is a strong version of Conjecture 2.3. It is known in the case where the groups G_i are sofic. This will be a part of a more general conjecture which we discuss in the next subsection.

2.6 The general Lück approximation conjecture

In this subsection we will introduce a new type of approximation that unify together the sofic approximation and the approximation in the space of marked groups. Then we will formulate the Lück approximation conjecture for this general situation.

As before, let F be a finitely generated free group, freely generated by a finite set S, N a normal subgroup of F and $G = F/N$. Let $\{H_k\}_{k\in\mathbb{N}}$ be a family of groups and X_k an (H_k, F)-set (i.e. H_k acts on the left, F acts on the right and these two actions commute) such that H_k acts freely on X_k and $H_k\backslash X_k$ is finite. We define

$$T_{k,s} = \{x \in X_k :\ x = x\cdot w \text{ if } w \in B_s(1_F)\cap N, \text{ and } \ x \neq x\cdot w \text{ if } w \in B_s(1_F)\setminus N\}.$$

Then we say that $\{X_k\}$ **approximates** G if for every s,

$$\lim_{k\to\infty} \frac{|H_k\backslash T_{k,s}|}{|H_k\backslash X_k|} = 1.$$

The sofic approximation is a particular case of the general approximation and corresponds to the case when the groups H_k are trivial. The approximation in the space of marked groups arises from the general approximation in the case when H_k and F act transitively on X_k for every k.

As in the case of sofic approximation, the general approximation has a geometric interpretation. We see X_k as an $S^{\pm 1}$-labeled graph. Since the action of H_k and F commutes, the elements of H_k act on X_k as $S^{\pm 1}$-labeled graph isomorphisms. Therefore, for every $s \in \mathbb{N}$ the balls of radius s centered in the vertices of an H_k-orbit in X_k are isomorphic. There are only finitely many H_k-orbits in X_k and the approximation condition says that when k tends to infinity, for almost all of them, the ball of radius s centered in a point of the orbit is isomorphic to $B_s(1_G)$.

Now, we can generalize the previous notation in the following way. Let $A \in \mathrm{Mat}_{n\times m}(\mathbb{C}[F])$ be a matrix over $\mathbb{C}[F]$. Let H be a group and let X be an (H, F)-set such that H acts freely on X and $H\backslash X$ is finite. By multiplication on the right side, A induces a linear operator $\phi_X^A : (l^2(X))^n \to (l^2(X))^m$. We put

$$\mathrm{rk}_X(A) = \frac{\dim_H \overline{\mathrm{Im}\,\phi_X^A}}{|H\backslash X|} = n - \frac{\dim_H \ker \phi_X^A}{|H\backslash X|}.$$

Conjecture 2.6 (**The general Lück approximation conjecture over K for a group G**) Let K be a subfield of $\mathbb{C}$, F a finitely generated free group and N a normal subgroup of F. For each natural number k, let X_k be an (H_k, F)-set such that H_k is a group that acts freely on X_k and $H_k\backslash X_k$ is finite. Assume that $\{X_k\}$ approximates $G = F/N$. Then for every $A \in \mathrm{Mat}_{n\times m}(K[F])$,

$$\lim_{k\to\infty} \mathrm{rk}_{X_k}(A) = \mathrm{rk}_G(A).$$

This conjecture generalizes all the previous variations of the Lück approximation conjecture over fields of characteristic 0. We will explain in Section 10 the proof of this conjecture over $\mathbb{C}$ in the case where all groups H_k are sofic.

If G is an arbitrary group, we say that G satisfies the general Lück approximation conjecture over K if all its finitely generated subgroups do.

3 Von Neumann regular and $*$-regular rings

3.1 Von Neumann regular rings

An element x of a ring R is called **von Neumann regular** if there exists $y \in R$ satisfying $xyx = x$. A ring $\mathcal{U}$ is called **von Neumann regular** if all the elements of $\mathcal{U}$ are von Neumann regular. In the following proposition we collect the properties of von Neumann regular rings that we will need later.

Proposition 3.1 *[46] Let $\mathcal{U}$ be a von Neumann regular ring. Then the following statements hold:*

1. *every finitely generated left ideal of $\mathcal{U}$ is generated by an idempotent;*
2. *every finitely generated left submodule of a projective module P of $\mathcal{U}$ is a direct summand of P (and, in particular, it is projective);*
3. *every finitely generated left projective module of $\mathcal{U}$ is a direct sum of left cyclic ideals of $\mathcal{U}$.*

3.2 The ring of unbounded affiliated operators of a group

The ring of unbounded affiliated operators $\mathcal{U}(G)$ of a group G is one of the main examples of a von Neumann regular ring that appear in this survey. The Ph.D thesis of H. Reich [98] is a good source to learn basic facts about the ring $\mathcal{U}(G)$. We briefly define it in this subsection and also introduce additional notions that will motivate further definitions.

Let G be a countable group. The **group von Neumann algebra** $\mathcal{N}(G)$ of G is the algebra of G-equivariant bounded operators on $l^2(G)$:

$$\mathcal{N}(G) = \{\phi \in \mathcal{B}(l^2(G)) :\ \phi(gv) = g\phi(v) \text{ for all } g \in G,\ v \in l^2(G)\}.$$

It can be defined also as the weak closure of $\mathbb{C}[G]$ in $\mathcal{B}(l^2(G))$ or, algebraically, as the second centralizer of $\mathbb{C}[G]$ in $\mathcal{B}(l^2(G))$.

The ring $\mathcal{N}(G)$ satisfies the left Ore condition (a result proved by S. K. Berberian in [13]). We recall this notion in Subsection 4.1. The left classical ring of fractions $Q_l(\mathcal{N}(G))$ is denoted by $\mathcal{U}(G)$. The ring $\mathcal{U}(G)$ can be also described as the ring of densely defined (unbounded) operators which commute with the left action of

G. Therefore, $\mathcal{U}(G)$ is called the **ring of unbounded affiliated operators** of G. The ring $\mathcal{U}(G)$ is a $*$-regular ring. We will consider such rings in more detail in Subsection 3.4.

We can define a rank function rk_G on $\mathcal{U}(G)$ in the following way

$$\mathrm{rk}_G(s^{-1}r) = \mathrm{rk}_G(r) = \dim_G(\overline{l^2(G)r}) = \langle \mathrm{proj}_{\overline{l^2(G)r}} \mathbf{1}, \mathbf{1}\rangle_{l^2(G)}, \tag{3}$$

where $r \in \mathcal{N}(G)$ and $s \in \mathcal{N}(G)$ is a non-zero-divisor in $\mathcal{N}(G)$. Note that if $u \in \mathcal{U}(G)$, then

$$\mathrm{rk}_G(u) = 1 \text{ if and only if } u \text{ is invertible in } \mathcal{U}(G). \tag{4}$$

The function rk_G can be extended to all matrices over $\mathcal{U}(G)$ and it is an example of a faithful Sylvester matrix rank function on a $*$-regular ring. We will consider the Sylvester rank functions in more detail in Section 5. The Sylvester matrix rank function rk_G induces a Sylvester module rank function $\dim_G$ on finitely presented left modules of $\mathcal{U}(G)$ (see Subsection 5.3 for more details) that satisfies

$$\dim_G(\mathcal{U}(G)u) = \mathrm{rk}_G(u),\ u \in \mathcal{U}(G).$$

3.3 Von Neumann regular elements in a proper $*$-ring

Let R be a $*$-ring. The involution $*$ is called **proper** if $x^*x = 0$ implies $x = 0$ and it is called n-**positive definite** if $\sum_{i=1}^n x_i^*x_i = 0$ implies $x_1 = \cdots = x_n = 0$. Thus, the involution is proper if and only if it is 1-positive definite. If the involution is n-positive definite for all n, then we say that it is **positive definite**. We say that a $*$-ring is **proper** if its involution is proper.

In general if x is a von Neumann regular element there are several elements y satisfying $xyx = x$. However, if R is a proper $*$-ring there is a canonical one. In the following proposition we collect the main properties of regular elements in a proper $*$-ring.

Proposition 3.2 *([46],[56]) Let R be a proper $*$-ring and let $x \in R$. Assume that x^*x and xx^* are von Neumann regular elements. Then the following holds.*

1. *$Rx = Rx^*x$.*
2. *x and x^* are von Neumann regular.*
3. *There exists a unique projection e in R such that $Re = Rx$ and there exists a unique projection f such that $fR = xR$ (we put $e = \mathrm{RP}(x)$ and $f = \mathrm{LP}(x)$).*
4. *There exists a unique $y \in eRf$ such that $yx = e$ and $xy = f$ (we put $x^{[-1]} = y$ and call it **the relative inverse of** x).*
5. *$\mathrm{RP}(x) = \mathrm{RP}(x^*x) = \mathrm{LP}(x)$ and $(x^*)^{[-1]} = (x^{[-1]})^*$.*
6. *$(x^*x)^{[-1]} = x^{[-1]}(x^*)^{[-1]}$ and $x^{[-1]} = (x^*x)^{[-1]}x^*$.*
7. *If x is self-adjoint, then x commutes with $x^{[-1]}$.*

3.4 Von Neumann $*$-regular rings

A $*$-ring $\mathcal{U}$ is called **von Neumann $*$-regular** (or simply **$*$-regular**) if it is von Neumann regular and its involution is proper. The ring $\mathrm{Mat}_n(\mathbb{C})$ is $*$-regular. The ring $\mathbb{C}[G]$ is $*$-regular if and only if G is locally finite. However, we can embed $\mathbb{C}[G]$ in the $*$-regular ring $\mathcal{U}(G)$ for an arbitrary group G.

A direct product of $*$-regular rings is again $*$-regular. If $\mathcal{U}$ is a $*$-regular ring, then $\mathrm{Mat}_n(\mathcal{U})$ is again a $*$-ring: if $M = (m_{ij})$ then $M^* = (n_{ij})$ with $n_{ij} = (m_{ji}^*)$. Also $\mathrm{Mat}_n(\mathcal{U})$ is von Neumann regular. However, in general $*$ is not proper in $\mathrm{Mat}_n(\mathcal{U})$. We say that $\mathcal{U}$ is a **positive definite** $*$-regular if $\mathrm{Mat}_n(\mathcal{U})$ is $*$-regular for every $n \in \mathbb{N}$. It is equivalent to the condition that $*$ is positive definite. For example, $\mathrm{Mat}_n(\mathbb{C})$ and $\mathcal{U}(G)$ are positive definite $*$-regular rings.

Although in the definition of a $*$-regular ring the properties to be von Neumann regular and to be proper do not interact, using them together we obtain many interesting consequences. For example, if I is an ideal of a $*$-regular ring $\mathcal{U}$, then I is automatically $*$-invariant and moreover $*$ induces a proper involution on $\mathcal{U}/I$.

The following proposition explains how to construct the minimal $*$-regular subring containing a given $*$-subring. This was proved first for positive definite $*$-regular rings by P. Linnell and T. Schick in [72] and by P. Ara and K. Goodearl in the form that we present here in [6, Proposition 6.2].

Let R be a $*$-subring of a $*$-regular ring $\mathcal{U}$. We denote by $\mathcal{R}_1(R,\mathcal{U})$ the subring of $\mathcal{U}$ generated by R and all the relative inverses of all the elements $x \in R$. Clearly $\mathcal{R}_1(R,\mathcal{U})$ is again a $*$-subring of $\mathcal{U}$. We put

$$\mathcal{R}_{n+1}(R,\mathcal{U}) = \mathcal{R}_1(\mathcal{R}_n(R,\mathcal{U}),\mathcal{U}).$$

Proposition 3.3 *[6, Proposition 6.2] Let $\mathcal{U}$ be a $*$-regular ring and let R be a $*$-subring of $\mathcal{U}$. Then there is a smallest $*$-regular subring $\mathcal{R}(R,\mathcal{U})$ of $\mathcal{U}$ containing R. Moreover,*

$$\mathcal{R}(R,\mathcal{U}) = \cup_{i=1}^{\infty}\mathcal{R}_i(R,\mathcal{U}).$$

The subring $\mathcal{R}(R,\mathcal{U})$ is called the **$*$-regular closure** of R in $\mathcal{U}$. It was observed in [56] that, in fact, $\mathcal{R}_1(R,\mathcal{U})$ can be also defined as the subring of $\mathcal{U}$ generated by R and all the relative inverses of the elements of the form x^*x for $x \in R$.

If K is a subfield of $\mathbb{C}$ closed under complex conjugation and G is a countable group, then the $*$-regular closure of $K[G]$ in $\mathcal{U}(G)$ is denoted by $\mathcal{R}_{K[G]}$. For an arbitrary group G, $\mathcal{R}_{K[G]}$ is defined as the direct union of $\{\mathcal{R}_{K[H]}$: H is a finitely generated subgroup of $G\}$.

4 The Cohn theory of epic division R-algebras

4.1 The Ore localization

In this subsection we recall the definition of the left Ore condition and the construction of the Ore ring of fractions.

An element $r \in R$ is a **non-zero-divisor** if there exists no non-zero element $s \in R$ such that $rs = 0$ or $sr = 0$. Let T be a multiplicative subset of non-zero-divisors of R. We say that (T, R) satisfies the **left Ore condition** if for every

$r \in R$ and every $t \in T$, the intersection $Tr \cap Rt$ is not trivial. If T consists of all the non-zero-divisors we simply say that R satisfies the **left Ore condition**.

The goal is to construct the **left Ore ring of fractions** $T^{-1}R$. Let us recall briefly this construction. For more details the reader may consult [84, Chapter 2]. As a set, $T^{-1}R$ coincides with the set of equivalence classes in $T \times R$ with respect to the following equivalence relation:

$$(t_1, r_1) \equiv (t_2, r_2) \text{ if and only if there are } \\ r_1', r_2' \in R \text{ such that } \ r_1't_1 = r_2't_2 \in T \text{ and } \ r_1'r_1 = r_2'r_2.$$

The equivalence class of (t, a) is denoted by $t^{-1}a$. Note that there is no obvious interpretation for the sum $s^{-1}a + t^{-1}r$ and the product $(t^{-1}r)(s^{-1}a)$ $(a, r \in R, s, t \in T)$. In order to sum $s^{-1}a$ and $t^{-1}r$, we observe that for every $s, t \in T$ there exists $s', t' \in R$ such that $s's = t't \in T$. Hence,

$$s^{-1}a + t^{-1}r = (s's)^{-1}s'a + (t't)^{-1}t'r = (s's)^{-1}(s'a + t'r)$$

In order to multiply $s^{-1}a$ and $t^{-1}r$, we rewrite rs^{-1} as a product $(s_0)^{-1}r_0$ with $r_0 \in R$ and $s_0 \in T$. The condition $Tr \cap Rs$ is not trivial implies exactly the existence of $s_0 \in T$ and $r_0 \in R$ such that $s_0r = r_0s$, and so $rs^{-1} = (s_0)^{-1}r_0$. Hence,

$$(t^{-1}r)(s^{-1}a) = (t^{-1})(s_0)^{-1}r_0a = (s_0t)^{-1}r_0a.$$

When T consists of all the non-zero-divisors of R and (T, R) satisfies the left Ore condition, we denote $T^{-1}R$ by $Q_l(R)$ and we call it **the left classical ring of fractions** of R.

An important result in the theory of classical rings of quotients is Goldie's theorem [47, Theorem 6.15]. One of its consequences (see [47, Corollary 6.16]) is that every semiprime left Noetherian ring has a semisimple Artinian classical left ring of fractions.

4.2 Rational closure

Let R be a subring of S. Denote by $\mathrm{GL}(R; S)$ the set of square matrices over R which are invertible over S. The **rational closure** of R in S is the subring of S generated by all the entries of the matrices M^{-1} for $M \in \mathrm{GL}(R; S)$ (in fact, the entries of the matrices M^{-1} for $M \in \mathrm{GL}(R; S)$, form a subring).

Let $f : R \to S$ be a map and let Σ be a set of matrices over R such that $f(\Sigma) \subset GL(f(R); S)$. Then there exists the **universal localization of** R **with respect to** Σ. It is an R-ring $\lambda : R \to R_\Sigma$ such that every element from $\lambda(\Sigma)$ is invertible over R_Σ and every Σ-inverting homomorphism from R to another ring can be factorized uniquely by λ (see [19, Theorem 4.1.3]). An Ore localization is a particular case of universal localization.

A useful result to study rational clousures is Cramer's rule ([19, Proposition 4.2.3], [20, Proposition 7.1.5]). One of its consequences is the following proposition.

Proposition 4.1 *Let S be a rational closure of R. Then for every matrix A over S there are $k \geq 1$, a matrix A' over R and matrices P and Q which are invertible over S such that*

$$A \oplus I_k = PA'Q.$$

4.3 Epic homomorphisms

Let $f : R \to S$ be a ring homomorphism. We say that f is **epic** if for every ring Q and homomorphisms $\alpha, \beta : S \to Q$, the equality $\alpha \circ f = \beta \circ f$ implies $\alpha = \beta$. An **epic** R-ring is a pair (S, f) where $f : R \to S$ is epic. For simplicity we will write S instead of (S, f) when f is clear from the context. For example, if S is the rational closure of $f(R)$ in S, then f is epic.

We will say that two epic R-rings (S_1, f_1) and (S_2, f_2) are isomorphic if there exists an isomorphism $\alpha : S_1 \to S_2$ for which the following diagram is commutative:

$$\begin{array}{ccc} R & \to^{\mathrm{Id}} & R \\ \downarrow f_1 & & \downarrow f_2 \\ S_1 & \to^{\alpha} & S_2. \end{array}$$

Epic homomorphisms can be characterized in the following way.

Proposition 4.2 *[107, Proposition XI.1.2] Let $f : R \to S$ be a ring homomorphism. Then f is epic if and only if the multiplication map*

$$m : S \otimes_R S \to S$$

is an isomorphism of S-bimodules.

More generally if $f : R \to S$ is a ring homomorphism, we say that $s \in S$ is **dominated** by f if for any ring Q and homomorphisms $\alpha, \beta : S \to Q$, the equality $\alpha \circ f = \beta \circ f$ implies $\alpha(s) = \beta(s)$. The set of elements of S dominated by f is a subring of S, called the **dominion** of f. The following result implies that an epic homomorphism from a von Neumann regular ring is always surjective.

Proposition 4.3 *[107, Proposition XI.1.4] Let $\mathcal{U}$ be a von Neumann regular ring. Then for every ring homomorphism $\gamma : \mathcal{U} \to S$, the dominion of γ is equal to $\gamma(U)$.*

4.4 A characterization of epic division *R*-rings

An **epic division R-ring** is an epic R-ring $f : R \to D$, where D is a division ring. Applying Proposition 4.3, it is not difficult to see that for an epic division R-ring (D, f), D is the rational closure of $f(R)$ in D.

If R is a commutative ring, then there exists a natural bijection between $\mathrm{Spec}(R)$ and the isomorphism classes of division R-rings: a prime ideal $P \in \mathrm{Spec}(R)$ corresponds to the field of fractions $Q(R/P)$ of R/P and $f : R \to Q(R/P)$ is defined as $f(r) = r + P$ for any $r \in R$.

If R is a domain and satisfies the left Ore condition then its classical left ring of fractions $Q_l(R)$ is a division ring. Moreover, as in the commutative case, the division R-ring $Q_l(R)$ is the unique (up to R-isomorphism) faithful division R-ring. Thus, if R is a left Noetherian ring, then there exists a natural bijection between the **strong prime** ideals of R (ideals P such that R/P is a domain) and the isomorphism classes of division R-rings.

For an arbitrary ring R, P. Cohn proposed the following approach to classify division R-rings. If D is a division ring, let $\mathrm{rk}_D(M)$ be the D-rank of a matrix M over D.

Theorem 4.4 *[19, Theorem 4.4.1] Let (D_1, f_1) and (D_2, f_2) be two epic division R-rings. Then (D_1, f_1) and (D_2, f_2) are isomorphic if and only if for each matrix M over R*

$$\mathrm{rk}_{D_1}(f_1(M)) = \mathrm{rk}_{D_2}(f_2(M)).$$

5 Sylvester rank functions

The functions $\mathrm{rk}_{G/N}$ and rk_X which have appeared in Sections 1 and 2 are examples of Sylvester matrix rank functions on the algebra $K[G]$. In this section we introduce the notion of Sylvester rank functions on an arbitrary algebra and study their properties.

5.1 Sylvester matrix rank functions

Let R be an algebra. A **Sylvester matrix rank function** rk on R is a function that assigns a non-negative real number to each matrix over R and satisfies the following conditions.

(SMat1) $\mathrm{rk}(M) = 0$ if M is any zero matrix and $\mathrm{rk}(1) = 1$;

(SMat2) $\mathrm{rk}(M_1M_2) \le \min\{\mathrm{rk}(M_1), \mathrm{rk}(M_2)\}$ for any matrices M_1 and M_2 which can be multiplied;

(SMat3) $\mathrm{rk}\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$ for any matrices M_1 and M_2;

(SMat4) $\mathrm{rk}\begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix} \ge \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$ for any matrices M_1, M_2 and M_3 of appropriate sizes.

If $\phi : F_1 \to F_2$ is an R-homomorphism between two free finitely generated R-modules F_1 and F_2, then $\mathrm{rk}(\phi)$ is $\mathrm{rk}(A)$ where A is the matrix associated with ϕ with respect to some R-bases of F_1 and F_2. It is clear that $\mathrm{rk}(\phi)$ does not depend on the choice of the bases.

The following elementary properties of a Sylvester matrix rank function can be obtained from its definition.

Proposition 5.1 *Let R be an algebra and let* rk *be a Sylvester matrix rank function on R. Let $A, B \in \mathrm{Mat}_{n\times m}(R)$, and $C \in \mathrm{Mat}_{m\times k}(R)$. Then*

1. $\mathrm{rk}(A + B) \le \mathrm{rk}(A) + \mathrm{rk}(B)$.
2. $\mathrm{rk}(AC) \ge \mathrm{rk}(A) + \mathrm{rk}(C) - m$.

Proof The first statement is proved in [56]. Let us show (2). Indeed, we have that

$$\operatorname{rk}(AC)+m \overset{\text{SMat3}}{=} \operatorname{rk}\begin{pmatrix} AC & 0_{n\times m}\\ 0_{m\times k} & I_m\end{pmatrix} \overset{\text{SMat2}}{\geq}$$
$$\operatorname{rk}\left(\begin{pmatrix} 0_{m\times n} & -I_m\\ I_n & A\end{pmatrix}\begin{pmatrix} AC & 0_{n\times m}\\ 0_{m\times k} & I_m\end{pmatrix}\begin{pmatrix} I_k & 0_{k\times m}\\ -C & I_m\end{pmatrix}\right)=$$
$$\operatorname{rk}\begin{pmatrix} C & -I_m\\ 0_{n\times k} & A\end{pmatrix} \overset{\text{SMat4}}{\geq} \operatorname{rk}(A)+\operatorname{rk}(C).$$

□

For any algebra R we denote by $\mathbb{P}(R)$ the set of the Sylvester matrix rank functions on R. The set $\mathbb{P}(R)$ is a compact convex subset of functions on matrices over R (with respect to the point convergence topology). It is hard to calculate $\mathbb{P}(R)$ for a general algebra R (see [57] where various examples of explicit calculations of $\mathbb{P}(R)$ are presented).

For a given homomorphism $f: R\to S$ of algebras, we define $f^\#:\mathbb{P}(S)\to\mathbb{P}(R)$ by

$$f^\#(\operatorname{rk})(M)=\operatorname{rk}(f(M)),\ \text{where } M \text{ is a matrix over } R.$$

5.2 Sylvester matrix rank functions and rational closures

Proposition 5.2 *Let $f: R\to S$ be a homomorphism of algebras. Assume that S is a rational closure of $f(R)$. Then $f^\#$ is injective.*

Moreover, if $S=R_\Sigma$ is a universal localization, then

$$\operatorname{Im} f^\#=\{\operatorname{rk}\in\mathbb{P}(R):\ \operatorname{rk}(A)=n \textit{ if } A\in\Sigma\cap\operatorname{Mat}_n(R)\}.$$

In particular, if T is a multiplicative set of non-zero-divisors of R, (T,R) satisfies the left Ore condition and $S=T^{-1}R$, then

$$\operatorname{Im} f^\#=\{\operatorname{rk}\in\mathbb{P}(R):\ \operatorname{rk}(t)=1 \textit{ for all } t\in T\}.$$

Proof The first part of the proposition follows from Proposition 4.1 and the second one is proved in [105, Theorem 7.4]. The proof of [105, Theorem 7.4] is quite technical. Let us present here the proof of the last statement of the proposition, which will also give an idea about the proof of the general case.

Let $\operatorname{rk}\in\mathbb{P}(R)$ be such that $\operatorname{rk}(t)=1$ for all $t\in T$. We want to extend rk on $T^{-1}R$. Given $A=t^{-1}B\in$ ($t\in T$, B a matrix over R) we put $\operatorname{rk}(A)=\operatorname{rk}(B)$. The main difficulty is to show that this definition does not depend on the choice of the pair (t,B). Assume that we can write A also as $t_1^{-1}B_1^{-1}$ ($t_1\in T$, B_1 a matrix over R). We have to show that $\operatorname{rk}(B)=\operatorname{rk}(B_1)$. Applying the definition of Ore condition, we obtain that there are $a,b\in R$ such that $at=bt_2\in T$ and $aB=bB_1$. Since $\operatorname{rk}(at)=\operatorname{rk}(bt_2)=1$, we have that $\operatorname{rk}(a)=\operatorname{rk}(b)=1$. Hence

$$\operatorname{rk}(B)\overset{\text{Proposition 5.1(2)}}{=}\operatorname{rk}(aB)=\operatorname{rk}(bB_1)\overset{\text{Proposition 5.1(2)}}{=}\operatorname{rk}(B_1).$$

Thus, the extension of rk on $T^{-1}R$ is well-defined. Now, it is not difficult to see that it is indeed a Sylvester matrix rank function on $T^{-1}R$. □

In view of this proposition, we will identify $\mathbb{P}(R_\Sigma)$ with the corresponding subset of $\mathbb{P}(R)$.

5.3 Sylvester module rank functions

A **Sylvester module rank function** dim is a function that assigns a non-negative real number to each finitely presented R-module and satisfies the following conditions.

(SMod1) $\dim(\{0\}) = 0$, $\dim(R) = 1$;

(SMod2) $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$;

(SMod3) if $M_1 \to M_2 \to M_3 \to 0$ is exact then

$$\dim M_1 + \dim M_3 \geq \dim M_2 \geq \dim M_3.$$

Given a matrix $A \in \mathrm{Mat}_{n\times m}(R)$ we put $M_A = R^m/(R^n)A$. It is clear that M_A is a finitely presented left R-module. Conversely, given a finitely presented left R-module M we can find a matrix $A \in \mathrm{Mat}_{n\times m}(R)$ such that $M_A \cong M$. This observation allows to construct a natural one-to-one correspondence between the Sylvester matrix rank functions and the Sylvester module rank functions.

Proposition 5.3 *([83],[105, Chapter 7]) Let R be an algebra.*

1. *Let* rk *be a Sylvester matrix rank function on R and let $A \in \mathrm{Mat}_{n\times m}(R)$. We put*
$$\dim(M_A) = m - \mathrm{rk}(A).$$
Then dim *is well defined and it is a Sylvester module rank function on R.*
2. *Let* dim *be a Sylvester module rank function on R and let $A \in \mathrm{Mat}_{n\times m}(R)$. We put*
$$\mathrm{rk}(A) = m - \dim(M_A).$$
Then rk *is a Sylvester module rank function on R.*

If rk and dim are related as described in Proposition 5.3 we will say that they are **associated**.

5.4 The pseudo-metric induced by a Sylvester matrix rank function

Given a Sylvester matrix rank function rk on R, we define

$$\delta(x, y) = \mathrm{rk}(x - y),\ \ x, y \in R.$$

Proposition 5.1(1) implies that the function δ is a pseudo-metric on R. Even though δ is not always a metric, we refer to it as rk**-metric** for convenient abbreviation. Observe that the set

$$\ker \mathrm{rk} = \{a \in R :\ \mathrm{rk}(a) = 0\}$$

is an ideal of R. We say that rk is **faithful** if $\ker \mathrm{rk} = 0$. By Proposition 5.1(1), rk may be seen as a faithful Sylvester matrix rank function on the quotient ring $R/\ker \mathrm{rk}$, and so, δ is a metric on $R/\ker \mathrm{rk}$. Since the multiplication and addition on R are uniformly continuous with respect to δ, the (Hausdorff) completion of $R/\ker \mathrm{rk}$, which we denote by $\overline{R_{\mathrm{rk}}}$ (or simply $\overline{R}$ when rk is clear from the context) is a ring. The kernel of the natural map $R \to \overline{R_{\mathrm{rk}}}$ is $\ker \mathrm{rk}$. The function rk can be extended by continuity on $\overline{R_{\mathrm{rk}}}$ and on matrices over $\overline{R_{\mathrm{rk}}}$ and one easily may check that this extension (denoted also by rk) is a Sylvester matrix rank function on $\overline{R_{\mathrm{rk}}}$.

If G is a group and K a subfield of $\mathbb{C}$, then the completion of $\mathcal{R}_{K[G]}$ with respect to the rk_G-metric is denoted by $\overline{\mathcal{R}_{K[G]}}$.

5.5 Exact Sylvester rank functions

We say that a Sylvester module rank function dim on R is **exact** if it satisfies the following condition

(SMod3$'$) given a surjection $\phi : M \twoheadrightarrow N$ between two finitely presented R-modules,

$$\dim M - \dim N = \inf\{\dim L : \ L \twoheadrightarrow \ker \phi \text{ and } L \text{ is finitely presented}\}.$$

The following result is proved by S. Virili in [112].

Proposition 5.4 *([112]) Let R be an algebra and let* dim *be an exact Sylvester module rank function on R. For every finitely generated R-module M put*

$$\dim M = \inf\{\dim L : \ L \twoheadrightarrow M \ \textit{and } L \textit{ is finitely presented}\},$$

and for every arbitrary R-module put

$$(LF1) \ \dim M = \sup\{\dim L : \ L \leq M \ \textit{and } L \textit{ is finitely generated}\}.$$

Then the extended function $\dim : R\text{-Mod} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ *satisfies the following condition.*

(LF2) if $0 \to M_1 \to M_2 \to M_3 \to 0$ *is exact then* $\dim M_1 + \dim M_3 = \dim M_2$.

A function on R-Mod satisfying (LF1) and (LF2) is called a **length function**. If a length function l satisfies $l(R) = 1$, then the restriction of l on finitely presented R-modules is an exact Sylvester module rank function on R. Moreover, l can be recovered from this restriction using the formulas which appear in Proposition 5.4.

Length functions were first considered by D. Northcott and M. Reufel [90], generalizing the composition length of modules. This concept was investigated later by P. Vámos [111]. For more recent results the reader may consult [100, 113] and references therein. Note that the most interesting examples of length functions l on an algebra R do not satisfy the condition $l(R)$ is finite, and so, do not induce Sylvester module rank functions on R. Thus, the theory of length functions is almost parallel to the theory of Sylvester module rank functions.

5.6 Sylvester rank functions on von Neumann regular rings

An arbitrary algebra may not have an exact Sylvester module rank function. However, if $\mathcal{U}$ is von Neumann regular, then, by Proposition 3.1(2), finitely presented $\mathcal{U}$-modules are projective, and so, all the exact sequences of finitely presented $\mathcal{U}$-modules split. Thus, every Sylvester module rank function on a regular algebra $\mathcal{U}$ is exact. Note also that, by Proposition 3.1(3), a Sylvester matrix rank function on a von Neumann regular algebra $\mathcal{U}$ is completely determined uniquely by its values on elements from $\mathcal{U}$. Thus, pseudo-rank functions studied in [46] are exactly our Sylvester matrix rank functions. Let us mention one result from this book.

Proposition 5.5 *[46] Let $\mathcal{U}$ be a von Neumann regular algebra and* rk *a Sylvester matrix rank function.*

1. *The algebra $\overline{\mathcal{U}_{\mathrm{rk}}}$ is also von Neumann regular.*
2. *The following conditions are equivalent:*
   (a) *$Z(\overline{\mathcal{U}_{\mathrm{rk}}})$ is a field;*
   (b) *$\overline{\mathcal{U}_{\mathrm{rk}}}$ is simple;*
   (c) rk *is the only Sylvester matrix rank function on $\overline{\mathcal{U}_{\mathrm{rk}}}$.*

The conditions of the previous proposition hold in the following example. Recall that a group G is called **ICC group** if all the non-trivial conjugacy classes of G are infinite.

Proposition 5.6 *[56] Let G be an ICC group and K a subfield of $\mathbb{C}$ closed under complex conjugation. Then $Z(\overline{\mathcal{R}_{K[G]}})$ is a subfield of $\mathbb{C}$.*

We finish this subsection with the following definition. A Sylvester matrix rank function rk on an arbitrary algebra R is called **regular** if there exists an algebra homomorphism $f: R \to \mathcal{U}$ such that $\mathcal{U}$ is von Neumann regular and $\mathrm{rk} \in \mathrm{Im}\, f^{\#}$. In this case $\mathcal{U}$ is called a **regular envelope** of rk. Clearly, rk may have many regular envelopes. Later we will see that in some cases we can speak about the canonical regular envelope attached to rk. At this moment all known examples of Sylvester rank functions on an algebra are regular.

Question 5.7 Let R be an algebra. Is it true that every Sylvester rank function on R is regular?

5.7 Ultraproducts of von Neumann regular rings

Given a set X, an **ultrafilter** on X is a set ω consisting of subsets of X such that

1. the empty set is not an element of ω;
2. if A and B are subsets of X, A is a subset of B, and A is an element of ω, then B is also an element of ω;
3. if A and B are elements of ω, then so is the intersection of A and B;
4. if A is a subset of X, then either A or $X \setminus A$ is an element of ω.

If $a \in X$, we can define $\omega_a = \{A \subseteq X : \ a \in A\}$. It is a ultrafilter, called a **principal** ultrafilter. It is a known fact that if X is infinite, then the axiom of choice implies the existence of a non-principal ultrafilter.

Let ω be a ultrafilter on X and $\{a_i \in \mathbb{R}\}_{i\in X}$ a family of real numbers. We write $a = \lim\limits_{\omega} a_i$ if for any $\epsilon > 0$ the set $\{i \in X : \ |a - a_i| < \epsilon\}$ is an element of the ultrafilter w. It is not difficult to see that for any bounded family $\{a_i \in \mathbb{R}\}_{i\in X}$ there exists a unique $a \in \mathbb{R}$ such that $a = \lim\limits_{\omega} a_i$.

Now, let $\{\mathcal{U}_i\}_{i\in X}$ be a family of von Neumann regular rings and for each $i \in X$ let rk_i be a Sylvester matrix rank function on $\mathcal{U}_i$. Then $\prod\limits_{i\in X} \mathcal{U}_i$ is a von Neumann regular ring. Let ω be a ultrafilter on X. We put

$$\mathrm{rk}_\omega(r) = \lim_{\omega} \mathrm{rk}_i(r_i), \text{ where } r = (r_1, r_2, \ldots) \in \prod_{i\in X} \mathcal{U}_i.$$

One easily obtains that rk_ω is a Sylvester matrix rank function on $\prod\limits_{i\in X} \mathcal{U}_i$. We define

$$\prod_{\omega} \mathcal{U}_i = (\prod_{i\in X} \mathcal{U}_i)/\ker(\mathrm{rk}_\omega).$$

Then $\prod\limits_{\omega} \mathcal{U}_i$ is a von Neumnn regular ring and rk_ω is a faithful Sylvester matrix rank function on $\prod\limits_{\omega} \mathcal{U}_i$.

For an algebra R, we denote by $\mathbb{P}_{reg}(R)$ the space of regular Sylvester matrix rank functions on R. The previous construction implies the following proposition.

Proposition 5.8 *[56] $\mathbb{P}_{reg}(R)$ is a closed convex subset of $\mathbb{P}(R)$.*

5.8 Sylvester rank functions on epic von Neumann regular R-rings

Let R be an algebra and let $f : R \to \mathcal{U}$ be an epic von Neumann regular R-ring. From the following proposition, proved in [56], we obtain that any Sylvester matrix rank function on $\mathcal{U}$ is completely determined by its values on matrices over $f(R)$.

Proposition 5.9 *[56] Let R be a subalgebra of a von Neumann regular algebra $\mathcal{U}$. Assume that the embedding of R in $\mathcal{U}$ is epic. Then for any $r_1, \ldots r_k \in \mathcal{U}$, there is a matrix M of size $a \times b$ over R and there are vectors $v_1, \ldots, v_k \in R^b$ such that for every $t_1, \ldots, t_k \in R$ and every Sylvester matrix rank function* rk *on $\mathcal{U}$,*

$$\mathrm{rk}(t_1 r_1 + \ldots + t_k r_k) = \mathrm{rk}\begin{pmatrix} M \\ t_1 v_1 + \ldots + t_k v_k \end{pmatrix} - \mathrm{rk}(M).$$

This proposition can be applied, for example, in the case where $\mathcal{U}$ is a division algebra. But in this case it follows already from Proposition 4.1. Another interesting application of this proposition is presented in Subsection 5.10.

In the proof of Proposition 5.9 the condition that $\mathcal{U}$ is regular plays an important role. Nevertheless, we want to raise the following question.

Question 5.10 Let $f : R \to S$ be an epic homomorphism between two algebras. Is it true that the map $f^{\#} : \mathbb{P}(S) \to \mathbb{P}(R)$ is injective?

If S is a rational closure of R, then a positive answer on the previous question follows from Proposition 5.2.

Proposition 5.9 suggests that if R is an algebra and rk is a Sylvester matrix rank function on R having an epic von Neumann regular envelope, then this envelope might be "canonical". As we have seen this happens in the case where the envelopes are division algebras. We formulate this precisely as the following question.

Question 5.11 Let rk be a Sylvester matrix rank function on R having two epic von Neumann regular envelopes $\mathcal{U}_1$ and $\mathcal{U}_2$. Is it true that $\mathcal{U}_1$ and $\mathcal{U}_2$ are isomorphic as R-rings? More generally, let $\mathcal{U}$ be another von Neumann regular envelope for rk. Is there an R-homomorphism $f : \mathcal{U}_1 \to \mathcal{U}$?

As we have mentioned before, the answer to both questions is positive if $\mathcal{U}_1$ is a division algebra.

5.9 Sylvester matrix rank functions on $*$-regular rings

Now consider Sylvester rank functions on $*$-regular rings. In the following proposition we see that a Sylvester matrix rank function on a $*$-regular ring is always $*$-invariant.

Proposition 5.12 *Let* rk *be a Sylvester matrix rank function on a $*$-regular ring $\mathcal{U}$ and $M \in \mathrm{Mat}_{n\times m}(\mathcal{U})$. Then* $\mathrm{rk}(M) = \mathrm{rk}(M^*)$.

Proof Without loss of generality we may assume that $n = m$ and $M \in \mathrm{Mat}_n(\mathcal{U})$.

It is clear that if $a, b \in \mathcal{U}$ and $a\mathcal{U} = b\mathcal{U}$ or $\mathcal{U}a = \mathcal{U}b$, then $\mathrm{rk}(a) = \mathrm{rk}(b)$. Hence for every $r \in \mathcal{U}$,

$$\mathrm{rk}(r) = \mathrm{rk}(\mathrm{RP}(r)) = \mathrm{rk}(\mathrm{LP}(r^*)) = \mathrm{rk}(r^*). \tag{5}$$

Observe that the function rk^* defined as

$$\mathrm{rk}^*(X) = \mathrm{rk}(X^*),\ X \text{ is a matrix over } R,$$

is also a Sylvester matrix rank function on $\mathcal{U}$. We want to show that $\mathrm{rk} = \mathrm{rk}^*$. This will follow immediately if we show that the Sylvester module rank functions dim and $\dim^*$ associated with rk and rk^* respectively, defined as in Proposition 5.3, coincide. Note that if $X \in \mathrm{Mat}_{n\times m}(\mathcal{U})$, then

$$\mathrm{rk}(X) = m - \dim(\mathcal{U}^m/\mathcal{U}^n \cdot X) = \dim(U^n \cdot X) \text{ and}$$
$$\mathrm{rk}^*(X) = m - \dim^*(\mathcal{U}^m/\mathcal{U}^n \cdot X) = \dim^*(\mathcal{U}^n \cdot X).$$

Now, from (5) we obtain that

$$\dim(\mathcal{U}r) = \mathrm{rk}(r) = \mathrm{rk}^*(r) = \dim^*(\mathcal{U}r).$$

Note also that, by Proposition 3.1, any left finitely presented $\mathcal{U}$-module is a direct sum of modules $\mathcal{U}r$ $(r \in \mathcal{U})$. Hence we are done.

□

5.10 $*$-regular Sylvester rank functions

Now we consider the representations of $*$-rings in $*$-regular algebras. In [56] the following proposition was proved.

Proposition 5.13 *[56] Let R be a $*$-ring, $\mathcal{U}$ a $*$-regular ring and $f : R \to \mathcal{U}$ a $*$-homomorphism. Then $f : R \to \mathcal{R}(f(R),\mathcal{U})$ is epic.*

By analogy with the notion of epic division R-rings, introduced by P. Cohn we propose the following definition. Let R be a $*$-ring. An **epic $*$-regular R-ring** is a triple $(\mathcal{U}, \mathrm{rk}, f)$, such that

1. $\mathcal{U}$ is $*$-regular ring;
2. rk is a faithful Sylvester matrix rank function on $\mathcal{U}$;
3. $f : R \to \mathcal{U}$ is a $*$-homomorphism;
4. $\mathcal{R}(f(R),\mathcal{U}) = \mathcal{U}$.

We will write simply $(\mathcal{U}, \mathrm{rk})$ or $\mathcal{U}$ instead of $(\mathcal{U}, \mathrm{rk}, f)$ if f or (rk, f) are clear from the context. Observe that if $\mathcal{U}$ is a division algebra, there is only one possibility for rk. But in general this is not the case.

We will say that two epic $*$-regular R-rings $(\mathcal{U}_1, \mathrm{rk}_1, f_1)$ and $(\mathcal{U}_2, \mathrm{rk}_2, f_2)$ are isomorphic if there exists an $*$-isomorphism $\alpha : \mathcal{U}_1 \to \mathcal{U}_2$ for which the following diagram

$$\begin{array}{ccc} R & \to^{\mathrm{Id}} & R \\ \downarrow f_1 & & \downarrow f_2 \\ \mathcal{U}_1 & \to^{\alpha} & \mathcal{U}_2 \end{array}$$

is commutative and $\mathrm{rk}_2(\alpha(a)) = \mathrm{rk}_1(a)$ for every $a \in \mathcal{U}_1$.

The following result, which follows from Proposition 5.9, shows that, as in the case of epic division R-rings, the values $\mathrm{rk}(f(M))$, where M is a matrix over R, determine the epic $*$-regular ring $(\mathcal{U}, f, \mathrm{rk})$ uniquely up to isomorphism.

Theorem 5.14 *[56] Let $(\mathcal{U}_1, \mathrm{rk}_1, f_1)$ and $(\mathcal{U}_2, \mathrm{rk}_2, f_2)$ be two epic $*$-regular R-rings. Then $(\mathcal{U}_1, \mathrm{rk}_1, f_1)$ and $(\mathcal{U}_2, \mathrm{rk}_2, f_2)$ are isomorphic if and only if for every matrix M over R*

$$\mathrm{rk}_1(f_1(M)) = \mathrm{rk}_2(f_2(M)).$$

A Sylvester matrix rank function rk on an a $*$-algebra R is called **$*$-regular** if there exists a $*$-algebra homomorphism $f : R \to \mathcal{U}$ such that $\mathcal{U}$ is $*$-regular and $\mathrm{rk} \in \mathrm{Im}\, f^{\#}$. The previous theorem shows that the epic $*$-regular R-ring $(\mathcal{R}(f(R),\mathcal{U}), \mathrm{rk}, f)$ is completely determined by rk. We say that $\mathcal{R}(f(R),\mathcal{U})$ is **the $*$-regular R-algebra associated with** rk.

We denote by $\mathbb{P}_{*reg}(R)$ the space of $*$-regular rank functions on R.

Proposition 5.15 *[56] Let R be a $*$-algebra. Then $\mathbb{P}_{*reg}(R)$ is a closed convex subset of $\mathbb{P}(R)$.*

By Theorem 5.14, every element of $\mathbb{P}_{*reg}(R)$ has a canonical envelope if we require that this envelope has a compatible $*$-structure. It will be interesting to understand whether the same holds without this additional assumption and whether Question 5.11 has a positive solution in this particular case.

Question 5.16 Let R be a $*$-ring and $\mathrm{rk} \in \mathbb{P}_{*reg}(R)$. Is it true that the two questions in Question 5.11 have a positive answer for rk?

6 Algebraic reformulation of the strong Atiyah and Lück approximation conjectures

6.1 An algebraic variation of the strong Atiyah conjecture

In this subsection we formulate an algebraic variation of the strong Atiyah conjecture inspired by results of A. Knebusch, P. Linnell and T. Schick from [63]. First let us present Linnell's reformulation of the strong Atiyah conjecture for torsion-free groups.

Theorem 6.1 *[68] Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Let G be a torsion-free group.Then G satisfies the strong Atiyah conjecture over K if and only if $\mathcal{R}_{K[G]}$ is a division algebra.*

Let R be an algebra. We denote by $K_0(R)$ the abelian group generated by the symbols $[P]$, where P runs over all finitely generated projective R-modules, with the relations $[P_1]+[P_2]=[P_3]$ if $P_1 \oplus P_2 \cong P_3$.

Every homomorphism $f: R \to S$ induces a map $f^{\#}: K_0(R) \to K_0(S)$ that sends $[P]$ to $[S \otimes_R P]$. For any finite subgroup H of a group G, the map $K_0(K[H]) \to K_0(\mathcal{R}_{K[G]})$ is injective. Therefore we will consider $K_0(K[H])$ as a subgroup of $K_0(\mathcal{R}_{K[G]})$.

Conjecture 6.2 (**The algebraic Atiyah conjecture for G over K.**) Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Let G be a group with $\mathrm{lcm}(G)$ finite. Then $\{K_0(K[H])\}_{H\in\mathcal{F}(G)}$ generate $K_0(\mathcal{R}_{K[G]})$.

In view of Theorem 6.1, if G is torsion-free, then the strong Atiyah conjecture and the algebraic Atiyah conjecture are equivalent, because for a von Neumann regular ring $\mathcal{U}$ the condition $K_0(\mathcal{U}) =< [\mathcal{U}] >$ is equivalent to $\mathcal{U}$ being a division algebra.

In general, the algebraic Atiyah conjecture implies the strong Atiyah conjecture. In this survey we will consider only the strong Atiyah conjecture, but it will be interesting to check whether the algebraic Atiyah conjecture holds in the cases where we know that the strong Atiyah conjecture holds.

6.2 A structural reformulation of the general Lück approximation conjecture

Let H be a countable group and let X be a set on which H acts on the left side. Assume that H acts freely on X and $H\backslash X$ is finite. We denote by $\mathcal{U}_H(l^2(X))$ the algebra of unbounded operators on $l^2(X)$ commuting with the left H-action.

If we fix a set of H-representatives $\bar{X} = \{x_1, \ldots, x_n\}$ in X ($n = |H\backslash X|$), we obtain a natural isomorphism of H-Hilbert modules $l^2(H)^n$ and $l^2(X)$:

$$(a_1, \ldots, a_n) \mapsto a_1x_1 + \cdots + a_nx_n \ (a_1, \ldots, a_n \in l^2(H)),$$

which induces a $*$-isomorphism $\Psi_{\bar{X}} : \mathcal{U}_H(l^2(X)) \to \mathrm{Mat}_n(\mathcal{U}(H))$.

Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Let F be a finitely generated free group. If F acts on X on the right and this action commutes with the H-action, we obtain a $*$-homomorphism $f_X : K[F] \to \mathcal{U}_H(l^2(X))$.

Now, let us use the notation of Conjecture 2.6. Fix a set of H_k-representatives $\bar{X}_k$ in X_k, put $n_k = |\bar{X}_k|$ and let

$$f_k = \Psi_{\bar{X}_k} \circ f_{X_k} : \mathbb{C}[F] \to \mathrm{Mat}_{n_k}(\mathbb{C}[H_k]).$$

Remark 6.3 Note that if $A \in K[F]$, then $f_k(A) \in \mathrm{Mat}_{n_k}(K[H_k])$. Thus, the $*$-regular closure $\mathcal{R}(f_k(K[F]), \mathrm{Mat}_{n_k}(\mathcal{U}(H_k))$ of $f_k(K[F])$ in $\mathrm{Mat}_{n_k}(\mathcal{U}(H_k))$ is contained in $\mathrm{Mat}_{n_k}(\mathcal{R}_{K[H_k]})$.

Conjecture 2.6 claims that $\lim_{k\to\infty} \mathrm{rk}_{X_k} = \mathrm{rk}_G$ as Sylvester matrix rank functions on $K[F]$. However, observe that in general we do not know whether $\lim_{k\to\infty} \mathrm{rk}_{X_k}$ exists. In order to avoid this difficulty we will work with $\mathrm{rk}_\omega = \lim_\omega \mathrm{rk}_{X_k}$ instead of $\lim_{k\to\infty} \mathrm{rk}_{X_k}$, where ω is a non-principal ultrafilter on $\mathbb{N}$. Note that equality $\lim_{k\to\infty} \mathrm{rk}_{X_k} = \mathrm{rk}_G$ is equivalent to the equality $\mathrm{rk}_\omega = \mathrm{rk}_G$ for every non-principal ultrafilter ω on $\mathbb{N}$.

Therefore, we fix a non-principal ultrafilter ω on $\mathbb{N}$. We can define

$$f_\omega : \mathbb{C}[F] \to \prod_\omega \mathrm{Mat}_{n_k}(\mathcal{U}(H_k))$$

by sending $A \in \mathbb{C}[F]$ to

$$f_\omega(A) = (f_k(A)).$$

Then, since $\{X_k\}$ approximates $G = F/N$, $\ker f_\omega$ is the ideal of $\mathbb{C}[F]$ generated by $\{g - 1 : \ g \in N\}$. In particular, $f_\omega(K[F]) \cong K[G]$. We put

$$\mathcal{R}_{K[G],\omega} = \mathcal{R}(f_\omega(K[F]), \prod_\omega \mathrm{Mat}_{n_k}(\mathcal{U}(H_k))).$$

Thus, $\mathcal{R}_{K[G],\omega}$ is a $*$-regular algebra associated with $\mathrm{rk}_\omega \in \mathbb{P}_{*reg}(K[G])$.

Now, we reformulate the general Lück approximation conjecture using Theorem 5.14. In the case where G is amenable this result was proven by G. Elek in [32] and in this general form it appears in [56].

Theorem 6.4 *[56] Let K be a subfield of $\mathbb{C}$ closed under complex conjugation, F a finitely generated free group and N a normal subgroup of F. For each natural number k, let X_k be an (H_k, F)-set such that H_k is a group that acts freely on X_k with finitely many orbits. Assume that the family $\{X_k\}$ approximates $G = F/N$. Then the following two conditions are equivalent:*

1. *For any matrix A over $K[F]$,*

$$\lim_{k\to\infty} \mathrm{rk}_{X_k}(A) = \mathrm{rk}_G(A).$$

2. *For every non-principal ultrafilter ω on $\mathbb{N}$,*

$$(\mathcal{R}_{K[G]}, \mathrm{rk}_G) \text{ and } (\mathcal{R}_{K[G],\omega}, \mathrm{rk}_\omega)$$

are isomorphic as epic $*$-regular $K[F]$-rings.

7 The solution of the sofic Lück approximation conjecture for amenable groups over fields of arbitrary characteristic

In this section we explain the proof of the following theorem.

Theorem 7.1 *Let K be a field and F a finitely generated free group. Let $\{X_k\}_{k\in\mathbb{N}}$ be a family of finite F-sets. Assume that $\{X_k\}$ approximates an amenable group $G = F/N$. Then*

(1) for every $A \in \mathrm{Mat}_{n\times m}(K[F])$, there exists the limit $\lim_{k\to\infty} \mathrm{rk}_{X_k}(A)$;

(2) the limit does not depend on the sofic approximation $\{X_k\}$ of G.

Moreover, if we put

$$\mathrm{rk}_G = \lim_{k\to\infty} \mathrm{rk}_{X_k} \in \mathbb{P}_{reg}(K[G])$$

(in view of Theorem 10.1 this is coherent with the previous definition of rk_G when K is a subfied of $\mathbb{C}$) and denote by $\dim_G$ the associated Sylvester module rank function, then $\dim_G$ is exact.

Observe that the most interesting case of Theorem 7.1 corresponds to the case where K is of positive characteristic, because in the case of characteristic 0 we will prove a much stronger result in Theorem 10.1.

In this general form the theorem is stated for the first time. Several particular cases were considered previously in the literature.

1. When $K = \mathbb{Q}$, in order to obtain the conclusions of the theorem, one can use the argument from [76]. A variation of this case appears also in [25].
2. Observe that Conjecture 2.3 for amenable groups is a direct consequence of Theorem 7.1. In [31] G. Elek proved Conjecture 2.3 for amenable groups. D. Pape gave an alternative proof of this case in [95].
3. In [30] the theorem is proved, by G. Elek, in the case when X_k are built from a Følner family. G. Elek also showed that the Sylvester module rank function $\dim_G$ associated with rk_G is exact.
4. In [2], it is proved a particular case of the theorem corresponding to the situation described in the parts (1) and (2) of Conjecture 1.2. This case is also considered in [14].

7.1 Sofic approximations of amenable groups

The main idea behind the proof of Theorem 7.1 is to show that any two sofic approximations of a given amenable group are very similar. This was proved by G. Elek and E. Szabó in [37]. Let us formulate their result.

Let X be a finite set. The **Hamming distance** on $\mathrm{Sym}(X)$ is defined as follows.

$$\mathrm{d_H}(\sigma, \tau) = \frac{|\{x \in X : \ \sigma(x) \neq \tau(x)\}|}{|X|}.$$

Assume now that F is a finitely generated free group and let $\{X_i\}$ be a sofic approximation of $G = F/N$. Fix a non-principal ultrafilter on $\mathbb{N}$ and let d_ω be the pseudo-distance on $\prod_i \mathrm{Sym}(X_i)$:

$$\mathrm{d}_\omega((\sigma_i), (\tau_i)) = \lim_\omega \mathrm{d_H}(\sigma_i, \tau_i).$$

We put $N_\omega = \{\sigma \in \prod_i \mathrm{Sym}(X_i) : \ \mathrm{d}_\omega(\sigma, 1) = 0\}$ and $\Sigma_\omega = \prod_i \mathrm{Sym}(X_i)/N_\omega$. The actions of F on X_i induce a homomorphism $\psi_{\{X_i\},\omega} : F \to \Sigma_\omega$. Clearly $\ker \psi_{\{X_i\},\omega} = N$.

Now, let $\{X_i^1\}$ and $\{X_i^2\}$ be two sofic approximations of $G = F/N$. We put $Y_i^1 = Y_i^2 = X_i^1 \times X_i^2$ and let F act on Y_i^1 by acting only on the first coordinate and F act on Y_i^2 by acting only on the second coordinate. Then $\{Y_i^1\}$ and $\{Y_i^2\}$ are two approximations of F/N.

Theorem 7.2 *([37, Theorem 2]) The representations $\psi_{\{Y_i^1\},\omega}$ and $\psi_{\{Y_i^2\},\omega}$ are conjugate.*

The proof of this theorem uses in an essential way the results of a fundamental work of D. Ornstein and B. Weiss [91] on amenable groups.

7.2 Proof of Theorem 7.1

Observe that an infinite subfamily of a family that approximates a group G also approximates G. Thus, if (1) or (2) does not hold we will be able to find two families $\{X_i^1\}_{i\in\mathbb{N}}$ and $\{X_i^2\}_{i\in\mathbb{N}}$ such that the limits $\lim\limits_{i\to\infty} \mathrm{rk}_{X_i^1}(A)$ and $\lim\limits_{i\to\infty} \mathrm{rk}_{X_i^2}(A)$ exist but they are different. Let us use the notation of Theorem 7.2. Then clearly

$$\mathrm{rk}_{X_i^1} = \mathrm{rk}_{Y_i^1} \ \text{ and } \ \mathrm{rk}_{X_i^2} = \mathrm{rk}_{Y_i^2}.$$

On the other hand, Theorem 7.2 implies that

$$\lim_\omega \mathrm{rk}_{Y_i^1}(A) = \lim_\omega \mathrm{rk}_{Y_i^2}(A)$$

for any non-principal ultrafilter ω on $\mathbb{N}$. Thus,

$$\lim_{i\to\infty} \mathrm{rk}_{X_i^1}(A) = \lim_\omega \mathrm{rk}_{Y_i^1}(A) = \lim_\omega \mathrm{rk}_{Y_i^2}(A) = \lim_{i\to\infty} \mathrm{rk}_{X_i^2}(A).$$

We have obtained a contradiction.

8 Natural extensions of Sylvester rank functions

Let $R \leq S$ be two algebras and let $\mathrm{rk} \in \mathbb{P}(R)$. In this section we consider the following question.

Question 8.1 When is it possible to extend rk to a Sylvester matrix rank function on S? If there are several extensions, can we define a canonical one?

We will see that if S is an "amenable" extension of R (we do not have a precise definition for this notion), then we can expect to be able to construct the "natural" extension of rk. It will be interesting to investigate further the examples presented in this section and produce a general definition for natural extensions.

8.1 A generalization of the construction of rk_G

The construction of rk_G may be generalized in the following may. Let $S = R*G$ be a crossed product of an algebra R and an amenable group G, that is $S = \oplus_{g\in G} S_g$ is a G-graded ring such that $S_e = R$ and for every $g \in G$ there exists an invertible $\bar{g} \in S_g$. Let dim be an exact Sylvester module rank function on R, satisfying

$$\dim L = \dim \bar{g}L, \text{ for every } \ g \in G,\ L \in R-mod. \tag{6}$$

Then we can construct a Sylvester module rank function $\widetilde{\dim}$ on S, which we will call **the natural extension of** dim.

Theorem 8.2 *([112]) Let $S = R*G$ be a crossed product of an algebra R and an amenable group G and let* dim *be an exact Sylvester module rank function on R satisfying (6). Let M be an S-module. Then*

1. *Let $\{F_i\}$ be a Følner family of G. For any finitely generated R-submodule K of M, there exists*
$$e(K) = \lim_{i\to\infty} \frac{\dim \sum_{g\in F_i} \bar{g}K}{|F_i|}.$$
2. *$e(K)$ does not depend on the Følner family $\{F_i\}$.*
3. *$\widetilde{\dim}\, M = \sup_K e(K)$ is an exact Sylvester module rank function on S.*

If $\widetilde{\mathrm{rk}}$ is associated with $\widetilde{\dim}$ and rk is associated with dim, we also say that $\widetilde{\mathrm{rk}}$ is **the natural extension** of rk (notice that, by [112], the restriction of $\widetilde{\mathrm{rk}}$ on R is indeed equal to rk). The compatibility condition (6) can be also expressed in terms of rk:

$$\mathrm{rk}(A) = \mathrm{rk}(\bar{g}^{-1}A\bar{g}), \text{ for every } g \in G \text{ and every matrix } A \text{ over } R. \tag{7}$$

We can also express $\widetilde{\mathrm{rk}}$ in terms of rk.

Proposition 8.3 *Let $S = R*G$ be a crossed product of an algebra R and an amenable group G and let* rk *be an exact Sylvester matrix rank function on R satisfying (7). Fix $\{F_i\}$ a Følner family of G. Let $A \in \mathrm{Mat}_{n\times m}(S)$ be a matrix* *over S and let T be a finite set of elements of G such that the entries of A lie in $\sum_{g\in T} S_g$. Denote by*

$$\phi_i : (\bigoplus_{g\in F_i} S_g)^n \to (\bigoplus_{g\in F_i T} S_g)^m$$

the R-homomorphim of free R-modules induced by lright multiplication by A. Then

$$\widetilde{\mathrm{rk}}(A) = \lim_{i\to\infty} \frac{\mathrm{rk}(\phi_i)}{|F_i|}.$$

Thus, in view of Theorem 7.1, if G is amenable, then $\mathrm{rk}_G \in \mathbb{P}(K[G])$ is the natural extension of $\mathrm{rk}_K \in \mathbb{P}(K)$. It seems logical to ask the following questions.

Question 8.4 Let $S = R * G$ be a crossed product of an algebra R and a group G and let $\mathrm{rk} \in \mathbb{P}(R)$ be G-invariant. Is it possible to extend rk on S? Assuming that rk is faithful, is it possible to find a faithful extension?

One of the motivations for these questions is Kaplansky's direct finiteness conjecture (see Subsection 13.4).

8.2 Other instances of natural extensions

There are other instances where we can speak about the notion of natural extension. They appeared in the proof of some results from [56]. We call them algebraic and transcendental natural extensions.

Let R be an algebra and rk a Sylvester matrix rank function on R. Let E/K be an algebraic extension of fields. Take a matrix $A \in \mathrm{Mat}_{n\times m}(R \otimes_K E)$. Then there exists a finite subextension E_0/K of E/K such that $A \in \mathrm{Mat}_{n\times m}(R \otimes_K E_0)$.

The action of $A \in \mathrm{Mat}_{n\times m}(R \otimes_K E_0)$ by right multiplication on $(R \otimes_K E_0)^n$ defines an R-homomorphism

$$\phi^A : (R \otimes_K E_0)^n \to (R \otimes_K E_0)^m$$

of free R-modules. We put

$$\widetilde{\mathrm{rk}}(A) = \frac{\mathrm{rk}(\phi^A)}{|E_0 : K|}.$$

Observe that $\widetilde{\mathrm{rk}}(A)$ does not depend on the choice of E_0. It is clear that $\widetilde{\mathrm{rk}}$ is a Sylvester matrix rank function on $R \otimes_K E$ and we call it **the natural algebraic extension** of rk on $R \otimes_K E$.

Now consider a matrix $A \in \mathrm{Mat}_{n\times m}(R[t])$ over the polynomial ring $R[t]$ and let

$$\phi^A_{R[t]/(t^i)} : (R[t]/(t^i))^n \to (R[t]/(t^i))^m,\ (a_1, \dots, a_n) \mapsto (a_1, \dots, a_n)A.$$

We put

$$\widetilde{\mathrm{rk}}_i(A) = \frac{\mathrm{rk}(\phi^A_{R[t]/(t^i)})}{i}.$$

Proposition 8.5 *[112, 56] Let* rk *be a regular Sylvester matrix rank function. Then for every matrix A there exists $\lim_{i\to\infty} \widetilde{\mathrm{rk}}_i(A)$, which we denote by $\widetilde{\mathrm{rk}}(A)$*

Note that $\widetilde{\mathrm{rk}}(p) = 1$ for every $0 \neq p \in K[t]$. Thus, taking into account a remark after Proposition 5.2, we can think about $\widetilde{\mathrm{rk}}$ as a Sylvester rank function on $R \otimes_K K(t)$. The Sylvester matrix rank function $\widetilde{\mathrm{rk}}$ on $R \otimes_K K(t)$ is called **the natural transcendental extension** of rk.

As we will see later the notions of natural algebraic and transcendental extension appear in the proof of Theorem 10.1. We will use them to prove the equality between some Sylvester matrix rank functions. We can recognize the natural transcendental extension using the following result.

Proposition 8.6 *[56] Let $\mathcal{U}$ be a von Neumann regular algebra and let* rk *be a Sylvester matrix rank function on $\mathcal{U}$. Let* rk' *be a Sylvester matrix rank function on $\mathcal{U}[t^{\pm 1}]$ which extends* rk*. Assume that for any n by n matrix A,*

$$\mathrm{rk}'(I_n + tA) = n.$$

Then rk' *is the natural transcendental extension of* rk*.*

We want to mention an interesting question, which arose when we were working on [56]. By Proposition 5.5, if $\mathcal{U}$ is a simple von Neumann regular ring and rk is a Sylvester matrix rank function on $\mathcal{U}$ such that $\mathcal{U}$ is complete with respect to the rk-metric, then $\mathbb{P}(\mathcal{U}) = \{\mathrm{rk}\}$. Thus, one can expect to be able to describe $\mathbb{P}(\mathcal{U}[t])$. In particular, we want to ask the following question.

Question 8.7 Let $\mathcal{U}$ be a simple von Neumann regular ring and rk a Sylvester matrix rank function on $\mathcal{U}$ such that $\mathcal{U}$ is complete with respect to rk-metric. Let $K = Z(\mathcal{U})$. Is it true that $\mathbb{P}(\mathcal{U} \otimes_K K(t)) = \{\widetilde{\mathrm{rk}}\}$?

In [57] we answer this question positively in the case where $\mathcal{U}$ is a simple Artinian ring.

We finish this subsection with the following general question.

Question 8.8 Let R be an algebra and $\mathrm{rk} \in \mathbb{P}(R)$. Let E/K a field extension. Is there a general definition for the natural extension $\widetilde{\mathrm{rk}} \in \mathbb{P}(R \otimes_K E)$ that unifies the notions of algebraic and transcendental natural extensions introduced in this subsection?

9 The solution of the strong Atiyah conjecture for elementary amenable groups over fields of arbitrary characteristic

9.1 A variation of Moody's induction theorem

Let R be an algebra. We denote by $G_0(R)$ the abelian group generated by the symbols $[M]$, where M runs over all finitely generated R-modules, with the relations $[M_1] + [M_3] = [M_2]$ if there exists an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$.

If dim is an exact Sylvester module rank function on a Noetherian ring R then dim can be extended to an homomorphism $\dim : G_0(R) \to \mathbb{R}$. Conversely any homomorphism $\phi : G_0(R) \to \mathbb{R}$ such that $\phi([R]) = 1$ can be viewed as an exact

Sylvester module rank function on R. Thus, the study of exact Sylvester module rank functions on R and of the group $G_0(R)$ are very related.

Clearly there exists a natural map $K_0(R) \to G_0(R)$. This is an isomorphism if any finitely generated R-module has a finite resolution consisting of finitely generated projective R-modules.

Any flat homomorphism $f : R \to S$ induces the natural induction map $f : G_0(R) \to G_0(S)$ that sends $[M]$ to $[S \otimes_R M]$. Recall that the embedding of an algebra in an Ore ring of fractions is flat. If $R * G$ is a crossed product and H is a subgroup of G, then the embedding of $R * H$ into $R * G$ is also flat.

In [86] J. Moody proved the following result.

Theorem 9.1 *Let R be a right Noetherian ring, let G be a polycyclic-by-finite group, and let $\mathcal{F}(G)$ denote the set of finite subgroups of G. Then the natural induction map*

$$\oplus_{H \in \mathcal{F}(G)} G_0(R * H) \to G_0(R * G)$$

is surjective.

Corollary 9.2 *[65] Let R be a left Artinian ring, let G be an elementary amenable group such that the orders of finite subgroups of G are bounded. Then the following holds.*

1. *$R * G$ satisfies the left Ore condition and the ring $Q_l(R * G)$ is left Artinian .*
2. *The natural induction map*

$$\oplus_{H \in \mathcal{F}(G)} G_0(R * H) \to G_0(Q_l(R * G))$$

is surjective.

Proof The first part of the corollary is proved in [65, Proposition 4.2]. Let us prove the second one.

We follow the proof of [65, Lemma 4.1]. First recall an alternative description for the class of elementary amenable groups given in [65]. Let $\mathcal{B}$ denote the class of all finitely generated abelian- by-finite groups. For any class of groups $\mathcal{C}$ we denote by $L\mathcal{C}$ the class of locally-$\mathcal{C}$ groups. For each ordinal α, define $\mathcal{E}_\alpha$ inductively as follows: $\mathcal{E}_0$ consists of trivial groups. $\mathcal{E}_\alpha = (L\mathcal{E}_{\alpha-1})\mathcal{B}$ if α is a successor ordinal. $\mathcal{E}_\alpha = \cup_{\beta<\alpha}\mathcal{E}_\beta$ if α is a limit ordinal. Now, $\cup_\alpha \mathcal{E}_\alpha$ is the class of elementary amenable groups.

The result will be proved by transfinite induction. Choose the least ordinal α such that $G \in \mathcal{E}_\alpha$, and assume that the result is true whenever $G \in \mathcal{E}_\beta$ and $\beta < \alpha$. Now α cannot be a limit ordinal, and the result is clearly true if $\alpha = 0$. Therefore we may assume that $\alpha = \gamma + 1$ for some ordinal γ.

Take $A \in L\mathcal{E}_\gamma$. Since $Q_l(R * A)$ is left Artinian, any finitely generated $Q_l(R * A)$-module is finitely presented. Hence we obtain that (2) holds for A because it holds for any finitely generated subgroup of A. Also recall that virtually $L\mathcal{E}_\gamma$-groups are in $L\mathcal{E}_\gamma$ too (see [68, Lemma 4.9]).

Since $G \in \mathcal{E}_\alpha$, there exists a normal subgroup $A \in L\mathcal{E}_\gamma$ such that $G/A \in \mathcal{B}$. Let S be the set of non-zero-divisors of $R * A$. Then for any normal subgroup N of G containing A such that N/A is finite we have that

$$\oplus_{H \in \mathcal{F}(N)} G_0(R * H) \to G_0(Q_l(R * N)) = G_0(S^{-1}(R * N))$$

is surjective. On the other hand, applying Moody's theorem we obtain

$$\oplus_{N/A \in \mathcal{F}(G/A)} G_0(S^{-1}(R * N)) \to G_0(S^{-1}(R * G))$$

is surjective. Therefore,

$$\oplus_{H \in \mathcal{F}(G)} G_0(R * H) \to G_0(S^{-1}(R * G))$$

is surjective. Since, by [65, Lemma 2.2], the map

$$G_0(S^{-1}(R * G)) \to G_0(Q_l(R * G))$$

is surjective, we obtain (2) for G.

□

9.2 The strong Atiyah conjecture for elementary amenable groups

We have the following immediate application of Corollary 9.2.

Corollary 9.3 *Let $S = R * G$ be a crossed product of an Artinian algebra R and an elementary amenable group G. Assume that* $\mathrm{lcm}(G)$ *is finite. Let* $\widetilde{\mathrm{rk}} \in \mathbb{P}(S)$ *be the natural extension of an exact Sylvester matrix rank function* rk *on R.*

1. *$Q_l(S)$ is an envelope of* $\widetilde{\mathrm{rk}}$.
2. *We have the following equality.*

$$\langle \widetilde{\mathrm{rk}}(A) : \ A \textit{ is a matrix over } \ S\rangle = \langle \widetilde{\mathrm{rk}}(A) : \ A \textit{ is a matrix over some } R * H, H \in \mathcal{F}(G)\rangle.$$

Proof Observe that if $s \in S$ is a non-zero-divisor, then since $\widetilde{\mathrm{rk}}$ is exact, $\widetilde{\mathrm{rk}}(s) = 1$. Hence, by Proposition 5.2, $\widetilde{\mathrm{rk}}$ is extended to $Q_l(S)$. This implies (1). The second statement follows from Corollary 9.2.

□

Applying the previous result to $K[G]$, we obtain the positive solution of Conjecture 1.2 (3) and Conjecture 2.2 for elementary amenable groups.

Corollary 9.4 *Let G be an elementary amenable group and let K be a field. Assume that* $\mathrm{lcm}(G)$ *is finite. Then for any matrix A over $K[G]$,* $\mathrm{rk}_G(A) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$.

Moreover, $K[G]$ satisfies the left Ore condition and $Q_l(K[G])$ is a left Artinian envelope of rk_G. *In particular, if $Q_l(K[G])$ is simple (for example, when $K[G]$ is prime), then* $\mathrm{rk}_G(A) = \mathrm{rk}_{Q_l(K[G])}(A)$ *for every matrix A over $K[G]$.*

Some variations of this result appear in [68] when K is of characteristic 0 and in [69] when K is of positive characteristic.

9.3 The Atiyah question in positive characteristic

If G is an amenable group we have constructed rk_G as a Sylvester matrix rank function not only on $\mathbb{C}[G]$ but also on $K[G]$ for every field K. In particular, we can formulate an analogue of Atiyah's question in characteristic p: is it true that rk_G takes only rational values as a Sylvester matrix rank function on $\mathbb{F}_p[G]$? This question was considered in [45] where it was shown that for every real number r there exists an amenable group G such that $r \in \mathcal{A}_{\mathbb{F}_p}(G)$. Again, as in the case of similar examples in characteristic 0, the examples of groups from [45] have finite subgroups of unbounded order.

10 The solution of the general Lück approximation conjecture for sofic groups in characteristic 0

In this section we present the main ideas of the proof of Conjecture 2.6 for sofic groups.

Theorem 10.1 *Let K be a subfield of $\mathbb{C}$, F a finitely generated free group and N a normal subgroup of F. For each natural k, let X_k be an (H_k, F)-set such that H_k is a sofic group that acts freely on X_k and $H_k \backslash X_k$ is finite. Assume that $\{X_k\}$ approximates $G = F/N$. Then for every $A \in \mathrm{Mat}_{n\times m}(K[F])$,*

$$\lim_{k\to\infty} \mathrm{rk}_{X_k}(A) = \mathrm{rk}_G(A).$$

The proof combines several different tools. The case where K is a number field is obtained using analytic methods. In particular, the proof of this case uses a partial solution of the determinant conjecture. We will explain this approach in Subsection 10.3. This idea has its origin in a very influential paper by W. Lück [76] and was developed later in [26, 35]. The passage from algebraic number fields to arbitrary fields K uses algebraic techniques presented in Sections 3, 4 and 5. These methods were introduced in [56].

10.1 Representations of operators

Let G be a countable group. The main results of this section are about the G-equivariant operators $\phi : l^2(G)^n \to l^2(G)^m$ that can be realized as the multiplication on the right side by some matrix $\bar{A} \in \mathrm{Mat}_{n\times m}(\mathbb{C}[G])$:

$$\phi(v_1, \ldots, v_n) = r_{\bar{A}}(v_1, \ldots, v_n) = (v_1, \ldots, v_n)\bar{A},\ v_i \in l^2(G).$$

More concretely, we are interested in the value $\mathrm{rk}_G(\bar{A})$. Let G_0 be the subgroup of G generated by the group elements of G involved in the coefficients of $\bar{A}$. Then

$$\mathrm{rk}_G(\bar{A}) = \mathrm{rk}_{G_0}(\bar{A}).$$

Thus, without loss of generality, we can assume that G is finitely generated.

Since we consider different approximations of the operator $r_{\bar{A}}$, it is convenient for us to consider $r_{\bar{A}}$ in the form ϕ_G^A (as defined in Subsection 2.1). Here $A \in$

$\mathrm{Mat}_{n\times m}(\mathbb{C}[F])$, F is a free finitely generated group, $G = F/N$ and $\bar{A}$ coincides with the image of A in $\mathrm{Mat}_{n\times m}(\mathbb{C}[G])$.

Different types of approximations, that we use in the paper, lead us not only to consider ϕ_F^A but also H-equivariant operators $\phi_X^A : l^2(X)^n \to l^2(X)^m$ (such as it has been defined in Subsection 2.6), where H is a countable group, X is an (H, F)-set such that H acts freely on X and $H\backslash X$ is finite.

For any $x \in X$ we denote by $\mathbf{x}_i$ the element of $(l^2(X))^n$ having x in the ith entry and 0 in the rest of the entries. Note that $l^2(X) \cong l^2(H)^{|H\backslash X|}$ as H-Hilbert modules, and so, we can, if we need it, represent ϕ_X^A again as the multiplication on the right side of $l^2(H)^{|H\backslash X|n}$ by some matrix over $\mathbb{C}[H]$. For this we fix a set of representatives $\bar{X}$ of H-orbits in X and denote by

$$\overline{A_{\bar{X}}} = (b_{\mathbf{x}_i,\mathbf{y}_j})_{x,y\in\bar{X},1\le i\le n,1\le j\le m}$$

a $|\bar{X}|n \times |\bar{X}|m$ matrix over $\mathbb{C}[H]$, such that if $x \in \bar{X}$ we have

$$\phi_X^A(\mathbf{x}_i) = \mathbf{x}_i A = \sum_{y\in\bar{X},1\le j\le m} b_{\mathbf{x}_i,\mathbf{y}_j}\mathbf{y}_j.$$

In the following lemma we collect the properties of the matrices $\overline{A_{\bar{X}}}$ which we will need later.

Lemma 10.2 *The following properties hold.*

1. *Let* $r_{\overline{A_{\bar{X}}}} : l^2(H)^{|\bar{X}|n} \to l^2(H)^{|\bar{X}|m}$ *be the operator that can be realized as multiplication on the right side by the matrix* $\overline{A_{\bar{X}}}$. *Then*
$$\mathrm{rk}_H(\overline{A_{\bar{X}}}) = |\bar{X}|\,\mathrm{rk}_X(A).$$
2. *If* $A, B \in \mathrm{Mat}_n(\mathbb{C}[F])$, *then* $\overline{(AB)_{\bar{X}}} = \overline{A_{\bar{X}}} \cdot \overline{B_{\bar{X}}}$.
3. *If* $A \in \mathrm{Mat}_{n\times m}(\mathbb{C}[F])$, *then* $\overline{(A^*)_{\bar{X}}} = (\overline{A_{\bar{X}}})^*$.

For any element $f = \sum_{h\in H} f_h h$ ($f_h \in \mathbb{C}$) of the group algebra $\mathbb{C}[H]$ we denote by
$$S(f) = |\{h : \ f_h \neq 0\}|$$
the size of the support $\mathrm{supp}\, f$ of f and we put

$$|f| = \sum_{h\in H} |f_h|.$$

If $M = (m_{ij})$ is a matrix over $\mathbb{C}[H]$, then we define

$$S(M) = \max_j \sum_i S(m_{ij}) \text{ and } \ |M| = \max_{i,j} |m_{ij}|.$$

The parameter $S(M)$ was introduced in [26]. The parameter $|M|$ is a variation of the parameter $|M|_\infty$ that appears also in [26]. The following lemma is a direct consequence of the definitions.

Lemma 10.3 *We have that $S(\overline{A_{\bar X}})$ and $|\overline{A_{\bar X}}|$ do not depend on $\bar X$ and moreover*

$$S(\overline{A_{\bar X}}) \le S(A) \text{ and } |\overline{A_{\bar X}}| \le |A|.$$

As corollary we obtain a uniform upper bound for the norm of ϕ_X^A that depends only on the matrix A and not on the set X (we follow the proof of [26, Lemma 3.15]).

Lemma 10.4 *We have that*

1. *For any $x, y \in X$, $1 \le i \le n$ and $1 \le j \le m$,*

$$|\langle \phi_X^A(\mathbf{x}_i), \mathbf{y}_j\rangle| \le |\overline{A_{\bar X}}|.$$

2. *For any $y \in X$ and $1 \le j \le m$,*

$$|\{(x,i) \in X \times \{1,\dots,n\} : \langle \phi_X^A(\mathbf{x}_i), \mathbf{y}_j\rangle \ne 0\}| \le S(\overline{A_{\bar X}}).$$

3. $\|\phi_X^A\| \le \sqrt{S(\overline{A_{\bar X}})S(\overline{A^*_{\bar X}})}|\overline{A_{\bar X}}| \le \sqrt{S(A)S(A^*)}|A|.$

Proof The first and the second statements are clear. Let us prove the third one. Let $v = \sum_{x\in X, 1\le i\le n} v_{\mathbf{x}_i}\mathbf{x}_i \in l^2(X)^n$ ($v_{\mathbf{x}_i} \in \mathbb{C}$). Then

$$\|\phi_X^A(v)\|^2 = \sum_{y\in X, 1\le j\le m} |\langle \phi_X^A(v), \mathbf{y}_j\rangle|^2 =$$

$$\sum_{y\in X, 1\le j\le m}\left| \sum_{x\in X, 1\le i\le n} v_{\mathbf{x}_i}\langle \phi_X^A(\mathbf{x}_i), \mathbf{y}_j\rangle\right|^2 \overset{\text{by (1) and (2)}}{\le}$$

$$S(\overline{A_{\bar X}})(|\overline{A_{\bar X}}|)^2 \sum_{y\in X, 1\le j\le m}\ \sum_{x\in X, 1\le i\le n, \langle \phi_X^A(\mathbf{x}_i), \mathbf{y}_j\rangle\ne 0} |v_{\mathbf{x}_i}|^2 =$$

$$S(\overline{A_{\bar X}})(|\overline{A_{\bar X}}|)^2 \sum_{x\in X, 1\le i\le n}\ \sum_{y\in X, 1\le j\le m, \langle \mathbf{x}_i, \phi_X^{A^*}(\mathbf{y}_j)\rangle\ne 0} |v_{\mathbf{x}_i}|^2 \overset{\text{by (2)}}{\le}$$

$$S(\overline{A_{\bar X}})S(\overline{A^*_{\bar X}})(|\overline{A_{\bar X}}|)^2 \sum_{x\in X, 1\le i\le n} |v_{\mathbf{x}_i}|^2 = S(\overline{A_{\bar X}})S(\overline{A^*_{\bar X}})(|\overline{A_{\bar X}}|)^2\|v\|^2 \overset{\text{Lemma 10.3}}{\le}$$

$$S(A)S(A^*)|A|^2\|v\|^2.$$

□

10.2 The probability measure associated with ϕ_X^A

Let F be a free finitely generated group, H a countable group and X an (H,F)-set such that H acts freely on X and $H\backslash X$ is finite. Let A be a matrix over $\mathbb{C}[F]$. The operator $\phi_X^{A^*}$ is adjoint to the operator ϕ_X^A. Observe also that $\ker \phi_X^A = \ker \phi_X^{AA^*}$. Thus, the change of A by AA^* allows us to assume that A in Conjecture 2.6

is square and hermitian ($*$-symmetric) and so the operators $\phi^A_{X_k}$ and ϕ^A_G are self-adjoint and positive. Therefore, from now on, we will assume always that $A = BB^*$ for some $B \in \mathrm{Mat}_{n\times m}(\mathbb{C}[F])$. Let us recall the following result from the theory of self-adjoint operators on a Hilbert space.

Proposition 10.5 *[64, Proposition 3.11] Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{B}(\mathcal{H})$ a self-adjoint operator and $v \in \mathcal{H}$ a fixed vector. There exists a unique positive Radon measure μ on $\mathrm{Spec}(T)$, depending on T and v, such that*

$$\int_{\mathrm{Spec}(T)} f d\mu = \langle f(T)v, v\rangle, \quad \textit{for all continuous functions} \quad f \quad \textit{on} \quad \mathrm{Spec}(T).$$

In particular, we have $\mu(\mathrm{Spec}(T)) = \|v\|^2$; so it is a finite measure.

The measure μ from the proposition is called the **spectral measure associated to v and T**. In a similar way we can associate to the operators ϕ^A_X probability Radon measures μ^A_X on $[0, a]$, where $a = \sqrt{S(A)S(A^*)}|A|$ (see Lemma 10.4). It can be done in the following way.

Fix a set $\bar{X}$ of representatives of H-action on X. For each $\bar{x} \in \bar{X}$ and $1 \le i \le n$, let $(\mu^A_X)_{\bar{x},i}$ be the Radon measure associated to $(0, \dots, \bar{x}, \dots, 0)$ ($\bar{x}$ is on the ith place) and ϕ^A_X. Now, we put

$$\mu^A_X = \frac{1}{|\bar{X}|} \sum_{\bar{x}\in\bar{X}, 1\le i\le n} (\mu^A_X)_{\bar{x},i}.$$

If G is a group, then μ^A_G will denote the measure associated with ϕ^A_G.

Let $\mathcal{S}$ be a metric space with its Borel σ-algebra Σ. We say that a sequence of positive probability measures μ_i ($i \in \mathbb{N}$) on $(\mathcal{S}, \Sigma)$ **converges weakly** to the measure μ, if

$$\int_{\mathcal{S}} f d\mu_i \to \int_{\mathcal{S}} f d\mu \text{ (when } i \to \infty)$$

for all bounded, continuous functions f on $\mathcal{S}$.

From now on, let F be a finitely generated free group and N a normal subgroup of F. For each natural k, let X_k be an (H_k, F)-set such that H_k is a countable group that acts freely on X_k and $H_k\backslash X_k$ is finite. Assume that $\{X_k\}$ approximates $G = F/N$. Let $A = BB^*$ for some $B \in \mathrm{Mat}_{n\times m}(K[F])$.

Lemma 10.6 *The measures $\mu^A_{X_k}$ converge weakly to μ^A_G.*

Proof We should check that for any continuous function f on $[0, a]$

$$\int_{[0,a]} f d\mu^A_{X_k} \to \int_{[0,a]} f d\mu^A_G.$$

Since, by the Weierstrass Approximation Theorem, any continuous function can be approximated by polynomials, we can assume that $f = x^i$. Note that

$$\int_{[0,a]} x^i d\mu^A_{X_k} = \frac{\mathrm{Tr}_{H_k}(\phi^A_{X_k})^i}{|H_k\backslash X_k|} = \frac{\mathrm{Tr}_{H_k}\,\phi^{A^i}_{X_k}}{|H_k\backslash X_k|}.$$

Now, since X_k approximate G, we obtain that

$$\frac{\mathrm{Tr}_{H_k}\, \phi_{X_k}^{A^i}}{|H_k \backslash X_k|} \xrightarrow{k\to\infty} \mathrm{Tr}_G\, \phi_G^{A^i} = \int_{[0,a]} x^i d\mu_G^A.$$

$\square$

Clearly the previous lemma does not imply directly that $\mu_{X_k}^A(\{0\})$ converges to $\mu_G^A(\{0\})$ (note that this is an equivalent reformulation of Conjecture 2.6). However, it implies one of the two inequalities of Conjecture 2.6.

Proposition 10.7 *(Kazhdan's inequality) The following inequality holds:*

$$\limsup_{k\to\infty} \dim_{X_k} \ker \phi_{X_k}^A \leq \dim_G \ker \phi_G^A.$$

Proof Note that by the Portmanteau theorem (see, for example, [27, Theorem 11.1.1]),

$$\mu_G^A(C) \geq \limsup_{k\to\infty} \mu_{X_k}^A(C) \text{ for all closed sets } C \text{ of } [0,a]. \tag{8}$$

Thus, we obtain the following

$$\dim_G \ker \phi_G^A = \mu_G^A(\{0\}) \geq \limsup_{k\to\infty} \mu_{X_k}^A(\{0\}) = \limsup_{k\to\infty} \dim_{X_k} \ker \phi_{X_k}^A.$$

$\square$

10.3 The determinant conjecture

Observe that the Portmanteau theorem implies also that for $\epsilon > 0$,

$$\mu_G^A(\{0\}) \leq \mu_G^A([0,\epsilon)) \leq \liminf_{k\to\infty} \mu_{X_k}^A([0,\epsilon)) \leq \\ \liminf_{k\to\infty} \mu_{X_k}^A(\{0\}) + \limsup_{k\to\infty} \mu_{X_k}^A((0,\epsilon)),$$

and so

$$\mu_G^A(\{0\}) - \liminf_{k\to\infty} \mu_{X_k}^A(\{0\}) \leq \limsup_{k\to\infty} \mu_{X_k}^A((0,\epsilon))$$

Thus, in order to prove Conjecture 2.6, it will be enough to show that $\mu_{X_k}^A((0,\epsilon))$ tends uniformly (in k) to zero when ϵ tends to zero.

With this aim, it was proposed to use the Fuglede-Kadison determinant of ϕ_X^A defined as follows.

$$\det{}^+(\phi_X^A) := \begin{cases} \exp\left(\int_{0+}^a \ln(x) d\mu_X^A\right) & \text{if the integral converges} \\ 0 & \text{otherwise} \end{cases}$$

This idea is contained implicitly in the paper of W. Lück [76]. It seems that explicitly it appeared first in [25, 103].

Proposition 10.8 *Assume that there exists a constant C such that*

$$\ln \det{}^+(\phi^A_{X_k}) \geq C \text{ for all } k.$$

Then

$$\mu^A_G(\{0\}) = \lim_{k\to\infty} \mu^A_{X_k}(\{0\}).$$

Moreover $\ln \det{}^+(\phi^A_G) \geq C$ *as well.*

Proof Assume that

$$\int_{0+}^a \ln(x) d\mu^A_{X_k} \geq C.$$

Hence, for any $\epsilon > 0$,

$$\mu^A_{X_k}((0,\epsilon)) \ln \epsilon \geq \int_{0+}^{\epsilon} \ln(x) d\mu^A_{X_k} \geq C - \int_{\epsilon}^a \ln(x) d\mu^A_{X_k} \geq C - a\ln a.$$

Thus, we obtain that

$$\mu^A_{X_k}((0,\epsilon)) \leq \frac{a\ln a - C}{-\ln \epsilon}.$$

Thus, $\mu^A_{X_k}((0,\epsilon))$ tends uniformly (in k) to zero and so

$$\mu^A_G(\{0\}) = \lim_{k\to\infty} \mu^A_{X_k}(\{0\}).$$

This proves the first statement of the proposition. The second statement follows from the Portmanteau theorem.

$$\ln \det{}^+(\phi^A_G) = \int_{0+}^a \ln(x) d\mu^A_G = \lim_{\epsilon\to 0^+} \int_{\epsilon}^a \ln(x) d\mu^A_G \overset{\text{by (8)}}{\geq} \lim_{\epsilon\to 0^+} \limsup_{k\to\infty} \int_{\epsilon}^a \ln(x) d\mu^A_{X_k} \geq C.$$

□

Thus, the previous proposition shows that Conjecture 2.6 is a consequence of the following conjecture.

Conjecture 10.9 (The determinant conjecture over K) Let K be a subfield of $\mathbb{C}$ closed under complex conjugation. Let F be a finitely generated free group and A a $*$-symmetric matrix over $K[F]$. Then there exists a constant C depending only on A such that for every countable group H and every (H,F)-set X such that H acts freely on X and $H\backslash X$ is finite,

$$\ln \det{}^+(\phi^A_X) \geq C.$$

As we have seen before the determinant conjecture is a way to control the measures μ_X^A uniformly in the small intervals around 0. A stronger form of the determinant conjecture is a conjecture of J. Lott and W. Lück (formulated only when $K=\mathbb{Q}$) about Shubin-Novikov invariants (see [78]). It was known for free groups ([101]) and free abelian groups ([73, 80]) but few years ago a counterexample was constructed by Ł. Grabowski [43]. Another related conjecture is the determinant approximation conjecture. We will not describe it here, but we recommend to read the introduction of [43], where the relation between all three conjectures is presented.

Unfortunately Conjecture 10.9 is not correct if $K=\mathbb{C}$. Our example is a modification of [78, Example 13.69].

Construct a sequence $n_1=1$ and $n_{j+1}=3^{n_j}$. Put $r=\sum_{j=1}^\infty \frac{1}{n_j}$. Consider $A=z-e^{2\pi i r}\in\mathbb{C}[z^{\pm 1}]=\mathbb{C}[\mathbb{Z}]$. Let $X_j=\mathbb{Z}/(n_j)$. Then

$$\ln\det{}^+\phi^A_{X_k}\le \ln|e^{2\pi i\sum_{j=k+1}^\infty\frac{1}{n_j}}-1|+(n_k-1)\ln 2\le\\ -\ln n_{k+1}+n_k\ln 2=(\ln 2-\ln 3)n_k.$$

Thus, $\ln\det^+\phi^A_{X_k}$ are not bounded from below.

10.4 The proof of the determinant conjecture for sofic groups over $\bar{\mathbb{Q}}$

The following theorem is a slight modification of [26, Theorem 3.2].

Theorem 10.10 *Let X be an (H,F)-set such that F is a finitely generated free group and H is a group acting freely on X with finite number of orbits. Let K be a number field of degree s over $\mathbb{Q}$. Denote by $\sigma_i: K\to\bar{\mathbb{Q}}$ the s different embeddings with $\sigma_1=\mathrm{Id}$. Let $A=BB^*$ with $B\in\mathrm{Mat}_{n\times m}(\mathcal{O}_K[F])$ ($\mathcal{O}_K$ is the ring of integers of K). If H is sofic, then*

$$\ln\det{}^+(\phi^A_X)\ge -n\sum_{i=2}^s\ln(\sqrt{S(\sigma_i(A))S(\sigma_i(A)^*)}|A|).$$

Proof First we consider the case when H is trivial (or finite). Let $C=\oplus_{i=1}^s\sigma_i(A)$ be the diagonal sum of the matrices $\sigma_i(A)$. Note that C is not a $*$-symmetric positive matrix, but we still can define $(\det^+\phi^C_X)^{|X|}$ as the product of the absolute values of all non-zero roots (counted with multiplicities) of the characteristic polynomial of ϕ^C_X and $(\det^+\phi^{\sigma_i(A)}_X)^{|X|}$ as the product of the absolute values of all non-zero roots (counted again with multiplicities) of the characteristic polynomial of $\phi^{\sigma_i(A)}_X$. In particular $(\det^+\phi^C_X)^{|X|}$ is a non-zero algebraic integer lying in $\mathbb{Q}$, and so, $(\det^+\phi^C_X)^{|X|}\ge 1$. Hence

$$1\le\det{}^+\phi^C_X=\prod_{i=1}^s\det{}^+\phi^{\sigma_i(A)}_X\le\det{}^+\phi^A_X\prod_{i=2}^s\|\phi^{\sigma_i(A)}_X\|^n\overset{\text{Lemma 10.4}}{\le}\\ \det{}^+\phi^A_X\prod_{i=2}^s(\sqrt{S(\sigma_i(A))S(\sigma_i(A)^*)}|A|)^n$$

Therefore, we conclude that

$$\ln \det{}^+ \phi_X^A \geq -n \sum_{i=2}^{s} \ln(\sqrt{S(\sigma_i(A))S(\sigma_i(A)^*)}|A|).$$

Now, assume that H is sofic. As we have explained in Subsection 10.1, we can represent the operator ϕ_X^A in the form $\phi_{\tilde{H}}^{A_{\bar{X}}}$. Let $\{Y_k\}$ be a family of finite $\tilde{F}$-sets that approximate $\tilde{H}$. Then from the finite case of the proposition we obtain that

$$\ln \det{}^+ \phi_{Y_k}^{A_{\bar{X}}} \geq -n \sum_{i=2}^{s} \ln(\sqrt{S(\sigma_i(A_{\bar{X}}))S(\sigma_i(A_{\bar{X}})^*)}|A_{\bar{X}}|) \overset{\text{by Lemma 10.3}}{\geq} - n \sum_{i=2}^{s} \ln(\sqrt{S(\sigma_i(A))S(\sigma_i(A)^*)}|A|).$$

Thus, by Proposition 10.8,

$$\ln \det{}^+ \phi_X^A = \ln \det{}^+ \phi_{\tilde{H}}^{\tilde{A}_{\bar{X}}} \geq -n \sum_{i=2}^{s} \ln(\sqrt{S(\sigma_i(A))S(\sigma_i(A)^*)}|A|).$$

□

Theorem 10.10 and Proposition 10.8 imply together Theorem 10.1 with $K = \bar{\mathbb{Q}}$.

Corollary 10.11 *Let F be a free finitely generated group, $A \in \mathrm{Mat}_{n\times m}(\bar{\mathbb{Q}}[F])$ and H_k ($k \in \mathbb{N}$) a family of sofic groups. For each natural number k, let X_k be an (H_k, F)-set such that H_k acts freely on X_k and $H_k \backslash X_k$ is finite. Assume that $\{X_k\}$ approximates $G = F/N$. Then*

$$\lim_{k\to\infty} \dim_{X_k} \ker \phi_{X_k}^A = \dim_G \ker \phi_G^A.$$

Proof Without loss of generality we may assume that $A = BB^*$ where B is a matrix over $\mathcal{O}_K[F]$ and K is a finite extension of $\mathbb{Q}$. Now we can apply Theorem 10.10 and Proposition 10.8. □

10.5 The proof of Theorem 10.1

In the previous section we have proved Theorem 10.1 in the case $K = \bar{\mathbb{Q}}$. Now let us explain how to proceed in the general case. This has been done recently in [56].

Assume, in addition, that K is closed under complex conjugation. Then rk_G is a $*$-regular Sylvester matrix rank function on $K[G]$. The $*$-regular algebra associated to rk_G is $\mathcal{R}_{K[G]}$. Let us fix a non-principal ultrafilter ω on $\mathbb{N}$. Then $\mathrm{rk}_\omega = \lim_\omega \mathrm{rk}_{X_k}$ is another $*$-regular Sylvester matrix rank function on $K[G]$. The $*$-regular $K[G]$-algebra associated with rk_ω is $R_{K[G],\omega}$.

A straightforward reformulation of Theorem 10.1 is to say that for every non-principal ultrafilter ω on $\mathbb{N}$,

$$\mathrm{rk}_G = \mathrm{rk}_\omega \text{ as Sylvester matrix rank functions on } K[G].$$

Our structural reformulation of the sofic Lück approximation conjecture over K (Theorem 6.4) implies that it is equivalent to the existence of a $K[G]$-*-isomorphism

$$\alpha_K : \mathcal{R}_{K[G]} \to \mathcal{R}_{K[G],\omega} \text{ such that } \quad \mathrm{rk}_G = \mathrm{rk}_\omega \circ \alpha_K.$$

At first glance, it seems that this reformulation cannot help us to prove Theorem 10.1, because to prove the existence of α_K is harder than to prove the equality between the Sylvester rank functions rk_G and rk_ω. However, we have already proved Theorem 10.1 when $K = \mathbb{Q}$ (and in fact, when $K = \bar{\mathbb{Q}}$). Thus, we know that $\alpha_{\mathbb{Q}}$ exists! This is the first brick in our construction of α_K for an arbitrary subfield K of $\mathbb{C}$.

It is clear that it is enough to prove Theorem 10.1 for finitely generated subfields K of $\mathbb{C}$. Any finitely generated subfield K of $\mathbb{C}$ of transcendental degree n over $\mathbb{Q}$ is a subfield of a field K_{2n}, where K_i are constructed inductively:

1. $K_1 = \mathbb{Q}$;
2. if $i \geq 1$, $K_{2i} = \overline{K_{2i-1}}$ is the algebraic closure of K_{2i-1} in $\mathbb{C}$;
3. if $i \geq 1$, $K_{2i+1} = K_{2i}(\lambda_i)$ for some $\lambda_i \in \mathbb{C} \setminus K_{2i}$ such that $|\lambda_i| = 1$.

Theorem 10.1 for K_i is proved by induction on i.

First we consider the inductive step for algebraic extensions. Thus, we assume that Theorem 10.1 holds for K_{2i-1}.

Given a Sylvester matrix rank function rk on an algebra R and an algebraic extension E/K we have defined in Subsection 8.2 the natural algebraic extension $\widetilde{\mathrm{rk}} \in \mathbb{P}(R \otimes_K E)$ of rk. It is proved in [56] that if G is sofic, K is a subfield of $\mathbb{C}$ closed under complex conjugation and the sofic Lück approximation holds over K, then

$$\mathcal{R}_{\bar{K}[G]} \cong \mathcal{R}_{K[G]} \otimes_K \bar{K} \text{ as } \bar{K}[G]\text{-*-rings} \tag{9}$$

and, moreover, the restriction of rk_G on $\mathcal{R}_{\bar{K}[G]}$ is the natural algebraic extension of the restriction of rk_G on $\mathcal{R}_{K[G]}$. This also implies a similar statement for rk_ω: the restriction of rk_ω on $\mathcal{R}_{\bar{K}[G],\omega}$ is the natural algebraic extension of the restriction of rk_G on $\mathcal{R}_{K[G],\omega}$ and

$$\mathcal{R}_{\bar{K}[G],\omega} \cong \mathcal{R}_{K[G],\omega} \otimes_K \bar{K} \text{ as } \bar{K}[G]\text{-*-rings.} \tag{10}$$

Using the induction assumption we have that there exists $\alpha_{K_{2i-1}}$. Taking into account (9) and (10), we construct $\alpha_{K_{2i}}$. Now the uniqueness of the natural extension implies that $\mathrm{rk}_\omega \circ \alpha_{K_{2i}} = \mathrm{rk}_G$ as Sylvester matrix rank functions on $K_{2i}[G]$. This proves Theorem 10.1 for K_{2i} .

Let us describe now the proof of the inductive step for transcendental extensions. We assume that Theorem 10.1 holds for K_{2i}.

Given a regular Sylvester matrix rank function rk on an algebra R we have defined in Subsection 8.2 the natural transcendental extension $\widetilde{\mathrm{rk}} \in \mathbb{P}(R \otimes_K K(t))$ of rk. If R is a von Neumann regular algebra then, by Proposition 8.6, $\widetilde{\mathrm{rk}}$ is characterized by the condition that for every n by n matrix A over R,

$$\widetilde{\mathrm{rk}}(I_n + tA) = n.$$

This leads us to consider the following conjecture.

Conjecture 10.12 (The strong algebraic eigenvalue conjecture over K for G) Let G be a countable group, K a subfield of $\mathbb{C}$ and $A \in \mathrm{Mat}_n(\mathcal{R}_{K[G]})$. Then for any $\lambda \in \mathbb{C}$ which is not algebraic over K, the matrix $A - \lambda I_n$ is invertible over $\mathcal{U}(G)$.

This conjecture generalizes the algebraic eigenvalue conjecture formulated in [26]. The proof of the strong algebraic eigenvalue conjecture for a sofic group G over an arbitrary subfield of $\mathbb{C}$ is presented in [56].

Observe that the strong algebraic eigenvalue conjecture over K_{2i} implies that

$$\mathrm{rk}_G(I_n + \lambda_i A) = \mathrm{rk}_G(A + \lambda_i^{-1} I_n) = n$$

for every n by n matrix A over $\mathcal{R}_{K_{2i}[G]}$. This means that the restriction of rk_G on $K_{2i+1}[G]$ is the natural transcendental extension of the restriction of rk_G on $K_{2i}[G]$. The same holds for rk_ω. The uniqueness of the natural transcendental extension implies that rk_G and rk_ω as Sylvester matrix rank functions on $K_{2i+1}[G]$ are equal. This proves Theorem 10.1 for K_{2i+1}.

10.6 Other variations of the Lück approximation

There are other variations of the Lück approximation considered in the literature.

The Lück approximation in the context of Benjamini-Schramm convergence of graphs is studied by M. Abért, A. Thom and B. Virag in [3].

In [61] S. Kionke describes a general construction of the Sylvester matrix rank function on $\mathbb{C}[G]$ associated with a representation of the group G in a finite von Neumann algebra and extends the Lück approximation to this more general situation.

In [96] H. D. Petersen, R. Sauer and A. Thom present a general Lück approximation theorem for normalised Betti numbers for Farber sequences of lattices in totally disconnected groups.

11 The Approximation and strong Atiyah conjecture for completed group algebras of virtually pro-p groups

Let $(R, \mathbf{m})$ be a commutative completed local domain such that $R/\mathbf{m}$ is finite of characteristic $p > 0$ and let K be the ring of fractions of R. In this section K may be of characteristic p or 0. Let G be a countably based virtually a pro-p group. We consider

$$\Lambda = \Lambda(R[[G]]) = K \otimes_R R[[G]]$$

(see [23] for the definition of $R[[G]]$ and its properties). For every open normal subgroup U of G we have the canonical map

$$\Lambda \to K[G/U],$$

which induces a Sylvester matrix rank function $\mathrm{rk}_{G/U}$ on Λ. Let us formulate the Lück approximation and the strong Atiyah conjectures in this situation.

Conjecture 11.1 Let G be virtually a pro-p group and let $G > G_1 > G_2 \ldots$ be a chain of open normal subgroups of G with trivial intersection. Then

1. there exists the limit $\lim_{i\to\infty} \mathrm{rk}_{G/G_i} \in \mathbb{P}(\Lambda)$;
2. the limit does not depend on the chain $G > G_1 > G_2 \ldots$;
3. if $\mathrm{lcm}(G) < \infty$, $\lim_{i\to\infty} \mathrm{rk}_{G/G_i}(A) \in \frac{1}{\mathrm{lcm}(G)}\mathbb{Z}$ for every matrix A over Λ.

If the parts (1) and (2) of conjecture hold we denote the limit by $\mathrm{rk}_{R[[G]]}$. Then $\dim_{R[[G]]}$ will denote the Sylvester module rank function associated with $\mathrm{rk}_{R[[G]]}$. We can also give an explicit formula for $\dim_{R[[G]]}$. If M is finitely presented Λ-module, we obtain

$$\dim_{R[[G]]} M = \lim_{i\to\infty} \frac{\dim_K K[G/G_i] \otimes_\Lambda M}{|G : G_i|}. \tag{11}$$

Observe that Conjecture 11.1 is stronger than Conjecture 1.2, because the first conjecture claims the approximation for matrices over Λ, which is larger than $K[G]$. However since G is virtually pro-p, the case of Conjecture 11.1 (1) and (2), where K is of characteristic p, is easier than the one where K is of characteristic 0.

Proposition 11.2 *[14, Lemma 4.1] Assume that K is of characteristic p. Then the parts (1) and (2) of Conjecture 11.1 hold.*

Proof When G_{i+1} is pro-p, $K[G_i/G_{i+1}]$ is a local ring. Thus, for this i, we obtain that for every finitely presented Λ-module M,

$$\dim_{G/G_i} M = \frac{\dim_K(K[G/G_i] \otimes_\Lambda M)}{|G : G_i|} =$$
$$\frac{\dim_K(K[G/G_i] \otimes_{K[G/G_{i+1}]} (K[G/G_{i+1}] \otimes_\Lambda M)}{|G : G_i|}) \geq$$
$$\frac{\dim_K(K[G/G_{i+1}] \otimes_\Lambda M)}{|G : G_{i+1}|} = \dim_{G/G_{i+1}} M.$$

This show that the limit (11) exists. A similar argument shows that it does not depend on the chain (see [55, Corollary 2.2]). □

Later we will use the following property of $\dim_{R[[G]]}$ when R is of characteristic p.

Proposition 11.3 *Assume that K has characteristic p. Let M be a proper quotient of Λ^n. Then $\dim_{R[[G]]} M < n$.*

Proof Since M is a proper quotient of Λ^n, there exists l such that if $i \geq l$, $\dim_{G/G_i} M < n$. As we have seen in the proof of the previous proposition $\{\dim_{G/G_i} M\}$ is virtually decreasing. Hence $\dim_{R[[G]]} M < n$

□

Recall that a p-adic analytic profinite group can be defined as a closed subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some n. These groups play a special role in the theory of pro-p groups and linear groups (see [23]). For example, A. Lubotzky [75] proved the following criterion of linearity of a finitely generated group.

Proposition 11.4 *[75] Let G be a finitely generated group. Then G is linear over a field of characteristic 0 if and only if G can be embedded into a p-adic analytic group for some prime p.*

If G is a p-adic profinite group and R is Noetherian, then $R[[G]]$ is a Noetherian ring ([23]). By a result of Lazard [66], if G is torsion free, then the ring $R[[G]]$ does not have non-trivial zero divisors (see also [87, 8, 108]). Thus, in this case the ring $\mathcal{D}_R(G) = Q_l(R[[G]])$ is a division algebra.

An arbitrary p-adic analytic profinite groups G contains an open torsion-free normal pro-p subgroup H. Therefore, we can speak about $\mathrm{rk}_{D_R(H)} \in \mathbb{P}(R[[H]])$. Observe that $R[[G]] = R[[H]]*G/H$. So, we define $rk_{D_R(G)}$ as the natural extension of $rk_{D_R(H)}$ (see Subsection 8.1).

The following result is attributed to M. Harris [52] (see also [39, 14]).

Proposition 11.5 *[52] Let G be a p-adic analytic profinite group. Then*

$$\mathrm{rk}_{R[[G]]} = rk_{D_R(G)}.$$

In particular,

1. *Conjecture 11.1 (1) and (2) hold for p-adic analytic profinite groups and*
2. *Conjecture 11.1 (3) holds for torsion-free p-adic analytic profinite groups.*

As a consequence we obtain that Conjecture 11.1 (3) holds over fields of characteristic p for a large class of pro-p groups which includes free pro-p groups.

Corollary 11.6 *[55] Let K be of characteristic p. Let G be an inverse limit of torsion-free p-adic pro-p groups. Then Conjecture 11.1 (3) holds for G.*

Proof We combine the approximation in characteristic p (Proposition 11.2) and Proposition 11.5. □

At this moment Corollary 11.6 describes all the examples for which Conjecture 11.1 (3) is known when K has characteristic p. It will be interesting to prove Conjecture 11.1 (3) over fields of characteristic p for virtually free pro-p groups and free-by-cyclic pro-p groups.

When K has characteristic 0, the obvious example to check is the case of free pro-p group. All three parts of Conjecture 11.1 are not known in this case.

12 Positive results on the strong Atiyah conjecture over fields of characteristic 0

12.1 Amenable extensions

Let F be a free group freely generated by a finite set S and N a normal subgroup of F. Put $G = F/N$. Let H be a normal subgroup of G such that G/H is amenable. Consider a transversal $\bar{X}$ of H in G. Since G/H is amenable we can find a family of finite subsets $\{\bar{X}_k\}_{k\in\mathbb{N}}$ of $\bar{X}$ satisfying

$$|\bar{T}_k| \geq (1-\frac{1}{k})|\bar{X}_k|, \text{ where } \bar{T}_k = \{\bar{x} \in \bar{X}_k : \ H\bar{x}(S\cup S^{-1}\cup\{1\})^{k+1} \subseteq H\bar{X}_k\}. \quad (12)$$

We put $X_k = H\bar{X}_k$ and $T_k = H\bar{T}_k$. Our aim is to define a right action of F on X_k that commutes with the left action of H and such that the (H,F)-sets X_k approximate G.

First for any $s \in S$ and any $\bar{x} \in \bar{X}_k$ we will define an element $\bar{x}\cdot s \in X_k$. If $\bar{x} \in T_k(1\cup S\cup S^{-1})^k$, then $\bar{x}\cdot s = \bar{x}s \in X_k$ is well defined by our conditions. If $\bar{x} \notin T_k(1\cup S\cup S^{-1})^k$ we define $\bar{x}\cdot s \in X_k$ in a such way that the induced action of s on $H\backslash X_k = H\backslash H\bar{X}_k$ is a bijection.

Now, if $x \in X_k$ is an arbitrary element we can write $x = h\bar{x}$ for some $h \in H$ and $\bar{x} \in \bar{X}_k$ and we define $x\cdot s = h(\bar{x}\cdot s)$. Thus, X_k is an (H,F)-set, H acts freely on X_k and $H\backslash X_k$ is finite.

Let $x \in T_k$ and w be a word in S of length $l \leq k$. Arguing by induction on l, we easily obtain that $x\cdot w = xw$. Hence we obtain that X_k approximate G.

In the following theorem we show that the general Lück approximation over $\mathbb{C}$ holds in the previous situation. Our proof is similar to the one of [76, Theorem 6.37].

Theorem 12.1 *Let $A \in \mathrm{Mat}_{n\times m}(\mathbb{C}[F])$. Then*

$$\lim_{k\to\infty} \dim_{X_k} \ker \phi^A_{X_k} = \dim_G \ker \phi^A_G.$$

Proof For simplicity we assume that $A \in \mathbb{C}[F]$. The general case can be proved similarly.

If Z is a subset of G we denote by $Z^c = G\setminus Z$ the complement of Z in G. For subsets $\bar{Y}$, $\bar{Y}_1 \subset \bar{Y}_2$ subsets of $\bar{X}$ we denote by $\mathrm{proj}^{\bar{Y}_2}_{\bar{Y}_1}$ the projection of $l^2(H\bar{Y}_2)$ onto $l^2(H\bar{Y}_1)$.

For every $k \geq 1$ we put $U_k = \ker\phi^G_A \cap l^2((X_k)^c)$, $W_k = \ker\phi^G_A \cap l^2(T_k)$ and $L_k = (U_k \oplus W_k)^\perp \cap \ker\phi^G_A$. Then U_k, W_k and L_k are (left) H-invariant closed subspaces and the following decomposition holds

$$\ker\phi^G_A = U_k \oplus W_k \oplus L_k.$$

Let $k_0 \geq 1$ be such that all the group elements involved in A lie in S^{k_0} and let $k \geq 2k_0$. By the definition of the sets $\bar{T}_k$ (see (12)), we obtain that $(X_k)^c\cdot S^{k_0} \cap T_k\cdot S^{k_0} = \emptyset$, and so,

$$\ker\phi^G_A \cap (l^2((X_k)^c) \oplus l^2(T_k)) = U_k \oplus W_k.$$

Hence the restriction of $\mathrm{proj}^{\bar X}_{\bar X_k\setminus \bar T_k}$ on L_k is injective and so

$$\dim_H \overline{\mathrm{proj}^{\bar X}_{\bar X_k}(L_k)} = \dim_H \overline{\mathrm{proj}^{\bar X}_{\bar X_k\setminus \bar T_k}(L_k)} \le |\bar X_k \setminus \bar T_k|. \tag{13}$$

The definition of the sets $\bar T_k$ also implies that $W_k \le \ker \phi_A^{X_k}$. We put $M_k = (W_k)^\perp \cap \ker \phi_A^{X_k}$. Then M_k is a (left) H-invariant closed subspace and

$$\ker \phi_A^{X_k} = W_k \oplus M_k.$$

Since the restriction of $\mathrm{proj}^{\bar X_k}_{\bar X_k\setminus \bar T_k}$ on M_k is injective, we obtain that

$$\dim_H M_k \le |\bar X_k \setminus \bar T_k|. \tag{14}$$

Observe that on the one hand

$$\dim_G \ker \phi_G^A = \langle \mathrm{proj}_{\ker \phi_G^A}(\mathbf{1}), \mathbf{1}\rangle = \frac{1}{|\bar X_k|}\sum_{g\in \bar X_k} \langle \mathrm{proj}_{\ker \phi_G^A}(g), g\rangle =$$

$$\frac{1}{|\bar X_k|}\sum_{g\in \bar X_k} \langle \mathrm{proj}_{U_k}(g) + \mathrm{proj}_{W_k}(g) + \mathrm{proj}_{L_k}(g), g\rangle =$$

$$\frac{\dim_H W_k + \dim_H \overline{\mathrm{proj}^{\bar X}_{\bar X_k}(L_k)}}{|\bar X_k|}$$

and on the other

$$\dim_{X_k} \ker \phi_{X_k}^A = \frac{1}{|\bar X_k|}\dim_H \ker \phi_{X_k}^A = \frac{1}{|\bar X_k|}\sum_{g\in \bar X_k} \langle \mathrm{proj}_{\ker \phi_{X_k}^A}(g), g\rangle =$$

$$\frac{1}{|\bar X_k|}\sum_{g\in \bar X_k} \langle \mathrm{proj}_{W_k}(g) + \mathrm{proj}_{M_k}(g), g\rangle = \frac{\dim_H W_k + \dim_H M_k}{|\bar X_k|}.$$

Thus, we have that

$$\left|\dim_G \ker \phi_G^A - \dim_{X_k} \ker \phi_{X_k}^A\right| =$$

$$\frac{\left|\dim_H \overline{\mathrm{proj}^{\bar X}_{\bar X_k}(L_k)} - \dim_H M_k\right|}{|\bar X_k|} \overset{\text{by (13) and (14)}}{\le} \frac{|\bar X_k \setminus \bar T_k|}{|\bar X_k|} \le \frac{1}{k}.$$

This finishes the proof of the proposition. □

Corollary 12.2 *Let G be a countable group and let A be a normal subgroup of G such that G/A is amenable. Then rk_G as a Sylvester matrix rank function on $\mathcal{R}_{K[A]} * G/A$ is a natural extension of the Sylvester matrix rank function rk_A on $\mathcal{R}_{K[A]}$.*

Proof By Proposition 5.13, $\mathcal{R}_{K[G]}$ is an epic $*$-regular $K[G]$-ring. Therefore, by Proposition 5.9, rk_G as a Sylvester matrix rank function on $\mathcal{R}_{K[G]}$ is completely determinated by its values on matrices over $K[G]$. Hence, Theorem 12.1 and Proposition 8.3 imply that rk_G as a Sylvester matrix rank function on $\mathcal{R}_{K[A]} * G/A$ is the natural extension of rk_A. □

Corollary 12.3 *Let G be a group such that $\operatorname{lcm}(G) < \infty$ and let H be a normal subgroup of G such that G/H is elementary amenable. Let K be a subfield of $\mathbb{C}$. Assume that for every finite subgroup P/H of G/H, P satisfies the strong Atiyah conjecture over K. Then G satisfies the strong Atiyah conjecture over K.*

Proof Let α be the least ordinal such that $G/H \in \mathcal{E}_\alpha$. We argue by transfinite induction on α as we have done in the proof of Corollary 9.2. We use the notation introduced there. We can assume that $\alpha = \gamma + 1$, and there exists a normal subgroup $A/H \in \mathcal{LE}_\gamma$ of G/H such that $G/A \in \mathcal{B}$.

Notice that if T/A is a finite subgroup of G/A, then $T \in \mathcal{LE}_\gamma$. Hence T satisfies the strong Atiyah conjecture over K. By Proposition 5.9,

$$\langle \operatorname{rk}_T(r) : r \in \mathcal{R}_{K[T]} \rangle \overset{\text{by Proposition 5.9}}{=} \mathcal{A}_K(T) = \frac{1}{\operatorname{lcm}(T)}\mathbb{Z} \le \frac{1}{\operatorname{lcm}(G)}\mathbb{Z}.$$

Thus, Corollary 12.2 and Corollary 9.3 imply that G satisfies the strong Atiyah conjecture over K.

□

12.2 The strong Atiyah conjecture for groups from the class $\mathcal{D}$

The class $\mathcal{D}$ is the smallest non-empty class of groups such that:

1. If G is torsion-free and A is elementary amenable, and we have a projection $p : G \to A$ such that $p^{-1}(E) \in \mathcal{D}$ for every finite subgroup E of A, then $G \in \mathcal{D}$.
2. $\mathcal{D}$ is subgroup closed.
3. Let $G_i \in \mathcal{D}$ be a directed system of groups and G its (direct or inverse) limit. Then $G \in \mathcal{D}$.

Theorem 12.4 *[26, 56] Let G be a group from the class $\mathcal{D}$. Then G satisfies the strong Atiyah conjecture over $\mathbb{C}$.*

Proof For ordinals α define the class of groups $\mathcal{D}_\alpha$ as follows:

1. $\mathcal{D}_0$ is the class of torsion-free elementary amenable groups.
2. $\mathcal{D}_{\alpha+1}$ is the class of groups G such that G is a subgroup of a direct or inverse limit of groups $G_i \in \mathcal{D}_\alpha$ or G is torsion-free and there exists an elementary amenable group T, and a map $p : G \to T$ such that $p^{-1}(E) \in \mathcal{D}_\alpha$ for every finite subgroup E of T.
3. $\mathcal{D}_\beta = \cup_{\alpha<\beta}\mathcal{D}_\alpha$, when β is a limit ordinal.

Clearly $\mathcal{D} = \cup \mathcal{D}_\alpha$. We prove the theorem using the transfinite induction. Let α be the smallest ordinal such that $G \in \mathcal{D}_\alpha$. If $\alpha = 0$, G satisfies the strong Atiyah conjecture over $\mathbb{C}$ by Corollary 12.3.

Now, assume that $\alpha \neq 0$. Since α is not a limit ordinal, there exists γ such that $\alpha = \gamma + 1$.

Assume first that G is a direct or inverse limit of groups $G_i \in \mathcal{D}_\gamma$. Without loss of generality, we may assume that G is generated by $d < \infty$ elements. Thus, we can choose G_i such that G is a limit of G_i in the space of marked d-generated groups. Observe that G_i are sofic. Thus, applying the Lück approximation in the space of marked sofic groups, which follows from Theorem 10.1, we obtain that G satisfies the strong Atiyah conjecture over $\mathbb{C}$, because the groups G_i do.

Now, let us assume that G is torsion-free and there exists an elementary amenable group T, and a map $p : G \to T$ such that $p^{-1}(E) \in \mathcal{D}_\gamma$ for every finite subgroup E of T. Since the groups in $\mathcal{D}_\gamma$ satisfy the strong Atiyah conjecture over $\mathbb{C}$, G also does by Corollary 12.3. $\square$

The proof of Theorem 12.4 for a smaller class $\mathcal{C}$, which includes free or surface groups, was given first in [68]. The argument did not use the Lück approximation. The proof of the free group case used the theory of Fredholm operators. The idea of using the Lück approximation in the Atiyah conjecture appeared first in a paper of T. Schick [104].

12.3 The base change and the strong Atiyah conjecture

Let K be a subfield of $\mathbb{C}$ and let G be a group. If $\mathrm{lcm}(G) < \infty$, then strong Atiyah conjecture predicts that the group $\mathcal{A}_K(G)$ does not depend on K. In fact, we do not know whether the condition $\mathrm{lcm}(G) < \infty$ is necessary.

Question 12.5 Let G be a group. Is it true that $\mathcal{A}_{\mathbb{C}}(G) = \mathcal{A}_{\mathbb{Q}}(G)$? If G is countable, is it true that $\mathcal{A}_{\mathbb{C}}(G)$ is countable?

We can answer the previous question in the case G is sofic and $\mathcal{A}_{\bar{\mathbb{Q}}}(G)$ is finitely generated.

Theorem 12.6 *([56]) Let G be a sofic group and assume that there exists $n \in \mathbb{N}$ such that $\mathcal{A}_{\bar{\mathbb{Q}}}(G) \le \frac{1}{n}\mathbb{Z}$. Then $\mathcal{A}_{\mathbb{C}}(G) = \mathcal{A}_{\bar{\mathbb{Q}}}(G)$. In particular, if G satisfies the strong Atiyah conjecture over $\bar{\mathbb{Q}}$, then it does over $\mathbb{C}$.*

Proof Let us give an idea of the proof. Since $\mathcal{A}_{\bar{\mathbb{Q}}}(G) \le \frac{1}{n}\mathbb{Z}$, $\mathcal{R}_{\bar{\mathbb{Q}}[G]}$ is semisimple Artinian. Hence there are division rings $D_1, \ldots, D_k$ and natural numbers $n_1 \ldots, n_k$ such that

$$\mathcal{R}_{\bar{\mathbb{Q}}[G]} \cong \mathrm{Mat}_{n_1}(D_1) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(D_k).$$

Using results of Section 10 from [56] we can show that

$$\mathcal{R}_{\mathbb{C}[G]} \cong \mathrm{Mat}_{n_1}(E_1) \oplus \ldots \oplus \mathrm{Mat}_{n_k}(E_k),$$

where E_i is the division algebra isomorphic to the classical Ore ring of fractions of $D_i \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$. This implies that $\mathcal{A}_{\mathbb{C}}(G) = \mathcal{A}_{\bar{\mathbb{Q}}}(G)$. $\square$

12.4 The strong Atiyah conjecture for torsion-free p-adic pro-p groups

Using Proposition 11.5, we can show the strong Atiyah conjecture over $\mathbb{C}$ for torsion-free compact p-adic groups.

Theorem 12.7 *[39, 56] Let G be a torsion-free p-adic analytic group. Then G satisfies the strong Atiyah conjecture over $\mathbb{C}$.*

Proof We want to show that $\mathrm{rk}_G(A)$ is an integer number. There exists a finitely generated subfield K of $\mathbb{C}$ such that A is a matrix over K. We embed K in a p-adic field F. Let R be the ring of integers of F. Since there exists $a \in K$ such that aA is a matrix over R, without loss of generality we may assume that A is a matrix over R. Now applying the Lück approximation (Theorem 10.1), we obtain that $\mathrm{rk}_G(A) = \mathrm{rk}_{R[[G]]}(A)$. Hence by Proposition 11.5, $\mathrm{rk}_G(A) \in \mathbb{Z}$. This implies the strong Atiyah conjecture for G. $\square$

Therefore, from Proposition 11.4 and Theorem 12.7 we obtain that a finitely generated group, which is linear over a field of characteristic 0, contains a torsion-free group satisfying the strong Atiyah conjecture over $\mathbb{C}$. Thus, it seems natural to investigate the following problem.

Question 12.8 Do groups, which are linear over a field of characteristic 0, satisfy the strong Atiyah conjecture?

12.5 Finite extensions and the strong Atiyah conjecture

It is clear that if G satisfies the strong Atiyah conjecture over a field K, then a subgroup H of G, satisfying $\mathrm{lcm}(H) = \mathrm{lcm}(G)$, does. The question whether the strong Atiyah conjecture holds for a group G if it holds for a subgroup of finite index is a very delicate one and it is wide open (a particular case is Question 12.8). Some progress in the solution of this question was obtained first by P. Linnell and T. Schick in [71] and later using similar ideas by different authors in [15, 70, 106]. The main technical idea is to reduce the problem to previously established results on extensions with certain amenable quotients.

Theorem 12.9 *[71, 106, 56] Let G have a normal subgroup H such that G/H is elementary amenable and such that H is one of the following groups:*

1. *H is an iterated semi-direct product of finitely generated free groups, surface groups, primitive one-relator groups, knot groups and primitive link groups and such that the quotient always acts trivially on the abelianization of the kernel.*
2. *H is a cocompact special group.*

Then G satisfies the strong Atiyah conjecture over $\mathbb{C}$.

Proof In view of Corollary 12.3 we may assume that G/H is finite. Applying the results of [71] in the first case and of [106] in the second we obtain that G contains a torsion-free subgroup $N \leq H$, which is normal in G, such that $\mathrm{lcm}(G) = \mathrm{lcm}(G/N)$ and G/N is elementary amenable. Now H, and so N, are from the class $\mathcal{D}$. Hence by Theorem 12.4, N satisfies the strong Atiyah conjecture over $\mathbb{C}$. Arguing as in the proof of Corollary 12.3, we obtain that G also satisfies the strong Atiyah conjecture over $\mathbb{C}$.

$\square$

Note that hyperbolic 3-manifold groups are virtually cocompact special groups ([4, 114]) and the full Artin braid groups contain the pure braid groups, which are isomorphic to iterated semi-direct products of finitely generated free groups.

13 Applications and motivations

There are many applications of L^2-Betti numbers in Algebra, Topology and Geometry. We discuss several of them, mostly those which are close to our research. But before we mention briefly some other applications which we do not consider in this survey.

The Hopf and the Singer conjectures are discussed by M. Gromov [51], W. Lück [78, Chapter 11] and R. Sauer and A. Thom [102]. An application to the Baum-Connes conjecture is studied by W. Lück in [77]. Relation between the first L^2-Betti numbers and acylindrical hyperbolicity of groups is investigated by D. Osin [93]. Also D. Osin [92] constructed the first examples of finitely generated, non-unitarizable groups without nonabelian free subgroups using L^2-Betti numbers.

13.1 The growth of Betti numbers in covers of a CW-complex

Let X be a CW-complex on which acts freely a group G. We assume that the quotient CW-complex X/G is of finite type i.e., there are only finitely many cells of every given dimension. Let $G > G_1 > G_2 > \dots$ be a descending chain of subgroups such that G_i is normal in G, the index $|G : G_i|$ is finite and $\cap_{i\geq 1} G_i = \{1\}$. We put $X_i = X/G_i$. Then X_i is a normal cover of X. We may ask the following natural questions.

Question 13.1 Let $K = \mathbb{Q}$ or $\mathbb{F}_p$ and let $p \in \mathbb{N}$.

1. How do the normalized pth Betti numbers over K $\{\frac{b_p(X_i,K)}{|G:G_i|}\}$ grow? Is there $\lim\limits_{i\to\infty} \dfrac{b_p(X_i,K)}{|G:G_i|}$?
2. How does the growth of the numbers $\{\frac{b_p(X_i,K)}{|G:G_i|}\}$ depend on the choice of the chain $G > G_1 > G_2 > \dots$?

Let us show that these two questions are reformulations of the first two questions of Question 1.1. Consider the cellular chain complex of X

$$\mathcal{C}(X): \quad \dots \mathbb{Z}[\mathcal{C}_{p+1}(X)] \overset{\partial_{p+1}}{\to} \mathbb{Z}[\mathcal{C}_p(X)] \overset{\partial_p}{\to} \mathbb{Z}[\mathcal{C}_{p-1}(X)] \dots \to \mathbb{Z} \to 0.$$

Since G acts freely and X/G is of finite type, we obtain that $\mathbb{Z}[\mathcal{C}_p(X)]$ is a free $\mathbb{Z}[G]$-module of finite rank and the connected morphisms ∂_p are represented by a multiplication by a matrix over $\mathbb{Z}[G]$. Hence we obtain the following representation of $\mathcal{C}(X)$

$$\mathcal{C}(X): \quad \dots\dots \mathbb{Z}[G]^{n_{p+1}} \overset{\times A_{p+1}}{\to} \mathbb{Z}[G]^{n_p} \overset{\times A_p}{\to} \mathbb{Z}[G]^{n_{p-1}} \dots \to \mathbb{Z} \to 0.$$

Therefore the normalized pth Betti numbers over K $\{\frac{b_p(X_i,K)}{|G:G_i|}\}$ can be computed as

$$\frac{b_p(X_i,K)}{|G:G_i|}=\frac{\dim_K \mathrm{H}_p(G_i\backslash\mathcal{C},K)}{|G:G_i|}=n_p-(\mathrm{rk}_{G/G_i}(A_p)+\mathrm{rk}_{G/G_i}(A_{p+1})).$$

Thus, the answer to Question 13.1 is known when $K=\mathbb{Q}$ (Theorem 10.1) and when $K=\mathbb{F}_p$ and G is amenable (Theorem 7.1). Moreover, in the case $K=\mathbb{Q}$ the limit $\lim_{i\to\infty}\frac{b_p(X_i,K)}{|G:G_i|}$ is equal to the pth L^2-Betti number of X:

$$b_p^{(2)}(X)=\dim_G \mathrm{H}_p(X,\mathcal{R}_{\mathbb{Q}[G]})=\dim_G \overline{\mathrm{H}}_p(X,l^2(G)).$$

13.2 A characterization of amenable groups G

When G is amenable we have seen in Theorem 7.1 that the Sylvester module rank function $\dim_G$ on $K[G]$ is exact. This implies the following result proved first by M. Cheeger and M. Gromov [18].

Theorem 13.2 *[18] Let X be an aspherical CW-complex and let an amenable group G acts freely on X. Then $b_p^{(2)}(X)=0$ for any $p\ge 1$.*

In [78, Conjecture 6.8] W. Lück conjectured that the property proved in the theorem characterizes amenable groups. This has been confirmed recently by L. Bartholdi [11]. In fact, his result implies the following elegant characterization of amenable groups.

Theorem 13.3 *[11] Let G be a group and let K be a field. Then G is amenable if and only if $K[G]$ has an exact Sylvester module rank function.*

Proof If G is amenable then $\dim_G$ is exact.

If G is not amenable, then L. Bartholdi [11] proved that there exists $n\in\mathbb{N}$ such that $K[G]^{n+1}$ is isomorphic to a submodule of $K[G]^n$. This clearly implies that $K[G]$ does not have an exact Sylvester module rank function.

□

13.3 The growth of the first $\mathbb{F}_p$-Betti numbers of subgroups of finite index of a finitely presented group

A particular case of the situation described in Subsection 13.1 is the study of the growth of the first Betti numbers of subgroups of finite index of a finitely presented group.

Conjecture 13.4 Let G be a finitely presented group and let $G>G_1>G_2>\dots$ be a descending chain of subgroups such that G_i is normal in G, the index $|G:G_i|$ is finite and $\cap_{i\ge1}G_i=\{1\}$. Then $\lim_{i\to\infty}\frac{b_1(G_i,\mathbb{F}_p)}{|G:G_i|}$ exists and, moreover,

$$\lim_{i\to\infty}\frac{b_1(G_i,\mathbb{F}_p)}{|G:G_i|}=b_1^{(2)}(G).$$

The number $b_1^{(2)}(G)$ is the first L^2-Betti number of G and it is defined as

$$b_1^{(2)}(G) = \dim_G \mathrm{H}_1(G, \mathcal{R}_{\mathbb{Q}[G]}) = \dim_G \overline{\mathrm{H}}_1(G, l^2(G)).$$

Conjecture 13.4 is discussed, for example in [38]. It is related to another interesting problem. Recall that $d(G)$ denotes the minimal number of generators of G.

Conjecture 13.5 Let G be a finitely presented group and let $G > G_1 > G_2 > \dots$ be a descending chain of subgroups such that G_i is normal in G, the index $|G : G_i|$ is finite and $\cap_{i\geq 1} G_i = \{1\}$. Then

$$\lim_{i\to\infty} \frac{d(G_i)}{|G : G_i|} = b_1^{(2)}(G).$$

We will not say much about this conjecture and recommend to look at the following paper of M. Abert and N. Nikolov [1].

A very interesting particular case of Conjecture 13.4 (proposed by F. Calegari and M. Emerton in [16]) arises when G is a lattice in $\mathrm{SL}_2(\mathbb{C})$ and $\{G_i\}$ is a p-adic chain, i.e. the completion of G with respect to $\{G_i\}$ is a p-adic group. The interest of F. Calegari and M. Emerton in this question was motivated by questions in the theory of automorphic forms [17]. Another motivation comes from the paper [49], where it is shown that if the Calegari-Emerton conjecture holds then the congruence kernel of any arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ is a projective profinite group.

13.4 Kaplansky's conjectures about group algebras

Let G be a group and K a field. I. Kaplansky proposed several conjectures about the group ring $K[G]$.

We say that a ring R is **directly finite** if $xy = 1$ in R implies that $yx = 1$ as well. Kaplansky's direct finiteness conjecture states that the group ring $K[G]$ is directly finite (see [28] for more details about this problem). The following observation provides a large source of rings which are directly finite.

Proposition 13.6 *Assume that an algebra R has a faithful Sylvester matrix rank function. Then $\mathrm{Mat}_n(R)$ is directly finite for all n.*

Proof We will prove the statement for R, the same proof works for $\mathrm{Mat}_n(R)$. Let rk be a faithful Sylvester matrix rank function on R. Assume $xy = 1$. In particular, $\mathrm{rk}(x) = 1$, Then, by Proposition 5.1(2),

$$0 = \mathrm{rk}(x(yx - 1)) \geq \mathrm{rk}(yx - 1).$$

Since rk is faithful, $yx = 1$.

□

Corollary 13.7 *[60, 34] Let G be a group and K a field. Assume that K is of characteristic 0 or K is of positive characteristic and G is sofic. Then $\mathrm{Mat}_n(K[G])$ is directly finite for all n.*

Proof Clearly, without loss of generality, we can assume that K and G are finitely generated. Therefore, K is a subfield of $\mathbb{C}$ if the characteristic of K is zero. The Sylvester matrix rank function rk_G on $K[G]$ is faithful, and so, we can apply Proposition 13.6.

Assume now that K is of positive characteristic and G is sofic. Represent G as $G = F/N$, where F is a finitely generated free group. Let $\{X_k\}$ be a family of finite F-sets approximating G. Fix a non-principal filter ω on $\mathbb{N}$ and let $\mathrm{rk}_\omega = \lim_\omega \mathrm{rk}_{X_i} \in \mathbb{P}(K[F])$. Since $\mathrm{rk}_\omega(g-1) = 0$ for every $g \in N$, rk_ω is also a Sylvester matrix rank function on $K[F/N] = K[G]$. In order to apply Proposition 13.6 we have to show that rk_ω is faithful on $K[G]$.

Let $a = \sum_{i=1}^l a_i g_i \in K[G]$ $(0 \neq a_i \in K, g_i \in G)$ be such that all g_i are different. Consider $f_i \in F$ such that $g_i = f_i N$. Put $A = \sum_{i=1}^l a_i f_i \in K[F]$.

Since $\{X_i\}$ approximate F/N, for any $\epsilon > 0$ there is $n \in \mathbb{N}$ such that for every $j \geq n$ there exists a subset L_j of X_j of size at least $(1-\epsilon)|X_j|$ such that $xf_i \neq xf_1$ for every $x \in L_j$ if $i \neq 1$.

Now let us construct inductively a subset $\{x_1, x_2, \ldots, x_m\}$ of L_j, where $m = \left[\frac{|L_j|}{l}\right]$. Let x_1 be any element of L_j. Assume we have constructed $x_1, \ldots, x_t$. Then take any

$$x_{t+1} \in L_j \setminus \cup_{i=1}^l \{x_1, \ldots, x_t\} f_i f_1^{-1}.$$

Then

$$x_{t+1} f_1 \notin \cup_{i=1}^l \{x_1, \ldots, x_t\} f_i \cup_{i=2}^l x_{t+1} f_i. \tag{15}$$

From(15) we obtain that $(\sum_{i=1}^{t+1} \alpha_i x_t) A \neq 0$ if $\alpha_{t+1} \neq 0$. Therefore,

$$\dim_K (\sum_{i=1}^m K x_i) A = m.$$

Hence,

$$\mathrm{rk}_\omega(a) = \mathrm{rk}_\omega(A) = \lim_\omega \mathrm{rk}_{X_i}(A) \geq \frac{1}{l}.$$

□

Now, assume that G is torsion-free. Kaplansky's zero divisor conjecture states that $K[G]$ does not contain non-trivial zero divisors, that is, it is a domain. In view of Proposition 5.1(2), an algebra, having a faithful Sylvester matrix rank function taking only integer values, is a domain. Thus, the strong Atiyah conjecture over K implies Kaplansky's zero divisor conjecture for $K[G]$. There are cases of groups G where we know Kaplansky's zero divisor conjecture for $\mathbb{C}[G]$, but we still do not know the strong Atiyah conjecture for G. This is the case of one-relator groups without torsion.

Problem 13.8 Show that the strong Atiyah conjecture holds for one-relator groups without torsion.

By a result of D. Wise [114], the one-relator groups with torsion are virtually cocompact special. Hence in this case the strong Atiyah conjecture follows from Theorem 12.9.

13.5 The Hanna Neumann conjecture

Let F be a free group and U and W two finitely generated subgroups of F. In 1954, A. G. Howson [54] showed that the intersection of U and W is finitely generated. Three years later H. Neumann [88] improved the Howson bound and proved that

$$\overline{d}(U \cap W) \leq 2\overline{d}(U)\overline{d}(W) \text{ where } \overline{d}(U) = \max\{d(U) - 1, 0\}.$$

She also conjectured that the factor of 2 in the above inequality is not necessary and that one always has

$$\overline{d}(U \cap W) \leq \overline{d}(U)\overline{d}(W).$$

This statement became known as the Hanna Neumann conjecture. It received a lot of attention since then.

In 1990, W. D. Neumann [89] conjectured that, in fact, the following inequality holds

$$\sum_{x \in U\backslash F/W} \overline{d}(U \cap xWx^{-1}) \leq \overline{d}(U)\overline{d}(W).$$

This conjecture received the name of the strengthened Hanna Neumann conjecture. It was proved independently by J. Friedman [40] and I. Mineyev [85] in 2011. Later W. Dicks gave a simplification for both proofs (see [21, 40]).

In [55] A. Jaikin-Zapirain gave a new proof of the strengthened Hanna Neumann conjecture. This new approach used the Lück approximation and the strong Atiyah conjecture for free groups. Recently Y. Antolin and A. Jaikin-Zapirain have been able to extend this new approach to non-abelian surface groups.

Theorem 13.9 *[5] Let G be a non-abelian surface group. Then for any finitely generated subgroups U and W of G*

$$\sum_{x \in U\backslash F/W} \overline{d}(U \cap xWx^{-1}) \leq \overline{d}(U)\overline{d}(W).$$

The strengthened Hanna Neumann conjecture can be also formulated for pro-p groups. The only difference is that we now consider closed subgroups U and W and $d(U)$ means the number of profinite generators of U. The Howson property for free pro-p groups was proved by A. Lubotzky [74] and the strengthened Hanna Neumann conjecture by A. Jaikin-Zapirain [55]. Again the proof of [55] uses in an essential way the pro-p analogue of the strong Atiyah conjecture for free pro-p-groups (Corollary 11.6).

Demushkin pro-p groups are Poincare duality pro-p groups of cohomological dimension 2 and can be seen as pro-p analogues of discrete surface groups. Applying the strategy developed in [55], A. Jaikin-Zapirain and M. Shusterman have proved in [58] the strengthened Hanna Neumann conjecture for non-solvable Demuskin pro-p groups.

Theorem 13.10 *[58] Let G be a non-solvable Demushkin pro-p group. Then for any closed finitely generated subgroups U and W of G*

$$\sum_{x \in U\backslash F/W} \overline{d}(U \cap xWx^{-1}) \leq \overline{d}(U)\overline{d}(W).$$

An important step of the proof of the previous theorem is to show that Conjecture 11.1(3) holds for Demushkin pro-p groups over $\mathbb{F}_p$.

13.6 J.-P. Serre's problem on torsion-free one-relator pro-p groups

R. Lyndon [82] proved that if a discrete group G is defined by a single relation $r = 1$, and r is not a power of an element in the free discrete group, then G is of cohomological dimension 2. J.-P. Serre asked whether the analogous statement holds for pro-p groups. D. Gildenhuys [41] found an easy counterexample and reformulated the Serre questions as the following conjecture.

Conjecture 13.11 Let G be a finitely generated one-relator torsion-free pro-p group. Then G is of cohomological dimension 2.

Proposition 13.12 *Let G be a finitely generated one-relator torsion-free pro-p group. Then Conjecture 11.1(3) over $\mathbb{F}_p$ implies Conjecture 13.11.*

Proof Recall that Conjecture 11.1 (1) and (2) hold over $\mathbb{F}_p$ by Proposition 11.2. Assume, in addition, that Conjecture 11.1 (3) holds over $\mathbb{F}_p$ for G.

Let I_G be the augmentation ideal of $\mathbb{F}_p[[G]]$ and R_G the relation module of G. Since G is one-relator group, R_G is generated by 1 element. We want to show that $R_G \cong \mathbb{F}_p[[G]]$. This would imply that G is of cohomological dimension 2.

Consider the exact sequence

$$0 \to R_G \to \mathbb{F}_p[[G]]^d \to I_G \to 0 \ (d = d(G)).$$

Since $R_G \neq \{0\}$, I_G is a proper quotient of $\mathbb{F}_p[[G]]^d$. Hence, by Proposition 11.3, $\dim_{\mathbb{F}_p[[G]]} I_G < d$. Since $\dim_{\mathbb{F}_p[[G]]}$ is an integer, $\dim_{\mathbb{F}_p[[G]]} I_G \leq d-1$. Thus

$$\dim_{\mathbb{F}_p[[G]]} R_G \geq \dim_{\mathbb{F}_p[[G]]} \mathbb{F}_p[[G]]^d - \dim_{\mathbb{F}_p[[G]]} I_G \geq d - (d-1) = 1.$$

Since R_G is a quotient of $\mathbb{F}_p[[G]]$, applying again Proposition 11.3, we conclude that $R_G \cong \mathbb{F}_p[[G]]$. $\square$

13.7 Properties of the operators in $\overline{\mathcal{R}_{K[G]}}$

Let K be a subfield of $\mathbb{C}$ and G a group. Recall that $\overline{\mathcal{R}_{K[G]}}$ has been defined as the completion of $\mathcal{R}_{K[G]}$ with respect to the rk_G-metric. If G is countable, we will identify $\overline{\mathcal{R}_{K[G]}}$ with the closure of $\mathcal{R}_{K[G]}$ in $\mathcal{U}(G)$ with respect to the rk_G-metric. In this subsection we discuss the properties of $\overline{\mathcal{R}_{K[G]}}$ and of its elements.

In Subsection 10.5 we have already mentioned the strong eigenvalue conjecture (Conjecture 10.12) which can be stated not only for the operators in $\mathcal{R}_{K[G]}$ but also for the operators in $\overline{\mathcal{R}_{K[G]}}$ (in fact, this two variations are equivalent). This conjecture was proved in [56] for sofic groups (see also [26] and [110][1] for previous results on this problem).

[1] We warm the reader that the proof of [110, Theorem 4.2] contains a gap, and so, the theorem holds only when $A = A^*$. This gap affects also [110, Theorem 4.3].

Recall that by Proposition 5.6, if G is an ICC countable group, then $Z(\overline{\mathcal{R}_{K[G]}})$ is a subfield of $\mathbb{C}$. Another consequence of Theorem 10.1 is the following result.

Corollary 13.13 *[56] Let K be a subfield of $\mathbb{C}$ closed under complex conjugation and let G be a countable sofic group. Then*

$$\overline{\mathcal{R}_{K[G]}} \cap \mathbb{C} = K.$$

In particular, if G is an ICC group, then $Z(\overline{\mathcal{R}_{K[G]}}) = K$.

For a division $*$-ring D, denote by $\mathcal{M}_D$ the completion of the direct limit $\varinjlim \mathrm{Mat}_{2^n}(D)$ with respect to the metric induced by its unique Sylvester matrix rank function. If G is ICC, all the known examples of $\overline{\mathcal{R}_{K[G]}}$ are either isomorphic to $\mathrm{Mat}_n(D)$ or to $\mathcal{M}_D$ for some division $*$-ring D. Moreover, in [33] G. Elek has shown that if H is countable and amenable, then $\overline{\mathcal{R}_{\mathbb{C}[C_2 \wr H]}}$ is isomorphic to $\mathcal{M}_{\mathbb{C}}$. It seems that his proof can be adapted to show that $\overline{\mathcal{R}_{K[C_2 \wr H]}}$ is isomorphic to $\mathcal{M}_K$ for any subfield K of $\mathbb{C}$. Interesting related results have been proved in [7] by P. Ara and J. Claramunt. All this together suggests the following question.

Question 13.14 Let G be an ICC group. Is it true that $\overline{\mathcal{R}_{K[G]}}$ is either isomorphic to $\mathrm{Mat}_n(D)$ or to $\mathcal{M}_D$ for some division $*$-ring D?

The next application of Theorem 10.1 shows that the von Neumann rank of a matrix $A \in \mathrm{Mat}_{n\times m}(K[G])$ does not depend on the embedding of K into $\mathbb{C}$ if G is sofic.

Corollary 13.15 *[56] Let G be a sofic group. Let K be a field and let $\phi_1, \phi_2 : K \to \mathbb{C}$ be two embeddings of K into $\mathbb{C}$. Then for every matrix $A \in \mathrm{Mat}_{n\times m}(K[G])$*

$$\mathrm{rk}_G(\phi_1(A)) = \mathrm{rk}_G(\phi_2(A)).$$

Clearly we expect that Corollaries 13.13 and 13.15 are still valid without the assumption that G is sofic.

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