# Omega subgroups of pro-$p$ groups *

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**Abstract**

Let $G$ be a pro-$p$ group and let $k \geq 1$. If $\gamma_{k(p-1)}(G) \leq \gamma_r(G)^{p^s}$ for some $r$ and $s$ such that $k(p-1) < r + s(p-1)$, we prove that the exponent of $\Omega_i(G)$ is at most $p^{i+k-1}$ for all $i$.

## 1 Introduction

The power structure of finite $p$-groups and pro-$p$ groups is analyzed with the help of two series of characteristic subgroups: the *power subgroups* $G^{p^i}$, gen-

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erated by all $p^i$-th powers of elements of $G$, and the *omega subgroups* $\Omega_i(G)$, generated by all elements $x \in G$ such that $x^{p^i} = 1$. In some circumstances, these sets of generators already form a subgroup, in other words, every element of $G^{p^i}$ is a $p^i$-th power and the exponent of $\Omega_i(G)$ is at most $p^i$. This is the case of regular finite $p$-groups, as P. Hall showed in his pioneering work [8], and of potent pro-$p$ groups for odd $p$, as proved by Arganbright [1] for power subgroups and by González-Sánchez and Jaikin-Zapirain [6] for omega subgroups. Recall that for $p > 2$ a pro-$p$ group is *potent* if $\gamma_{p-1}(G) \le G^p$.

However, it is possible that an element of $G^{p^i}$ is not even a $p$-th power and that $\Omega_i(G)$ has arbitrarily large exponent, in fact $\Omega_i(G)$ may have elements of infinite order in a pro-$p$ group. For example, if $P$ is the unique pro-$p$ group of maximal class then $\Omega_1(P) = P$, but $P$ has a torsion-free maximal subgroup. Furthermore, for every $k \ge 1$ the quotient $Q = P/\gamma_{k(p-1)+2}(P)$ is a finite $p$-group and the exponent of $\Omega_1(Q)$ is $p^{k+1}$. Thus it seems reasonable to ask for conditions which guarantee that $G^{p^i}$ and $\Omega_i(G)$ are not far from the situation of the previous paragraph, in the sense that for some fixed $\ell$ every element of $G^{p^i}$ is a $p^{i-\ell}$-th power for $i \ge \ell$ or that the exponent of $\Omega_i(G)$ is at most $p^{i+\ell}$ for all $i$. The kind of conditions we are thinking of are inclusions of a certain power-commutator subgroup (that is, a subgroup formed by taking commutators and powers in any order) in another power-commutator subgroup, such as $\gamma_{p-1}(G) \le G^p$ or $\gamma_{2p-1}(G) \le (G')^{p^2}$. We use the term *power-commutator condition* to refer to a condition of this type.

For power subgroups, González-Sánchez and Wilson [7] have proved that if $\gamma_{p^k}(G) \le \gamma_r(G)^{p^s}$ with $p^k < rp^s$ then the elements of $G^{p^i}$ are $p^{i-(k-1)}$-st powers. In particular, this holds if $\gamma_{p^k}(G) \le G^{p^{k+1}}$. In the dual case of omega subgroups, Easterfield [3] showed that the condition $\gamma_{k(p-1)+1}(G) = 1$ implies that the exponent of $\Omega_i(G)$ is at most $p^{i+k-1}$ for all $i$. On the other hand, if $\gamma_{k(p-1)}(G) \le G^{p^k}$ and $p$ is odd, Wilson [14] has proved that $\exp \Omega_1(G) \le p^k$. Furthermore, if $\lambda = \lceil \log_p k \rceil$ then he also obtains that $\exp \Omega_i(G) \le p^{s(i)}$, where

$$s(i) = \begin{cases} k + \displaystyle\sum_{j=0}^{i-2} \left\lceil \frac{k - p^j}{p^j(p-1)} \right\rceil, & \text{if } 2 \le i \le \lambda, \\ i - \lambda + s(\lambda), & \text{if } i > \lambda. \end{cases}$$

Note however that

$$s(i) \ge i + k - 1 + \sum_{j=1}^{r-1} \frac{p^{\lambda-j} - 1}{p - 1}, \qquad \text{where } r = \min\{i, \lambda\},$$

so Wilson's bound for the exponent of the omega subgroups can be much bigger than Easterfield's bound. In the main result of this paper, we prove that, under Wilson's assumption, Easterfield's bound is valid for every $i$ and for every prime $p$, including in particular the case $p = 2$.

**Main Theorem.** *Let $G$ be a pro-$p$ group and let $k \geq 1$. If $\gamma_{k(p-1)}(G) \leq \gamma_r(G)^{p^s}$ for some $r$ and $s$ such that $k(p-1) < r + s(p-1)$, then the exponent of $\Omega_i(G)$ is at most $p^{i+k-1}$ for all $i$. In particular, $\exp \Omega_i(G) \leq p^{i+k-1}$ if $\gamma_{k(p-1)}(G) \leq G^{p^k}$.*

One could ask whether it is possible to obtain the same conclusion of this theorem by imposing a less restrictive power-commutator condition, but the answer is negative:

(i) The subgroup in the right-hand side of the power-commutator condition cannot be made larger, in the following sense: if $\gamma_{k(p-1)}(G) \leq \gamma_r(G)^{p^s}$ for some $r$ and $s$ such that $k(p-1) = r + s(p-1)$, then the result is false, even if we ask that the inclusion holds for *every* $r$ and $s$ satisfying $k(p-1) = r + s(p-1)$. To see this, consider the infinite pro-$p$ group of maximal class.

(ii) If we want a smaller subgroup of the lower central series in the left-hand side of the power-commutator condition, we cannot obtain anything better than Easterfield's result. Indeed, if $V$ is a set of words such that the condition $\gamma_{k(p-1)+1}(G) \leq V(G)$ implies that $\exp \Omega_1(G) \leq p^k$ then necessarily $V(G) = 1$, and therefore $\gamma_{k(p-1)+1}(G) = 1$. (See Corollary 4.4 below.) Thus the condition $\gamma_{k(p-1)+1}(G) \leq V(G)$ does not bound the exponent of the omega subgroups for more groups than Easterfield's condition does.

In order to prove our main theorem, we introduce the new concept of *potent filtration* of a pro-$p$ group $G$, which is a central series $\{N_i\}_{i\in\mathbb{N}}$ of subgroups with trivial intersection and with the property that $[N_i, G, \overset{p-1}{\ldots}, G] \leq N_{i+1}^p$. If $G$ has a potent filtration with $N_1 = G$, we say that $G$ is a *PF-group*. We think that these concepts may be an important tool in the study of pro-$p$ groups, with applications to other problems apart from the one considered in this paper. It turns out that PF-groups are closely related to Lazard's $p$-saturable pro-$p$ groups, which are fundamental in his seminal paper on $p$-adic analytic groups [11]. More precisely, as proved in [5], $p$-saturable pro-$p$ groups are a particular type of PF-groups, namely those finitely generated and torsion-free.

Let us finally describe the structure of this paper. We devote Section 2 to the power-commutator calculus with closed normal subgroups of pro-$p$ groups, obtaining formulas that will be used extensively throughout the paper. In Section 3 we establish the main properties of potent filtrations and

PF-groups. Then we set this machinery to work and prove our main theorem in Section 4.

*Notation.* We use standard notation in group theory. If $G$ is a pro-$p$ group then all subgroups of $G$ considered will be understood in a topological sense, so when we write a subgroup generated by a subset of $G$, a verbal subgroup of $G$, etc., we will always mean the topological closure of the corresponding abstract subgroup. In particular, we have to modify slightly the definition of power subgroups and omega subgroups given at the beginning of this introduction in the pro-$p$ setting. Also, every power-commutator condition in a pro-$p$ group is understood in a topological sense. Note however that if an inclusion $V(G) \leq W(G)$ between two abstract verbal subgroups holds in a pro-$p$ group $G$, then it is also satisfied in all finite quotients of $G$, and consequently it also holds in $G$ between the corresponding topological verbal subgroups.

*Remark.* Part of the results of this paper are included in the Ph.D. thesis of the second author [4].

# 2 Power-commutator calculus

In this preliminary section, we collect a number of formulas that express general relations holding between commutators and powers of normal subgroups in a finite $p$-group or, equivalently, of closed normal subgroups in a pro-$p$ group. All these formulas are a consequence of one single result, the so-called *Philip Hall's collection formula*. If $F$ is the free group freely generated by two symbols $x$ and $y$, Hall's formula states that there exist words $c_i(x,y) \in \gamma_i(F)$ such that

$$(xy)^n = x^n y^n c_2(x,y)^{\binom{n}{2}} c_3(x,y)^{\binom{n}{3}} \dots c_{n-1}(x,y)^{\binom{n}{n-1}} c_n(x,y) \tag{1}$$

for all $n \in \mathbb{N}$. (See Theorem 9.4 in Chapter III of [9].) It follows that this formula holds generally for any elements $x$, $y$ of any group $G$, with every $c_i(x,y)$ lying in $\gamma_i(\langle x,y \rangle)$. Recalling that the binomial coefficient $\binom{p^k}{i}$ is divisible by $p^{k-j}$ for $p^j \leq i < p^{j+1}$, we get the following result.

**Theorem 2.1.** *Let $G$ be a group and let $x$, $y$ be two elements of $G$. Then, for all $k \geq 0$,*

$$(xy)^{p^k} \equiv x^{p^k} y^{p^k} \pmod{\gamma_2(H)^{p^k} \gamma_p(H)^{p^{k-1}} \gamma_{p^2}(H)^{p^{k-2}} \dots \gamma_{p^k}(H)}, \tag{2}$$

*where $H = \langle x,y \rangle$, and*

$$[x,y]^{p^k} \equiv [x^{p^k}, y] \pmod{\gamma_2(L)^{p^k} \gamma_p(L)^{p^{k-1}} \gamma_{p^2}(L)^{p^{k-2}} \dots \gamma_{p^k}(L)}, \tag{3}$$

*where $L = \langle x, [x,y] \rangle$.*

*Proof.* Simply note that (ii) is a consequence of (i), since $[x, y] = x^{-1}x^y$. □

We can use formulas (2) and (3) for elements in order to derive congruences for power-commutator subgroups of closed normal subgroups in a pro-$p$ group. When we write a congruence $H \equiv K \pmod{L}$, with $L$ a normal subgroup, we mean that $HL = KL$. Now if $H$, $K$ and $L$ are closed subgroups of a pro-$p$ group $G$, the equality $HL = KL$ holds if and only if $HLN = KLN$ for every open normal subgroup $N$ of $G$, which in turn is equivalent to the congruence $\overline{H} \equiv \overline{K} \pmod{\overline{L}}$ in the finite $p$-group $\overline{G} = G/N$. For this reason, in the proof of the following theorems we will always set ourselves from the beginning in the case that $G$ is a finite $p$-group, which has the advantage of allowing induction arguments. First we need a simple lemma.

**Lemma 2.2.** *Let $G$ be a finite $p$-group and let $N$ and $M$ be normal subgroups of $G$. If $N \leq M[N, G]N^p$ then $N \leq M$.*

*Proof.* We may assume that $M = 1$. If $N \neq 1$ then we can take a normal subgroup $K$ of $G$ such that $|N : K| = p$. Then $[N, G]N^p \leq K$ is properly contained in $N$, which is a contradiction. □

**Theorem 2.3.** *Let $G$ be a pro-$p$ group. If $G = \langle X \rangle$ then, for all $k \geq 0$,*

$$G^{p^k} \equiv \langle x^{p^k} \mid x \in X \rangle \pmod{\gamma_2(G)^{p^k} \gamma_p(G)^{p^{k-1}} \ldots \gamma_{p^k}(G)}.$$

*Proof.* If $x_1, \ldots, x_r \in X$, we only need to show that $(x_1 \ldots x_r)^{p^k}$ and $x_1^{p^k} \ldots x_r^{p^k}$ are congruent with respect to the modulus in the statement of the theorem. This follows immediately by induction on $r$, with the help of formula (2). □

In the remainder, we use the symbol $[H,_n K]$ to denote the commutator subgroup $[H, K, \ldots, K]$, with $K$ appearing $n$ times.

**Theorem 2.4.** *Let $G$ be a pro-$p$ group and let $N$ and $M$ be closed normal subgroups of $G$. Then*

$$[N^{p^k}, M] \equiv [N, M]^{p^k} \pmod{[M,_p N]^{p^{k-1}} [M,_{p^2} N]^{p^{k-2}} \ldots [M,_{p^k} N]}.$$

*Proof.* By formula (3), for every $n \in N$ and every $m \in M$ we have

$$[n^{p^k}, m] \in [N, M]^{p^k} [M,_p N]^{p^{k-1}} [M,_{p^2} N]^{p^{k-2}} \ldots [M,_{p^k} N],$$

which implies that $[N^{p^k}, M]$ is contained in this product of subgroups.

In order to prove the other inclusion of the congruence, we use induction on the order of $N$. Since $[N, M]$ is generated by the commutators $[n, m]$ with $n \in N$ and $m \in M$, by combining Theorem 2.3 and (3) we get that

$$[N, M]^{p^k} \leq [N^{p^k}, M][M, N, N]^{p^k} [M,_p N]^{p^{k-1}} \ldots [M,_{p^k} N].$$

Now, by the induction hypothesis,

$$\begin{aligned}[M,N,N]^{p^k} &\le [[M,N]^{p^k},N]\prod_{r=1}^{k}[N,_{p^r}[M,N]]^{p^{k-r}}\\ &\le [[M,N]^{p^k},N]\prod_{r=1}^{k}[M,_{p^r}N]^{p^{k-r}}.\end{aligned}$$

Consequently

$$[N,M]^{p^k} \le [N^{p^k},M][[N,M]^{p^k},N]\prod_{r=1}^{k}[M,_{p^r}N]^{p^{k-r}},$$

and the result follows from Lemma 2.2. □

If we want to work with commutators of arbitrary length, the formulas get too complicated and it is easier to suppose that all but one of the subgroups involved are equal to the whole group. We give two congruences, one with powers outside commutators in the modulus, and the other one with powers inside commutators.

**Theorem 2.5.** *Let $G$ be a pro-$p$ group and let $N$ be a closed normal subgroup of $G$. Then the following congruences hold for every $k,\ell \ge 0$:*

(i) $[N^{p^k},_\ell G] \equiv [N,_\ell G]^{p^k} \pmod{\prod_{r=1}^{k}[N,_{p^r+\ell-1}G]^{p^{k-r}}}$.

(ii) $[N^{p^k},_\ell G] \equiv [N,_\ell G]^{p^k} \pmod{\prod_{r=1}^{k}[N^{p^{k-r}},_{r(p-1)+\ell}G]}$.

*Proof.* (i) We argue by induction on $\ell$, the case $\ell=0$ being trivial. Assume then $\ell \ge 1$. Let $T=\prod_{r=1}^{k}[N,_{p^r+\ell-1}G]^{p^{k-r}}$. By the induction hypothesis, $[N^{p^k},_{\ell-1}G] \equiv [N,_{\ell-1}G]^{p^k} \pmod{U}$, where $U=\prod_{r=1}^{k}[N,_{p^r+\ell-2}G]^{p^{k-r}}$, and consequently $[N^{p^k},_\ell G] \equiv [[N,_{\ell-1}G]^{p^k},G] \pmod{[U,G]}$. On the other hand, by applying Theorem 2.4, we obtain that $[[N,_{\ell-1}G]^{p^k},G] \equiv [N,_\ell G]^{p^k}$ (mod $T$), thus it suffices to prove that $[U,G]\le T$. For this purpose, consider a general factor $[[N,_{p^r+\ell-2}G]^{p^{k-r}},G]$ in $[U,G]$. Again by Theorem 2.4, we have

$$\begin{aligned}[[N,_{p^r+\ell-2}G]^{p^{k-r}},G] &\le \prod_{s=0}^{k-r}[G,_{p^s}[N,_{p^r+\ell-2}G]]^{p^{k-r-s}}\\ &= \prod_{s=0}^{k-r}[[N,_{p^r+\ell-1}G],_{p^s-1}[N,_{p^r+\ell-2}G]]^{p^{k-r-s}}\\ &\le \prod_{s=0}^{k-r}[[N,_{p^r+\ell-1}G],_{p^s-1}\gamma_{p^r+\ell-1}(G)]^{p^{k-r-s}}.\end{aligned}$$

Since $[K, \gamma_i(G)] \leq [K,_i G]$ for any normal subgroup $K$ of $G$ (see Lemma 4.9 in [10]), it follows that

$$[[N,_{p^r+\ell-1} G],_{p^s-1} \gamma_{p^r+\ell-1}(G)] \leq [N,_{p^s(p^r+\ell-1)} G] \leq [N,_{p^{r+s}+\ell-1} G].$$

Therefore $[[N,_{p^r+\ell-2} G]^{p^{k-r}}, G] \leq T$ and we are done.

(ii) We use induction on $k$. Let $V = \prod_{r=1}^{k} [N^{p^{k-r}},_{r(p-1)+\ell} G]$. According to (i), we only need to show that $T \leq V$. For $1 \leq r \leq k$, the induction hypothesis yields that

$$[N,_{p^r+\ell-1} G]^{p^{k-r}} \equiv [N^{p^{k-r}},_{p^r+\ell-1} G] \pmod{\prod_{s=1}^{k-r} [N^{p^{k-r-s}},_{s(p-1)+p^r+\ell-1} G]}.$$

Since $s(p-1)+p^r+\ell-1 \geq (r+s)(p-1)+\ell$, it follows that $[N,_{p^r+\ell-1} G]^{p^{k-r}} \leq V$. Thus $T \leq V$, as desired. □

In the last theorem, one could expect that the congruence in (ii) could be improved by changing the subgroup in the modulus for a smaller one, following the same rule as in (i), where the number of times that $G$ appears in the commutators varies exponentially and not linearly with $r$. In other words, it seems reasonable to ask whether the congruence $[N^{p^k},_\ell G] \equiv [N,_\ell G]^{p^k}$ $(\text{mod } \prod_{r=1}^{k} [N^{p^{k-r}},_{p^r+\ell-1} G])$ holds or not. To see that the answer is negative, consider the following example.

**Example 2.6.** Let $p$ be an arbitrary prime and consider a homocyclic group $H = \langle x_1 \rangle \times \cdots \times \langle x_{2p} \rangle$ of exponent $p^2$. For convenience, write $x_i = 1$ for $i > 2p$. Then the rule $x_i \mapsto x_i x_{i+1}$ defines an automorphism $\alpha$ of $H$ of order $p^3$. Let $G$ be the corresponding semidirect product of $\langle \alpha \rangle$ and $H$. Thus $G$ has order $p^{4p+3}$.

The subgroup $N$ generated by $\alpha$ and all the $x_i$ with $i \geq 2$ is then a normal subgroup of $G$. It is clear that $[N, G]^{p^2} = 1$, and one can check without much difficulty that $[N^{p^2}, G] = \langle x_{p+1}^p, \ldots, x_{2p}^p \rangle$, $[N^p,_p G] \leq \langle x_{p+1}^p x_{2p}, x_{p+2}^p, \ldots, x_{2p}^p \rangle$ and $[N,_{p^2} G] = 1$. Thus $[N, G]^{p^2}$ and $[N^{p^2}, G]$ are not congruent modulo $[N^p,_p G][N,_{p^2} G]$.

However, in the next sections, when we use the power-commutator formulas of Theorem 2.4 and Theorem 2.5 (i) in the context of potent filtrations, it will be enough to work with bigger moduli in which the number of times that $G$ appears in the $r$-th subgroup is basically $r(p-1)$ instead of $p^r$, as happens in Theorem 2.5 (ii). For example, most of the times that we use Theorem 2.4, we do it in the following form:

$$[N^{p^k}, M] \equiv [N, M]^{p^k} \pmod{\prod_{r=1}^{k} [M,_{r(p-1)+1} N]^{p^{k-r}}}.$$

**Theorem 2.7.** *Let $G$ be a pro-$p$ group and let $N$ be a closed normal subgroup of $G$. If $N = \langle X \rangle$ then*

$$[N,_\ell G]^{p^k} \leq \prod_{r=0}^{k} [P_{k-r},_{r(p-1)+\ell} G],$$

*where $P_i = \langle x^{p^i} \mid x \in X \rangle^G$.*

*Proof.* We argue by induction on $k$. By (ii) of Theorem 2.5,

$$[N,_\ell G]^{p^k} \leq \prod_{r=0}^{k} [N^{p^{k-r}},_{r(p-1)+\ell} G].$$

For this reason, it suffices to prove the theorem for the case $\ell = 0$.

According to Theorem 2.3, $N^{p^k} \leq P_k T$, where

$$T = [N,N]^{p^k} \prod_{r=1}^{k} \gamma_{p^r}(N)^{p^{k-r}} \leq [N,N]^{p^k} \prod_{r=1}^{k} [N,_{r(p-1)} G]^{p^{k-r}}.$$

We now apply part (ii) of Theorem 2.5 to all the commutators in the last expression in order to get that

$$T \leq [N^{p^k}, N] \prod_{r=1}^{k} [N^{p^{k-r}},_{r(p-1)} G].$$

Then, by the induction hypothesis,

$$T \leq [N^{p^k}, N] \prod_{r=1}^{k} [P_{k-r},_{r(p-1)} G].$$

Thus

$$N^{p^k} \leq [N^{p^k}, N] \prod_{r=0}^{k} [P_{k-r},_{r(p-1)} G]$$

and the result follows from Lemma 2.2. □

For our purposes, the most important instance of Theorem 2.7 corresponds to the case when $N$ is an omega subgroup.

**Corollary 2.8.** *Let $G$ be a pro-$p$ group. Then*

$$[\Omega_i(G),_\ell G]^{p^k} \leq \prod_{r=0}^{k} [\Omega_{i-r}(G^{p^r}),_{(k-r)(p-1)+\ell} G].$$

*Proof.* We have $\Omega_i(G) = \langle X \rangle$, where $X = \{g \in G \mid g^{p^i} = 1\}$, and for this set of generators, $P_r = \langle x^{p^r} \mid x \in X \rangle \leq \Omega_{i-r}(G^{p^r})$. □

# 3 Potent filtrations

In this section we develop the basics of potent filtrations, a concept which is key to the proof of our main theorem.

Let $G$ be a pro-$p$ group. We say that a family $\{N_i\}_{i\in\mathbb{N}}$ of closed subgroups of $G$ is a *potent filtration* of $G$ if the following conditions hold:

(i) $N_i \le N_j$ for all $i \ge j$,

(ii) $\bigcap_{i\in\mathbb{N}} N_i = 1$,

(iii) $[N_i, G] \le N_{i+1}$ for all $i$,

(iv) $[N_i, {}_{p-1} G] \le N_{i+1}^p$ for all $i$.

For easiness of notation, we will write $\{N_i\}$ instead of $\{N_i\}_{i\in\mathbb{N}}$. If there is a potent filtration of $G$ beginning at a subgroup $N$, we say that $N$ is *PF-embedded* in $G$, and we call $G$ a *PF-group* if it is PF-embedded in itself.

Note that the concept of PF-group is a generalization of that of potent pro-$p$ group (and consequently also of powerful pro-$p$ group). More generally, the pro-$p$ groups satisfying $\gamma_p(G) \le \Phi(G)^p$ are PF-groups: by Lemma 4.4 in [7], if $\gamma_{kp}(G) \le D_{kp+1}(G)$ then $[D_n(G), {}_{p-1} D_k(G)] \le D_{n+1}(G)^p$ for all $n \ge k$, so that $D_n(G)$ is PF-embedded in $D_k(G)$. Recall that the subgroups $D_k(G)$ are defined by means of the formula

$$D_k(G) = \prod_{ip^j \ge k} \gamma_i(G)^{p^j}.$$

These are called the *dimension subgroups* of $G$, and they play an important role in the theory of pro-$p$ groups, see Chapter 11 of [2]. If there is no possible confusion, we will write simply $D_k$ instead of $D_k(G)$.

Interestingly, the property of being a PF-embedded subgroup is hereditary for quotient groups.

**Proposition 3.1.** *Let $G$ be a pro-$p$ group and let $N$ be a PF-embedded subgroup of $G$. Then $N/K$ is PF-embedded in $G/K$ for every closed normal subgroup $K$ of $G$.*

*Proof.* It suffices to note that $\cap_{i\in\mathbb{N}} N_i K = (\cap_{i\in\mathbb{N}} N_i)K = K$, where the first equality follows from Proposition 2.1.4 (a) of [13]. □

In the following two propositions we prove some basic properties of potent filtrations.

**Proposition 3.2.** *Let $G$ be a pro-$p$ group and let $\{N_i\}$ be a potent filtration of $G$. Then:*

(i) $[N_i^p, G] = [N_i, G]^p$ *for all $i$.*

(ii) $\{[N_i, G]\}$ *is a potent filtration of $G$.*

(iii) $\{N_i^p\}$ *is a potent filtration of $G$.*

*Thus if $N$ is PF-embedded in $G$ then $[N, G]$ and $N^p$ are also PF-embedded in $G$.*

*Proof.* (i) We may assume that $G$ is a finite $p$-group, and in this context we may argue by reverse induction on $i$. According to Theorem 2.4, the subgroups $[N_i^p, G]$ and $[N_i, G]^p$ are congruent modulo $[N_i,_p G]$. Since $\{N_i\}$ is a potent filtration, we have $[N_i,_p G] \leq [N_{i+1}^p, G]$. Now, by the induction hypothesis, $[N_{i+1}^p, G] = [N_{i+1}, G]^p$ and we are done.

(ii) The first three conditions in the definition of potent filtration are clear, and the fourth condition follows from (i): $[[N_i, G],_{p-1} G] \leq [N_{i+1}^p, G] = [N_{i+1}, G]^p$.

(iii) Simply note that, by parts (i) and (ii), $[N_i^p, G] = [N_i, G]^p \leq N_{i+1}^p$ and $[N_i^p,_{p-1} G] = [N_i,_{p-1} G]^p \leq (N_{i+1}^p)^p$. □

**Proposition 3.3.** *Let $G$ be a pro-$p$ group and let $\{N_i\}$ be a potent filtration of $G$. Then for all $g \in G$ and all $x \in N_i$,*

$$(gx)^p \equiv g^p x^p \pmod{N_{i+1}^p}.$$

*Proof.* This is a direct consequence of Hall's collection formula and the definition of potent filtration. □

We next see that PF-groups and PF-embedded subgroups have a nice behaviour with respect to taking $p$-powers.

**Theorem 3.4.** *Let $G$ be a PF-group. Then $G^{p^i} = \{g^{p^i} \mid g \in G\}$ for all $i$.*

*Proof.* Again, it suffices to deal with the case where $G$ is a finite $p$-group. Note also that, by Proposition 3.2, we only need to prove that $G^p = \{g^p \mid g \in G\}$.

Let $\{N_i\}$ be a potent filtration such that $G = N_1$. It suffices to prove, by reverse induction on $i$, that $x^p y^p$ is a $p$-th power for every $x \in G$ and $y \in N_i$. This follows immediately from the last proposition by applying (twice) the induction hypothesis. □

**Corollary 3.5.** *Let $G$ be a pro-$p$ group and let $N$ be a PF-embedded subgroup of $G$. Then $[N^{p^i}, G^{p^j}]^{p^k} = [N,G]^{p^{i+j+k}}$ for all $i,j,k \geq 0$.*

*Proof.* First of all, it is clear from the last theorem and Proposition 3.2 that $[N^{p^i}, G]^{p^k} = [N,G]^{p^{i+k}}$. Then it suffices to prove that $[N, G^{p^j}] = [N,G]^{p^j}$. We prove this equality in the case that $G$ is a finite $p$-group, by induction on the order of $N$. Let $\{N_i\}$ be a potent filtration of $G$ beginning at $N$. By Theorem 2.4,

$$[N, G^{p^j}] \equiv [N,G]^{p^j} \pmod{[N,_p G]^{p^{j-1}} \dots [N,_{j(p-1)+1} G]}. \tag{4}$$

According to the first mentioned property, this modulus is contained in $[N_2, G]^{p^j}$. By the induction hypothesis $[N_2, G]^{p^j} = [N_2, G^{p^j}]$, and consequently (4) implies that $[N, G^{p^j}] = [N,G]^{p^j}$. □

We can also use Theorem 3.4 to provide an alternative proof to the result of González-Sánchez and Wilson for power subgroups mentioned in the introduction. Now that we have defined dimension subgroups, we can state this result as in Theorem 4.7 of [7]: if $G$ is a pro-$p$ group such that $\gamma_{p^k}(G) \leq D_{p^k+1}$ then every element of $G^{p^i}$ is a $p^{i-(k-1)}$-st power for all $i \geq k-1$. For the proof, simply recall that, as indicated before Proposition 3.1, the condition $\gamma_{p^k}(G) \leq D_{p^k+1}$ implies that $D_{p^{k-1}}$ is a PF-group.

The following theorem is a fundamental tool in our study of the power structure of pro-$p$ groups via PF-embedded subgroups. It is inspired in a similar result for potent groups, see Theorem 4.1 in [6].

**Theorem 3.6.** *Let $G$ be a pro-$p$ group. Then for every closed normal subgroup $K$ of $G$ there exists a closed subgroup $T$ of $G$, which contains $K$, satisfying the following two properties:*

(i) *If $N$ is PF-embedded in $G$ then $N \cap T$ is PF-embedded in $T$. More precisely, if $\{N_i\}$ is a potent filtration of $G$ then $\{N_i \cap T\}$ is a potent filtration of $T$.*

(ii) *For every PF-embedded subgroup $M$ of $T$, we have $[M^{p^i}, T^{p^j}]^{p^k} = [M^{p^r}, K^{p^s}]^{p^t}$ whenever $i+j+k = r+s+t \geq 1$.*

*Proof.* Let $\mathcal{T}$ be the family of all closed subgroups of $G$ containing $K$ and satisfying property (i). We claim that $\mathcal{T}$ has minimal subgroups. By Zorn's Lemma, we only need to consider a chain $\{T_j\}$ of subgroups in $\mathcal{T}$ (ordered by reverse inclusion) and prove that $T = \cap_j T_j$ also belongs to $\mathcal{T}$. Choose then a potent filtration $\{N_i\}$ of $G$ and let us see that $\{N_i \cap T\}$ is a potent filtration of $T$. For this purpose, we take $x \in [N_i \cap T,_{p-1} T]$ and prove that

$x = t^p$ for some $t \in N_{i+1} \cap T$. Since $x \in [N_i \cap T_j,_{p-1} T_j] \leq (N_{i+1} \cap T_j)^p$, we can write $x = t_j^p$ with $t_j \in N_{i+1} \cap T_j$ (recall Theorem 3.4). We now define $F_j = \{t_j \in N_{i+1} \cap T_j \mid t_j^p = x\}$. Since $N_{i+1}$ and $T_j$ are closed in $G$ and taking powers is continuous on $G$, it follows that $F_j$ is a non-empty closed subset of $G$. Note that the family $\{F_j\}$ of closed subsets has the finite intersection property. Indeed, since $\{T_j\}$ is a chain, for any finite collection of indices $j_1, \ldots, j_n$ there exists an index $j_k$ such that $T_{j_1} \cap \cdots \cap T_{j_n} = T_{j_k}$, and consequently $F_{j_1} \cap \cdots \cap F_{j_n} = F_{j_k}$ is non-empty. Since $G$ is compact, there exists at least one element $t$ in the intersection of all $F_j$. It follows that $t \in N_{i+1} \cap T$ and $t^p = x$, as desired.

Let $T$ denote in the remainder a minimal subgroup in the family $\mathcal{T}$. We only need to show that $T$ satisfies (ii). First we prove that $[M^{p^r}, T] = [M^{p^r}, K]$ for every $M$ PF-embedded in $T$ and every $r \geq 1$. Since $M^{p^{r-1}}$ is also PF-embedded in $T$ and $K$ is contained in $T$, it suffices to see that $[M^p, T] \leq [M^p, K]$. Suppose by way of contradiction that this inclusion does not hold for some $M$. Then it must fail in some quotient $G/V$, where $V$ is a normal open subgroup of $G$. Let $\{M_i\}$ be a potent filtration whose first term is $M$. Since $G/V$ is finite, there must be an index $j$ such that $[M_j^p, T] \not\leq [M_j^p, K]V$ but $[M_{j+1}^p, T] \leq [M_{j+1}^p, K]V$. Define then

$$T^* = \{t \in T \mid [M_j^p, t] \leq [M_j^p, K]V\},$$

in other words, $T^*$ is the centralizer in $T$ of $M_j^p V/[M_j^p, K]V$. Thus $T^*$ is a proper closed normal subgroup of $T$ containing $K$. Given a potent filtration $\{N_i\}$ of $G$, we next prove that $\{N_i \cap T^*\}$ is a potent filtration of $T^*$, which is a contradiction with the definition of $T$. Choose $x \in [N_i \cap T^*,_{p-1} T^*]$ and let us see that $x \in (N_{i+1} \cap T^*)^p$. Since $[N_i \cap T^*,_{p-1} T^*] \leq [N_i \cap T,_{p-1} T] \leq (N_{i+1} \cap T)^p$, we can write $x = t^p$ with $t \in N_{i+1} \cap T$ and it suffices to see that $t \in T^*$. Now, by congruence (3) in Theorem 2.1,

$$[M_j^p, t] \leq [M_j, t^p][M_j, T, T]^p[M_j,_p T] \leq [M_j, x][M_{j+1}^p, T] \leq [M_j, x][M_{j+1}^p, K]V,$$

and on the other hand, since $x \in \gamma_p(T)$,

$$[M_j, x] \leq [M_j, \gamma_p(T)] \leq [M_j,_p T] \leq [M_{j+1}^p, T] \leq [M_{j+1}^p, K]V.$$

Hence $[M_j^p, t] \leq [M_{j+1}^p, K]V$ and $t \in T^*$, as desired.

Next we show that $[M^{p^r}, K] = [M, K]^{p^r} = [M, K^{p^r}]$ for every $r \geq 0$. Of course, we may assume $r \geq 1$ and that $G$ is a finite $p$-group. Now

$$[M^{p^r}, K] \equiv [M, K]^{p^r} \equiv [M, K^{p^r}] \pmod{[M,_p T]^{p^{r-1}} \ldots [M,_{r(p-1)+1} T]},$$

and this modulus is contained in $[M_2^{p^r}, T]$, which coincides with $[M_2^{p^r}, K]$ as seen above. Thus the result follows by induction on the order of $M$.

Now we can conclude the proof of (ii), since for $i+j+k=r+s+t\geq 1$, we have

$$[M^{p^r},K^{p^s}]^{p^t}=[M^{p^{r+s+t}},K]=[M^{p^{r+s+t}},T]=[M^{p^i},T^{p^j}],$$

by the results in the last two paragraphs and Corollary 3.5. □

# 4 The exponent of the omega subgroups

In this final section we prove the main theorem of our paper, and also Easterfield's result mentioned in the introduction, with the help of the following theorem, which shows the influence of a PF-embedded subgroup on the exponent of the omega subgroups of a pro-$p$ group.

**Theorem 4.1.** *Let $G$ be a pro-$p$ group having a PF-embedded subgroup $N$ such that $\gamma_{k(p-1)}(G)\leq N$, where $k\geq 1$. Then $\Omega_i(G)^{p^{i+k}}=1$ for all $i$.*

*Proof.* Choose a closed subgroup $T$ of $G$ containing $\Omega_i(G)$ and satisfying properties (i) and (ii) of Theorem 3.6. Thus $\Omega_i(G)=\Omega_i(T)$. Call $M=N\cap T$, which is PF-embedded in $T$. Note also that $\gamma_{k(p-1)}(T)\leq\gamma_{k(p-1)}(G)\cap T\leq N\cap T=M$.

By Corollary 2.8 and Theorem 2.5, for every $j\geq i$ we have

$$\begin{aligned}\Omega_i(T)^{p^j}&\leq\prod_{r=0}^{i-1}[\Omega_{i-r}(T^{p^r}),_{(j-r)(p-1)}T]\leq\prod_{r=0}^{i-1}[T^{p^r},_{(j-r)(p-1)}T]\\&\leq\prod_{r=0}^{i-1}\gamma_{(j-r)(p-1)+1}(T)^{p^r}.\end{aligned}\tag{5}$$

In particular,

$$\begin{aligned}\Omega_i(T)^{p^{i+k}}&\leq\prod_{r=0}^{i-1}\gamma_{(i+k-r)(p-1)+1}(T)^{p^r}\leq\prod_{r=0}^{i-1}[M,_{(i-r)(p-1)+1}T]^{p^r}\\&\leq\prod_{r=0}^{i-1}[M^{p^{i-r}},T]^{p^r}=[M,\Omega_i(T)^{p^i}],\end{aligned}$$

where the last equality follows from (ii) in Theorem 3.6. Now let $\{M_i\}$ be a potent filtration of $T$ whose first term is $M$. By using (5) again, we have

$$\begin{aligned}[M,\Omega_i(T)^{p^i}]&\leq\prod_{r=0}^{i-1}[M,\gamma_{(i-r)(p-1)+1}(T)^{p^r}]\leq\prod_{r=0}^{i-1}[M,_{(i-r)(p-1)+1}T]^{p^r}\\&\leq\prod_{r=0}^{i-1}[M_{i-r+1}^{p^{i-r}},T]^{p^r}=[M_2,T]^{p^i}=[M_2,\Omega_i(T)^{p^i}].\end{aligned}$$

By repeating this argument, we get that $[M, \Omega_i(T)^{p^i}] \leq [M_j, \Omega_i(T)^{p^i}]$ for all $j$. Since the subgroups $M_j$ intersect trivially, it follows that $[M, \Omega_i(T)^{p^i}] = 1$ and therefore also that $\Omega_i(T)^{p^{i+k}} = 1$. □

**Corollary 4.2.** *Let $G$ be a pro-$p$ group. If $G$ is a PF-group then $\Omega_i(G)^{p^{i+1}} = 1$ for all $i$.*

As the following example shows, we cannot assure in general that $\Omega_i(G)^{p^i} = 1$ in a PF-group, even if we know by Theorem 3.4 that every element of $G^{p^i}$ is a $p^i$-th power. Thus the previous corollary is best possible.

**Example 4.3.** Let $p$ be an arbitrary prime and let $n$ be any positive integer. Consider the abelian group $H = \langle x_1 \rangle \times \cdots \times \langle x_p \rangle$, where $x_i$ is of order $p^n$ for $1 \leq i \leq p-1$ and $x_p$ is of order $p^{n+1}$. Then the rules $x_i \mapsto x_i x_{i+1}$ for $1 \leq i \leq p-2$, $x_{p-1} \mapsto x_{p-1} x_p^p$ and $x_p \mapsto x_p$ define an automorphism $\alpha$ of $H$ of order $p^n$. Let $G$ be the corresponding semidirect product of $\langle \alpha \rangle$ and $H$. Then

$$(\alpha x_1)^{p^n} = \alpha^{p^n} x_1^{p^n} x_2^{\binom{p^n}{2}} \ldots x_{p-1}^{\binom{p^n}{p-1}} x_p^{p\binom{p^n}{p}} = x_p^{\lambda p^n},$$

with $\lambda$ not divisible by $p$. Thus $\Omega_n(G)^{p^n} \neq 1$. Note however that $G$ is a PF-group, since the series $N_1 = G$, $N_i = \langle x_i, \ldots, x_p \rangle$ for $2 \leq i \leq p$, and $N_{p+1} = 1$ defines a potent filtration of $G$.

Next we prove Easterfield's result and see that it is best possible if we want to use the subgroup $\gamma_{k(p-1)+1}(G)$ in the left-hand side of the power-commutator condition.

**Corollary 4.4.** *Let $k \geq 1$. Then:*

(i) *If $G$ is a pro-$p$ group and $\gamma_{k(p-1)+1}(G) = 1$ then $\Omega_i(G)^{p^{i+k-1}} = 1$ for all $i$.*

(ii) *If $V$ is a set of words such that all pro-$p$ groups satisfying $\gamma_{k(p-1)+1}(G) \leq V(G)$ have the property that $\exp \Omega_1(G) \leq p^k$, then necessarily $V(G) = 1$ in these groups.*

*Proof.* (i) Again it suffices to deal with finite $p$-groups. If $k = 1$ then $G$ has class less than $p$. Thus $G$ is regular and $\Omega_i(G)^{p^i} = 1$.

Suppose in the remainder that $k > 1$. Since $\gamma_{k(p-1)}(G)$ is central in $G$, we can choose an abelian group $A$ such that $A^p \cong \gamma_{k(p-1)}(G)$ and then construct the central product $P$ of $G$ and $A$ corresponding to this isomorphism. Define $N_i = \gamma_{(k-1)(p-1)+i-1}(G)A$. It is straightforward to check that $\{N_i\}$ is a potent filtration of $P$. Since $\gamma_{(k-1)(p-1)}(P) \leq N_1$, we deduce from Theorem 4.1 that $\Omega_i(P)^{p^{i+k-1}} = 1$ for all $i$, and the result follows.

(ii) Suppose, by way of contradiction, that there exists a pro-$p$ group $G$ such that $\gamma_{k(p-1)+1}(G) \le V(G)$ and $V(G) \ne 1$. We may assume that $G$ is a finite $p$-group. Choose a normal subgroup $N$ of $G$ such that $|V(G) : N| = p$ and define $L = G/N$. Then $V(L)$ has order $p$ and $\gamma_{k(p-1)+1}(L) \le V(L)$. Let now $Q$ be the group of maximal class of order $p^{k(p-1)+2}$ defined in the introduction, and consider the central product $P$ of $L$ and $Q$ that identifies $V(L)$ with $Z(Q)$. Then $\gamma_{k(p-1)+1}(P) = \gamma_{k(p-1)+1}(L)\gamma_{k(p-1)+1}(Q) \le V(L)Z(Q) = V(L) \le V(P)$, but $\Omega_1(P)^{p^k}$ is non-trivial, since it contains $\Omega_1(Q)^{p^k}$. This contradiction proves the result. □

Now we introduce a family $\{E_k^r\}$ of subgroups in a pro-$p$ group, which is similar in its definition to the dimension subgroup series. In the same way that dimension subgroups have applications to the study of power subgroups, the subgroups $E_k^r$ will reveal very useful in the dual study of omega subgroups.

**Definition 4.5.** Let $G$ be a pro-$p$ group. For any pair $k$, $r$ of positive integers, we define the subgroup

$$E_k^r(G) = \prod_{\substack{i+j(p-1)\ge k \\ i\ge r}} \gamma_i(G)^{p^j}.$$

If there is no risk of confusion, we will simply write $E_k^r$ instead of $E_k^r(G)$. Also, we usually write $E_k$ in place of $E_k^1$.

It is easy to check that $E_k \le D_k$ and $D_{p^k} \le E_{k(p-1)+1}$. These inclusions will be used later on.

In our next lemma, we study the behaviour of the subgroups $E_k^r$ with respect to powers and commutators.

**Lemma 4.6.** *Let $G$ be a pro-$p$ group. Then:*

(i) $[E_k^r,_\ell G] \le E_{k+\ell}^{r+\ell}$.

(ii) $(E_k^r)^p \le E_{k+p-1}^r$ *and* $E_{k+p-1}^r \le (E_k^r)^p\gamma_{k+p-1}(G)$. *Furthermore, the last inclusion is an equality if* $r \le k+p-1$.

(iii) *If* $\gamma_k(G) \le E_{k+1}^r$ *then* $\gamma_n(G) \le E_{n+1}^r$ *for all* $n \ge k$ *and* $E_n^r = (E_{n-p+1}^r)^p$ *for* $n \ge k$ *and* $n \ge p$.

(iv) $[E_k^r, E_\ell^s] \le E_{k+\ell}^{r+s}$.

*Proof.* By Theorem 2.4, we have

$$[\gamma_i(G)^{p^j}, G] \le \prod_{m=0}^{j} [\gamma_i(G),_{m(p-1)+1} G]^{p^{j-m}} = \prod_{m=0}^{j} \gamma_{m(p-1)+i+1}(G)^{p^{j-m}},$$

which is contained in $E_{k+1}^{r+1}$ if $i + j(p-1) \geq k$ and $i \geq r$. This proves that $[E_k^r, G] \leq E_{k+1}^{r+1}$, and (i) follows.

For the first inclusion in (ii), consider the group $H = E_k^r / E_{k+p-1}^r$ and observe that $H^p = 1$: $H$ is clearly generated by elements of order $p$ and on the other hand $\gamma_p(H) = 1$, since $[E_k^r, {}_{p-1} G] \leq E_{k+p-1}^r$ by (i). The second inclusion is clear, and it readily follows that we have an equality if $r \leq k + p - 1$.

The inclusion $\gamma_n(G) \leq E_{n+1}^r$ is immediate from (i). Let us see that $E_n^r \leq (E_{n-p+1}^r)^p$. We may assume that $G$ is a finite $p$-group. Then it suffices to apply reverse induction on $n$, since by (ii),

$$E_n^r \leq (E_{n-p+1}^r)^p \gamma_n(G) \leq (E_{n-p+1}^r)^p E_{n+1}^r.$$

Let us finally check that (iv) holds. By using Theorem 2.4 together with (i) and (ii), we have

$$\begin{aligned}
[E_k^r, \gamma_i(G)^{p^j}] &\leq \prod_{m=0}^{j} [E_k^r, {}_{m(p-1)+1} \gamma_i(G)]^{p^{j-m}} \leq \prod_{m=0}^{j} [E_k^r, {}_{im(p-1)+i} G]^{p^{j-m}} \\
&\leq \prod_{m=0}^{j} E_{k+im(p-1)+i+(j-m)(p-1)}^{r+im(p-1)+i} \leq E_{k+i+j(p-1)}^{r+i}.
\end{aligned}$$

If $i + j(p-1) \geq \ell$ and $i \geq s$, this subgroup is contained in $E_{k+\ell}^{r+s}$, and (iv) follows. □

Lazard proved in [11] that a finitely generated pro-$p$ group such that $\gamma_k(G) \leq G^{p^n}$ with $k < p^n$ is $p$-adic analytic. In a similar fashion, we have the following result.

**Proposition 4.7.** *Let $G$ be a finitely generated pro-p group. If $\gamma_k(G) \leq E_{k+1}$ for some $k$ then $G$ is p-adic analytic.*

*Proof.* By applying the previous lemma, we have $\gamma_n(G) \leq E_{n+1} \leq D_{n+1}$ for all $n \geq k$. If $L$ is the graded Lie algebra over $\mathbb{F}_p$ associated to the dimension subgroup series of $G$, it follows from Lemma 11.11 of [2] that $L$ is nilpotent. By Interlude A of [2], this means that $G$ is $p$-adic analytic. □

We can finally proceed to the proof of our main theorem, stated in a somewhat more general version than in the introduction.

**Theorem 4.8.** *Let $G$ be a pro-p group such that $\gamma_{k(p-1)}(G) \leq E_{k(p-1)+1}$. Then $\Omega_i(G)^{p^{i+k-1}} = 1$ for all $i$. In particular, the torsion elements of $G$ form a subgroup, which is a finite p-group if $G$ is finitely generated.*

*Proof.* As usually, we deal with finite $p$-groups. If $k = 1$ the assumption is that $\gamma_{p-1}(G) \le E_p = G^p\gamma_p(G)$. According to Lemma 2.2, this is equivalent to $\gamma_{p-1}(G) \le G^p$. Thus $G$ is trivial for $p = 2$ and $G$ is potent if $p > 2$. In the latter case, the result that $\Omega_i(G)^{p^i} = 1$ has been established in Theorem 1.1 of [6].

Suppose then that $k > 1$. By Theorem 4.1, it suffices to show that $E_{(k-1)(p-1)}$ is PF-embedded in $G$. We will use without further mention the properties of the subgroups $E_k^r$ given in Lemma 4.6. Let $c$ be the nilpotency class of $G$. For every $\ell \ge (k-1)(p-1)$, consider the following refinement of $E_\ell \ge E_{\ell+1}$:

$$E_\ell = E_\ell^1 E_{\ell+1} \ge E_\ell^2 E_{\ell+1} \ge E_\ell^3 E_{\ell+1} \ge \cdots \ge E_\ell^c E_{\ell+1} \ge E_\ell^{c+1} E_{\ell+1} = E_{\ell+1}. \tag{6}$$

Since $[E_\ell^r, G] \le E_{\ell+1}^{r+1}$, this series is central. On the other hand,

$$\begin{aligned}[E_\ell^r,_{p-1} G] &\le E_{\ell+p-1}^{r+p-1} \le (E_\ell^{r+p-1})^p \gamma_{\ell+p-1}(G) \\ &\le (E_\ell^{r+p-1})^p E_{\ell+p} = (E_\ell^{r+p-1})^p (E_{\ell+1})^p \le (E_\ell^{r+1} E_{\ell+1})^p,\end{aligned}$$

and

$$[E_{\ell+1},_{p-1} G] \le E_{\ell+p} = (E_{\ell+1})^p.$$

Hence the series (6) satisfies properties (i), (iii) and (iv) of the definition of potent filtration. Thus if we connect these series for all $\ell \ge (k-1)(p-1)$, we get a potent filtration of $G$ beginning at $E_{(k-1)(p-1)}$, as desired.

It is now clear that the torsion elements of $G$ form a subgroup, call it $T$. If $G$ is finitely generated, we know from Proposition 4.7 that $G$ is $p$-adic analytic. According to Interlude A of [2], it follows that $G$ has an open uniformly powerful subgroup $U$. In particular $U$ is torsion-free, and consequently $|T| = |T : T \cap U| \le |G : U|$ is finite. Thus $T$ is a finite $p$-group, which concludes the proof. □

What can we say about the power subgroups of a pro-$p$ group satisfying the condition $\gamma_{k(p-1)}(G) \le E_{k(p-1)+1}$ of the previous theorem? Let $\ell = \lceil \log_p k \rceil$. Then $p^{\ell+1} > k(p-1)$ and, according to Lemma 4.6 (iii), $\gamma_{p^{\ell+1}}(G) \le E_{p^{\ell+1}+1} \le D_{p^{\ell+1}+1}$. It follows from Theorem 4.7 of [7] that every element of $G^{p^i}$ is a $p^{i-\ell}$-th power for $i \ge \ell$. Thus power subgroups behave better than omega subgroups, a phenomenon that one also encounters in other situations: for example with powerful pro-2 groups, and more generally with PF-groups for an arbitrary prime, or in Mann's paper [12].

**Open question.** The remark made in the introduction after the statement of our main theorem shows that, if we want to assure that $\exp \Omega_1(G) \le$

$p^k$, $E_{k(p-1)+1}$ cannot be replaced in the power-commutator condition by a bigger subgroup $E_\ell$ (that is, with $\ell \leq k(p-1)$). Inspired by Corollary 4.4 (ii), the following question arises: is it possible to replace $E_{k(p-1)+1}$ by any other bigger verbal subgroup $V(G)$? In other words, if all pro-$p$ groups satisfying $\gamma_{k(p-1)}(G) \leq V(G)$ have the property that $\Omega_1(G)^{p^k} = 1$, does it follow that $V(G) \leq E_{k(p-1)+1}$ in these groups? Note that Wilson has proved [14, Proposition 3.6] that $\exp \Omega_1(G) \leq p^k$ under the condition $\gamma_{k(p-1)}(G) \leq D_{p^k}$, but this is worse than our main theorem, since $D_{p^k} \leq E_{k(p-1)+1}$ in any group.

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## 5 Appendix: checking Example 2.6

Let $p$ be an arbitrary prime and consider a homocyclic group $H = \langle x_1 \rangle \times \cdots \times \langle x_{2p} \rangle$ of exponent $p^2$. For convenience, write $x_i = 1$ for $i > 2p$ and $H_i = \langle x_k \mid k \geq i \rangle$. Then the rule $x_i \mapsto x_i x_{i+1}$ defines an automorphism $\alpha$ of $H$. Note that

$$x_i \overset{\alpha^n}{\longmapsto} x_i x_{i+1}^{\binom{n}{1}} x_{i+2}^{\binom{n}{2}} \dots x_{i+n}^{\binom{n}{n}}. \tag{7}$$

Since $p^2$ does not divide $\binom{p^2}{p}$ but $p^2$ divides $\binom{p^3}{k}$ for all $1 \leq k \leq 2p-1$, it follows that $\alpha$ has order $p^3$. Let $G$ be the corresponding semidirect product of $\langle \alpha \rangle$ and $H$. Thus $G$ has order $p^{4p+3}$.

Note that $[h, G] = \langle [h, \alpha, \overset{k}{\ldots}, \alpha] \mid k \geq 1 \rangle$ for every $h \in H$, since $H$ is abelian. In particular, if $h \in H_i$ then $[h, G] \leq \langle [h, \alpha] \rangle H_{i+2}$. On the other hand, if $K$ and $L$ are subgroups of $H$ then $[KL, G] = [K, G][L, G]$. We will also need to calculate powers inside $G$. For this purpose, the next lemma is very convenient (see Lemma 10.9 in [9]).

**Lemma 5.1.** *Let $G$ be a group and let $A$ be an abelian normal subgroup of $G$. Then*

$$(ga)^n = g^n a^n [a, g]^{\binom{n}{2}} [a, g, g]^{\binom{n}{3}} \dots [a, g, \overset{n-2}{\ldots}, g]^{\binom{n}{n-1}} [a, g, \overset{n-1}{\ldots}, g]^{\binom{n}{n}}$$

*for all $g \in G$, $a \in A$ and $n \in \mathbb{N}$.*

Put now $N = \langle \alpha \rangle H_2$, which is a normal subgroup of $G$. Let us see that $[N^{p^2}, G] \not\equiv [N, G]^{p^2} \pmod{[N^p,_p G][N,_{p^2} G]}$. It is clear that $[N, G]^{p^2} = 1$, thus we have to see that $[N^{p^2}, G]$ is not contained in $[N^p,_p G][N,_{p^2} G]$.

Let us calculate $[N^{p^2}, G]$. If $\beta \in \langle \alpha \rangle$ and $h \in H_2$, we may apply the previous lemma to get that

$$(\beta h)^{p^2} = \beta^{p^2} h^{p^2} [h, \beta]^{\binom{p^2}{2}} \dots [h, \beta, \overset{p^2-1}{\ldots}, \beta]^{\binom{p^2}{p^2}} = \beta^{p^2} [h, \beta, \overset{p-1}{\ldots}, \beta]^{\binom{p^2}{p}},$$

since $[h, \beta, \overset{k-1}{\ldots}, \beta] = 1$ for $k \geq 2p$ and $p^2$ divides $\binom{p^2}{k}$ for $1 \leq k \leq 2p-1$, $k \neq p$. It follows that $N^{p^2}$ is contained in $\langle \alpha^{p^2} \rangle H_{p+1}^p$. Since

$$(\alpha x_i)^{p^2} = \alpha^{p^2} x_{i+p-1}^{\binom{p^2}{p}} \qquad \text{for every } i \geq 2,$$

the reverse inclusion also holds and $N^{p^2} = \langle \alpha^{p^2} \rangle H_{p+1}^p$. Then $[N^{p^2}, G] = [\alpha^{p^2}, G][H_{p+1}^p, G] = [\alpha^{p^2}, G] H_{p+2}^p$. Now, arguing as above, it follows from (7) that

$$[x_i, \alpha^{p^2}] = x_{i+p}^{\binom{p^2}{p}},$$

and consequently $[\alpha^{p^2}, G] = H_{p+1}^p$. Thus $[N^{p^2}, G] = H_{p+1}^p$.

We now consider the modulus $[N^p,_p G][N,_{p^2} G]$. Clearly $[N,_{p^2} G] = 1$. On the other hand, if $\beta \in \langle \alpha \rangle$ and $h \in H_2$ then, again by the previous lemma,

$$(\beta h)^p = \beta^p h^p [h, \beta]^{\binom{p}{2}} \dots [h, \beta, \overset{p-1}{\dots}, \beta]^{\binom{p}{p}}.$$

Therefore $N^p \leq \langle \alpha^p \rangle H_2^p H_{p+1}$ and $[N^p,_p G] \leq [\alpha^p,_p G] H_{p+2}^p$. Now by (7),

$$[\alpha^p, G] \leq \langle [x_1, \alpha^p] \rangle H_3^p H_{p+2}$$

and consequently

$$[\alpha^p,_p G] \leq [[x_1, \alpha^p],_{p-1} G]\, H_{p+2}^p = \langle [x_1, \alpha^p, \alpha, \overset{k}{\dots}, \alpha] \mid k \geq p-1 \rangle\, H_{p+2}^p.$$

Since

$$[x_1, \alpha^p, \alpha, \overset{k}{\dots}, \alpha] = x_{k+2}^p x_{k+3}^{\binom{p}{2}} \dots x_{k+p}^{\binom{p}{p-1}} x_{k+p+1},$$

it follows that $[N^p,_p G][N,_{p^2} G] \leq \langle x_{p+1}^p x_{2p} \rangle H_{p+2}^p$.

Now if $[N^{p^2}, G] \leq [N^p,_p G][N,_{p^2} G]$ then in particular $x_{p+1}^p \in \langle x_{p+1}^p x_{2p} \rangle H_{p+2}^p$, which is not the case, and we are done.