Normal Subgroups of Profinite Groups of Non-negative Deficiency *

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September 12, 2013

Abstract

The principal focus of the paper is to show that the existence of a finitely generated normal subgroup of infinite index in a profinite group G of non-negative deficiency gives rather strong consequences for the structure of G. To make this precise we introduce the notion of p-deficiency (p a prime) for a profinite group G. We prove that if the p-deficiency of G is positive and N is a finitely generated normal subgroup such that the p-Sylow subgroup of G/N is infinite and p divides the order of N then we have $\mathrm{cd}_p(G) = 2$, $\mathrm{cd}_p(N) = 1$ and $\mathrm{vcd}_p(G/N) = 1$ for the cohomological p-dimensions; moreover either the p-Sylow subgroup of G/N is virtually cyclic or the p-Sylow subgroup of N is cyclic. If G is a profinite Poincaré duality group of dimension 3 at a prime p (PD^3-group at p) we show that for N and p as above either N is PD^1 at p and G/N is virtually PD^2 at p or N is PD^2 at p and G/N is virtually PD^1 at p.

We apply this results to deduce structural information on the profinite completions of ascending HNN-extensions of free groups and 3-manifold groups. We prove that the arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ are cohomologically good and give some implications of our theory to their congruence kernels.

1 Introduction

Let G be a group. The *deficiency* $\mathrm{def}(G)$ of G is the largest integer k such that there exists a (finite) presentation of G with the number of generators minus the number of relations equal to k. The groups of non-negative deficiency form an important class of finitely presented groups. It contains many important families of examples coming from geometry: fundamental groups of compact 3-manifolds, knot groups, arithmetic groups of rank 1, etc.

In this paper we study profinite groups of non-negative deficiency. The deficiency for profinite groups is defined in the same way as for discrete groups. We consider presentations in the category of profinite groups, i.e., saying generators we mean topological generators. We note that any presentation of a group is a presentation of its profinite completion and so the profinite completion of a group of non-negative deficiency is a profinite group of non-negative deficiency. Therefore, the profinite completions of groups mentioned above are in the range of our study. Moreover, as was shown by Lubotzky [21], Corollary 1.2, all projective groups (i.e. profinite groups of cohomological dimension 1) have non-negative deficiency as well.

*The second author was partially supported by the Spanish Ministry of Education, grant MTM2011-28229-C02-01. The third and forth authors were partially supported by "bolsa de produtividade de pesquisa" from CNPq, Brazil.

The principal focus of our study here is to show that the existence of a finitely generated normal subgroup of infinite index in a profinite group G of non-negative deficiency gives rather strong consequences for the structure of G. We are ready to state the principal results for groups of positive deficiency.

Theorem 1.1. *Let G be a finitely generated profinite group with positive deficiency and N a finitely generated normal subgroup such that the p-Sylow subgroup $(G/N)_p$ is infinite and p divides the order of N. Then either the p-Sylow subgroup of G/N is virtually cyclic or the p-Sylow subgroup of N is cyclic. Moreover, $\mathrm{cd}_p(G) = 2$, $\mathrm{cd}_p(N) = 1$ and $\mathrm{vcd}_p(G/N) = 1$, where cd_p and vcd_p stand for cohomological p-dimension and virtual cohomological p-dimension respectively.*

To prove this theorem we introduce in Section 2.2 the concept of p-deficiency $\mathrm{def}_p(G)$ for a prime p and a profinite group G. These new invariants are more suitable to the study of profinite groups than just the deficiency. The example of pro-p groups (which all have non-positive deficiency as profinite groups, see [21]) already makes it clear that our result requires a more subtle approach. Section 2.2 contains also results interconnecting the deficiency of a profinite group with its various p-deficiencies.

Theorem 1.1 has the following immediate consequence.

Corollary 1.2. *Let G be a finitely generated profinite group of positive deficiency and N a finitely generated normal subgroup of G such that the p-Sylow subgroup $(G/N)_p$ is infinite whenever the prime p divides $|N|$. Then N is projective.*

The pro-p version of Theorem 1.1 is proved in the paper [10] of Hillmann and Schmidt and its abstract version for a discrete group Γ is known only under further restrictions either on N or on Γ/N (see [2]).

A group Γ is called knot-like if $\Gamma/[\Gamma,\Gamma]$ is infinite cyclic and the deficiency satisfies $\mathrm{def}(\Gamma) = 1$. These two properties are possessed by any knot group, i.e., the fundamental group of the complement of a knot in the 3-sphere S^3. It was conjectured by E. Rapaport-Strasser in [28] that if the commutator group $\Gamma' = [\Gamma,\Gamma]$ of a knot-like group Γ is finitely generated then Γ' should be free. This conjecture is true as proved by D.H. Kochloukova in [13].

The next corollary shows that the profinite version of the Rapaport-Strasser conjecture is also true. In fact our result is stronger then just the profinite version of the conjecture since we do not assume $G/[G,G]$ to be cyclic and assume positive deficiency rather than for deficiency one.

Corollary 1.3. *Let G be a finitely generated profinite group of positive deficiency whose commutator subgroup $[G,G]$ is finitely generated. Then $\mathrm{def}(G) = 1$ and $[G,G]$ is projective. Moreover, $\mathrm{cd}(G) = 2$ unless $G = \widehat{\mathbb{Z}}$.*

The next class of groups that we study here are profinite Poincaré duality groups of dimension 3.

We establish in Section 4 the following profinite version of the result of Hillman in [9]:

Theorem 1.4. *Let G be a finitely generated profinite PD^3-group at a prime p and N be a finitely generated normal subgroup of G such that the p-Sylow $(G/N)_p$ is infinite and p divides $|N|$. Then either N is PD^1 at p and G/N is virtually PD^2 at p or N is PD^2 at p and G/N is virtually PD^1 at p.*

The pro-p version of this theorem reads as follows:

Corollary 1.5. *Let G be a finitely generated pro-p PD^3-group and N be a non-trivial finitely generated normal subgroup of G of infinite index. Then either N is infinite cyclic and G/N is virtually Demushkin or N is Demushkin and G/N is virtually infinite cyclic.*

We apply our results to ascending HNN-extensions of free groups (also known as mapping tori of free group endomorphisms). Groups of this type often appear in group theory and topology and were extensively studied (see [6], [3] for example). In particular, many one-relator groups are ascending HNN-extensions of free groups and many of such groups are hyperbolic. Corollary 1.2 allows us to establish the structure of the profinite completion of this important class of groups.

Theorem 1.6. *Let $F = F(x_1, \dots x_n)$ be a free group of finite rank n and $f : F \longrightarrow F$ an injective endomorphism. Let $\Gamma = \langle F, t \mid x_i^t = f(x_i)\rangle$ be the HNN-extension. Then the profinite completion of Γ is $\widehat{\Gamma} = P \rtimes \widehat{\mathbb{Z}}$, where P is projective. Moreover, P is free profinite of rank n if and only if f is an automorphism.*

As a corollary we obtain the following surprising consequence.

Theorem 1.7. *An ascending HNN-extension Γ of a free group is good.*

The group Γ is called *p-good* if the homomorphism of cohomology groups

$$H^n(\widehat{\Gamma}, M) \to H^n(\Gamma, M)$$

induced by the natural homomorphism $\Gamma \to \widehat{\Gamma}$ of Γ to its profinite completion $\widehat{\Gamma}$ is an isomorphism for every finite p-primary Γ-module M. The group Γ is called *good* if it is p-good for every prime p. This important concept was introduced by J-P. Serre in [31, Section I.2.6]. In his book Serre explains the fundamental role that goodness plays in the comparison of properties of a group and its profinite completion. In Section 5 we prove that all arithmetic Kleinian groups are good.

Theorem 1.8. *Let Γ be an arithmetic Kleinian group in $\mathrm{SL}_2(\mathbb{C})$. Then Γ is good.*

2 Preliminaries

This section contains certain preliminary lemmas which will be of use later. Also we fix the following standard notations for this paper.

- $\mathbb{F}_p$ - stands for the field of p elements;
- $\mathbb{Q}_p$ - is the field of p-adic numbers;
- $\mathbb{Z}_p$ - is the ring of p-adic integers;
- $\widehat{\Gamma}$ - is the the profinite completion of a group Γ;
- $\Gamma_{\hat{p}}$ - is the pro-p completion of a group Γ;
- G_p - is a p-Sylow subgroup of a profinite group G;
- $G_{[p]}$ - is the maximal pro-p quotient of a profinite group G;
- $\mathbb{Z}_p[[G]]$ - is the completed group ring, i.e., $\mathbb{Z}_p[[G]] = \varprojlim \mathbb{Z}_p[G_j]$ which is the inverse limit of ordinary group rings with G_j ranging over all the finite quotients of G;
- $[G : H]_p$ - is the largest power of p dividing the index $[G : H]$ of H in G;
- $\dim M$ - is the length of M as $\mathbb{Z}$-module.

2.1 Homology and cohomology of profinite groups

In this section we collect some notation and well known facts concerning the homology and cohomology of profinite groups. For the details see for example, [31, 30]. If we do not say the contrary module means left module.

Let G be a profinite group and B a profinite right $\mathbb{Z}_p[[G]]$-module. The ith homology group $H_i(G,B)$ of G with coefficients in B is defined by

$$H_i(G,B) = \mathrm{Tor}_i^{\mathbb{Z}_p[[G]]}(B,\mathbb{Z}_p),$$

where $\mathrm{Tor}_i^{\mathbb{Z}_p[[G]]}(-,\mathbb{Z}_p)$ is the ith left derived functor of $-\widehat{\otimes}_{\mathbb{Z}_p[[G]]}\mathbb{Z}_p$ from the category of right profinite $\mathbb{Z}_p[[G]]$-modules to profinite $\mathbb{Z}_p$-modules.

Similarly, given a discrete $\mathbb{Z}_p[[G]]$-module A, the ith cohomology group $H^i(G,A)$ of G with coefficients in A is defined by

$$H^i(G,A) = \mathrm{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p,A),$$

where $\mathrm{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p,-)$ is the ith derived functor of the left exact covariant functor $\mathrm{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p,-)$ from the category of left discrete $\mathbb{Z}_p[[G]]$-modules to discrete $\mathbb{Z}_p$-modules. It can be calculated by using either projective resolutions of the trivial module $\mathbb{Z}_p$ in the category of profinite $\mathbb{Z}_p[[G]]$-modules or injective resolutions of the discrete $\mathbb{Z}_p[[G]]$-module A.

The categories of profinite and torsion discrete $\mathbb{Z}_p[[G]]$-modules are dual via the Pontryagin duality ([30, 5.1]) and so are $H_i(G,-)$ and $H^i(G,-^*)$, where * stands for $\mathrm{Hom}(-,\mathbb{Q}/\mathbb{Z})$. Hence we have that $H_i(G,\mathbb{F}_p)$ and $H^i(G,\mathbb{F}_p)$ are vector spaces over $\mathbb{F}_p$ of the same dimension.

By definition, a profinite group H is of type p-FP_m if the trivial profinite $\mathbb{Z}_p[[H]]$-module $\mathbb{Z}_p$ has a profinite projective resolution over $\mathbb{Z}_p[[H]]$ with all projective modules in dimensions $\leq m$ finitely generated. We say that H is of type p-FP_∞ if H is of type p-FP_m for every m. If G is a profinite group of type p-FP_m and M is a G-module of p-power order, then the cohomology groups $H^i(G,A)$ (and therefore $H_i(G,A)$) are finite for all $i \leq m$. If G is a pro-p group, the fact that $\mathbb{Z}_p[[G]]$ is a local ring implies that G is of type FP_m if and only if $H^i(G,\mathbb{F}_p)$ are finite for $i \leq m$ (see Theorem 2.14 for an analogous result for an arbitrary profinite group).

The cohomological p-dimension of a profinite group G is the lower bound of the integers n such that for every discrete torsion G-module A, and for every $i > n$, the p-primary component of $H^i(G,A)$ is null. We shall use the standard notation $\mathrm{cd}_p(G)$ for cohomological p-dimension of the profinite group G. The cohomological dimension $\mathrm{cd}(G)$ of G is defined as the supremum $\mathrm{cd}(G) = \sup_p(\mathrm{cd}_p(G))$ where p varies over all primes p.

The next proposition gives a well-known characterization for cd_p (see [31, Prop. I.§3.1.11 and I.§4.1.21] and [30, Proposition 7.1.4]).

Proposition 2.1. *Let G be a profinite group, p a prime and n an integer. The following properties are equivalent:*

1. $\mathrm{cd}_p(G) \leq n$;
2. $H^i(G,A) = 0$ *for all $i > n$ and every discrete G-module A which is a p-primary torsion module;*
3. $H^{n+1}(G,A) = 0$ *when A is simple discrete G-module annihilated by p;*
4. $H^{n+1}(H,\mathbb{F}_p) = 0$ *for any open subgroup H of G.*

   Moreover, if G is a pro-p group these conditions are also equivalent to

5. *The projective dimension of the trivial $\mathbb{F}_p[[G]]$-module $\mathbb{F}_p$ is $\leq n+1$.*

Note that if G is pro-p then there is only one simple discrete G-module annihilated by p, namely the trivial module $\mathbb{F}_p$.

The Lyndon-Hochschild-Serre spectral sequence will be the most important tool in this paper. We will give its brief description and the most important consequences. For the details see [30].

Theorem 2.2 ([30, Thm. 7.2.4])**.** *Let N be a normal closed subgroup of a profinite group G, and let A be a discrete G-module. Then there exists a spectral sequence $\mathbb{E} = (E_t^{r,s})$ such that*

$$E_2^{r,s} \cong H^r(G/N, H^s(N, A)) \Rightarrow H^n(G, A).$$

Corollary 2.3. *Let N be a normal closed subgroup of a profinite group G, and let A be a discrete G-module. Then the following holds.*

(1) *There exists always a five term exact sequence*

$$\begin{aligned} &0 \to H^1(G/N, A^N) \to H^1(G, A) \to \\ &H^1(N, A)^{G/N} \to H^2(G/N, A^N) \to H^2(G, A). \end{aligned}$$

(2) *If $H^i(N, A) = 0$ for all $i \geq 1$, then $H^i(G, A) \cong H^i(G/N, A^N)$.*

(3) *If $H^1(N, A) = 0$, then $H^1(G/N, A) \cong H^1(G, A)$ and $H^2(G/N, A) \hookrightarrow H^2(G, A)$.*

(4) *If $\mathrm{cd}_p(G/N) \leq 1$ and A is a p-primary torsion, then there exists the following exact sequence*

$$0 \to H^1(G/N, H^i(N, A)) \to H^{i+1}(G, A) \to H^{i+1}(N, A)^{G/N} \to 0.$$

(5) *If $G/N^p[N, N]$ splits as the direct product $G/N \times N/N^p[N, N]$ then*

$$H^2(G/N, \mathbb{F}_p) \oplus H^1(G/N, H^1(N, \mathbb{F}_p)) \hookrightarrow H^2(G, \mathbb{F}_p).$$

Proof. Items (1) and (2) can be found in [30, Corollary 7.2.5]; (3) follows directly from (1). We sketch the proofs of items (4) and (5).

(4) Consider the Lyndon-Hochschild-Serre spectral sequences $(E_\bullet^{\bullet,\bullet}, d_\bullet)$ for the G-module A associated to the extension $1 \to N \to G \to G/N \to 1$. From the hypothesis it follows that $E_2^{t,\bullet} = 0$ for $t \geq 2$. Hence the spectral sequence $(E_\bullet^{\bullet,\bullet}, d_\bullet)$ collapses at the E_2-term.

(5) Consider the Lyndon-Hochschild-Serre spectral sequences $(E_\bullet^{\bullet,\bullet}, d_\bullet)$ for the trivial G-module $\mathbb{F}_p$ associated to the extension $1 \to N \to G \to G/N \to 1$, and $(\bar{E}_\bullet^{\bullet,\bullet}, d_\bullet)$ for the trivial $G/N^p[N, N]$-module $\mathbb{F}_p$ associated to the extension $1 \to N/N^p[N, N] \to G/N^p[N, N] \to G/N \to 1$. Note that by construction, the second extension is a direct product, and thus the spectral sequence $(\bar{E}_\bullet^{\bullet,\bullet}, d_\bullet)$ collapses at the $\bar{E}_2$-term, i.e., $\bar{d}_t = 0$ for all $t \geq 2$ ([27, p. 96, Exercise 7]). Denote by $\pi_2^{n,k}$ the natural map $\bar{E}_2^{n,k} \to E_2^{n,k}$. Then for each pair n, k we have the following commutative diagram

$$\begin{array}{ccc} \bar{E}_2^{n,k} & \xrightarrow{0} & \bar{E}_2^{n+2,k-1} \\ \downarrow{\scriptstyle \pi_2^{n,k}} & & \downarrow{\scriptstyle \pi_2^{n+2,k-1}} \\ E_2^{n,k} & \xrightarrow{d_2^{n,k}} & E_2^{n+2,k-1} \end{array}$$

Since $\pi_2^{0,1}$ is an isomorphism, it follows from the diagram that $d_2^{0,1} = 0$ and so $E_3^{2,0} = E_2^{2,0} \cong H^2(G/N, \mathbb{F}_p)$. In the same way, as $\pi_2^{1,1}$ is an isomorphism, we get that $d_2^{1,1} = 0$ and so $E_3^{1,1} = E_2^{1,1} \cong H^1(G/N, H^1(N, \mathbb{F}_p))$.

□

We will use the following result that relates the cohomological p-dimensions of G, N and G/N.

Theorem 2.4 ([34, Thm. 1.1])**.** *Let G be a profinite group of finite cohomological p-dimension $cd_p(G) = n$ and let N be a closed normal subgroup of G of cohomological p-dimension $cd_p(N) = k$ such that $H^k(N, \mathbb{F}_p)$ is nonzero and finite. Then G/N is of virtual cohomological p-dimension $n - k$.*

In the case when N is of cohomological p-dimension 0 or 1 we have the following corollary.

Corollary 2.5. *Let G be a profinite group of finite cohomological p-dimension $cd_p(G)$ and let N be a finitely generated closed normal subgroup of G of cohomological p-dimension $cd_p(N) \leq 1$. Then G/N is of virtual cohomological p-dimension $cd_p(G) - cd_p(N)$.*

Proof. If p does not divide $|N|$, then Corollary 2.3(2) implies the isomorphisms $H^k(U/N, \mathbb{F}_p) \cong H^k(U, \mathbb{F}_p)$ for all open subgroups U of G which contain N and all k. Now from Proposition 2.1 it follows that $cd_p(G/N) = cd_p(G)$.

If p divides $|N|$ then, by Proposition 2.1, $cd_p(N) = 1$ and so there exists an open subgroup V of N such that $H^1(V, \mathbb{F}_p) \neq 0$. We can find an open subgroup U of G such that $U \cap N = V$ and apply Theorem 2.4 to U and V. □

2.2 The deficiency

If G is a finitely generated group (profinite group), then we say that G is of deficiency k if there exist a presentation $\pi : F \to G$ of G of deficiency k, i.e. such that the rank of the free group (profinite group) F minus the number of generators of $\ker \pi$ as a (closed) normal subgroup of F is k. If $r > 0$, it is possible to add further relations that are consequences of the original ones, so a presentation of smaller deficiency is obtained. Thus by our definition a group of deficiency $k+1$ is also of deficiency k. We denote by $\mathrm{def}(G)$ the greatest k such that G is of deficiency k. Note that for any group G one has $\mathrm{def}(G) \leq \mathrm{def}(\widehat{G})$.

Below we shall introduce other invariants that help to describe more precisely properties of profinite groups; some of them are taken from [21]. The necessity of this already can be seen from the fact that any pro-p group (including free pro-p groups) as a profinite group is of non-positive deficiency.

Let G be a profinite group. Denote by $d(G) \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ its minimal number of generators. If M is a non zero finite G-module we denote by $\dim M$ the length of M as $\mathbb{Z}$-module and, moreover, if $\dim H^1(G, M)$ is finite we put

$$\overline{\chi}_2(G, M) = \frac{-\dim H^2(G, M) + \dim H^1(G, M) - \dim H^0(G, M)}{\dim(M)} \in \mathbb{Q} \cup \{-\infty\}.$$

Also we introduce a cohomological variation of the minimal number of generators. For a non zero finite G-module M we put

$$\overline{\chi}_1(G, M) = \frac{\dim H^1(G, M) - \dim H^0(G, M)}{\dim(M)} \in \mathbb{Q}_{\geq -1} \cup \{+\infty\}.$$

If $M = \{0\}$, then we agree that $\overline{\chi}_2(G, M) = +\infty$ and $\overline{\chi}_1(G, M) = -\infty$. In fact, $\overline{\chi}_k(G, M) = -\chi_k(G, M)/\dim(M)$, where $\chi_k(G, M)$ is the partial Euler characteristic of the G-module M.

Example 2.6. a) If G is a finitely generated profinite group and $d(G)$ denotes the minimal number of generators of G, we have

$$\dim H^0(G, M) = \dim M^G \text{ and } \overline{\chi}_1(G, M) \leq d(G) - 1;$$

b) If F is a free profinite group of rank $d(F)$, then

$$\overline{\chi}_2(F,M) = \overline{\chi}_1(F,M) = d(F) - 1.$$

Indeed, for any finitely generated profinite group G, the augmentation ideal of $\mathbb{Z}_p[[G]]$ can be generated as $\mathbb{Z}_p[[G]]$-module by $d(G)$ elements. So we write a partial free $\mathbb{Z}_p[[G]]$-resolution

$$\mathcal{F}: \quad F_2 \to \mathbb{Z}_p[[G]]^{d(G)} \xrightarrow{\delta} \mathbb{Z}_p[[G]] \to \mathbb{Z}_p \to 0$$

and apply $Hom_{\mathbb{Z}_p[[G]]}(-,M)$ to the complex $\mathcal{F}_{del}$ obtained by suppressing $\mathbb{Z}_p$. The cohomology of G (in small dimensions) is the cohomology of the complex

$$Hom_{\mathbb{Z}_p[[G]]}(\mathcal{F}_{del},M): \qquad 0 \to M \xrightarrow{\phi} M^{d(G)} \xrightarrow{\psi} Hom_{\mathbb{Z}_p[[G]]}(F_2,M).$$

Then

$$\dim H^0(G,M) = \dim\ker(\phi) = \dim M^G$$

and

$$\begin{aligned}\dim(M)\overline{\chi}_1(G,M) &= \dim H^1(G,M) - \dim H^0(G,M)\\ &= \dim\ker(\psi) - \dim\ \mathrm{Im}(\phi) - \dim\ \ker(\phi)\\ &= \dim\ker(\psi) - \dim M\\ &\le d(G)\dim(M) - \dim(M) = \dim(M)(d(G)-1).\end{aligned}$$

This proves item a). Now, if $G = F$ is free, the augmentation ideal of $\mathbb{Z}_p[[F]]$ is free as $\mathbb{Z}_p[[F]]$-module of rank $d(F)$. So we can take $F_2 = 0$ in $\mathcal{F}$ and the above inequality is an equality. The first equality in b) we obtain from $H^2(F,M) = 0$.

Lemma 2.7. *Let G be a finitely generated profinite group. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite G-modules. Then*

$$\overline{\chi}_2(G,M) \ge \min\{\overline{\chi}_2(G,M'), \overline{\chi}_2(G,M'')\}$$

and

$$\overline{\chi}_1(G,M) \le \max\{\overline{\chi}_1(G,M'), \overline{\chi}_1(G,M'')\}.$$

Proof. We will prove the first inequality. The second one is proved using the same method. The short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

gives a long exact sequence in cohomology

$$0 \to H^0(G,M') \to H^0(G,M) \to \ldots \to H^2(G,M) \to H^2(G,M'') \xrightarrow{\delta_3} \ldots$$

Counting dimension we have

$$\begin{aligned}\dim H^0(G,M) = \ & \dim H^0(G,M') + \dim H^0(G,M'') - \dim H^1(G,M')\\ & + \dim H^1(G,M) - \dim H^1(G,M'') + \dim H^2(G,M')\\ & - \dim H^2(G,M) + \dim H^2(G,M'') - \dim \mathrm{Im}\,\delta_3\end{aligned}$$

from which we obtain

$$\dim(M')\overline{\chi}_2(G,M') - \dim(M)\overline{\chi}_2(G,M) + \dim(M'')\overline{\chi}_2(G,M'') \le 0.$$

Hence,

$$\overline{\chi}_2(G,M) \ge \tfrac{\dim(M')\overline{\chi}_2(G,M') + \dim(M'')\overline{\chi}_2(G,M'')}{\dim(M)} \ge \min\{\overline{\chi}_2(G,M'), \overline{\chi}_2(G,M'')\}.$$

□

Let G be a finitely generated profinite group and let

$$1 \to R \to F \to G \to 1$$

be a presentation of G, where F is a $d(G)$-generated free profinite group. Then $\bar{R} := R/[R,R] \cong H_1(R, \hat{\mathbb{Z}})$ as G-modules and it is called the relation module of G. By Shapiro's lemma $\bar{R} \cong H_1(F, \hat{\mathbb{Z}}[[G]])$ and, as is shown in [21], it does not depend on the presentation.

If we decompose $\bar{R}$ as the product of its p-primary components $\bar{R} = \prod_p \bar{R}_p$, one gets $\bar{R}_p \cong H_1(F, \mathbb{Z}_p[[G]])$. Since the augmentation ideal of $\mathbb{Z}_p[[F]]$ is a free profinite $\mathbb{Z}_p[[F]]$-module of rank $d(G)$, we have the exact sequence

$$0 \to \mathbb{Z}_p[[F]]^{d(G)} \to \mathbb{Z}_p[[F]] \to \mathbb{Z}_p \to 0$$

of $\mathbb{Z}_p[[F]]$-modules. Applying the functor $\mathbb{Z}_p[[F/R]]\widehat{\otimes}_{\mathbb{Z}_p[[F]]}-$, we obtain the following exact sequence

$$0 \to \bar{R}_p \to \mathbb{Z}_p[[G]]^{d(G)} \xrightarrow{\varphi} \mathbb{Z}_p[[G]] \to \mathbb{Z}_p \to 0, \tag{2.1}$$

since $\ker(\varphi) = H_1(F, \mathbb{Z}_p[[G]])$. If M is a (left or right) $\mathbb{Z}_p[[G]]$-module, we denote by $d_G(M)$ its minimal number of generators. Thus, the last exact sequence may be rewritten as follows:

$$\mathbb{Z}_p[[G]]^{d_G(\bar{R}_p)} \to \mathbb{Z}_p[[G]]^{d(G)} \to \mathbb{Z}_p[[G]] \to \mathbb{Z}_p \to 0. \tag{2.2}$$

In particular, since $H_1(G, \mathbb{F}_p[[G]]) = 0$ if $i \geq 1$, we also obtain the following exact sequence of $\mathbb{F}_p[[G]]$-modules:

$$\mathbb{F}_p[[G]]^{d_G(\bar{R}_p)} \to \mathbb{F}_p[[G]]^{d(G)} \to \mathbb{F}_p[[G]] \to \mathbb{F}_p \to 0. \tag{2.3}$$

The following result is proved in [21].

Theorem 2.8. *Let G be a finitely generated profinite group. Then, for a fixed prime p,*

$$d_G(\bar{R}_p) = \inf_M \{d(G) - 1 - \overline{\chi}_2(G, M)\}$$

where M runs over all irreducible $\mathbb{Z}_p[[G]]$-modules, and

$$d_G(\bar{R}) = \max_p d_G(\bar{R}_p).$$

Moreover, $\mathrm{def}(G) = d(G) - d_G(\bar{R})$ *unless $\bar{R} = 0$ and G is not free, in which case* $\mathrm{def}(G) = d(G) - 1$.

Combining this theorem with Lemma 2.7, we obtain that if $\bar{R} \neq 0$, then

$$\mathrm{def}(G) = \inf\{1 + \overline{\chi}_2(G, M) | M \text{ is a finite } G\text{-module}\}. \tag{2.4}$$

If N is a closed subgroup of G and $\mathcal{M}_p(N)$ is the set of all finite $\mathbb{Z}_p[[G]]$-modules on which N acts trivially, we introduce the following invariants:

$$\mathrm{def}_p(G, N) = \inf_{M \in \mathcal{M}_p(N)} \{1 + \overline{\chi}_2(G, M)\}$$

and

$$d_p(G, N) = \max_{M \in \mathcal{M}_p(N)} \{1 + \overline{\chi}_1(G, M)\}.$$

For simplicity, we put

$$\mathrm{def}_p(G) = \mathrm{def}_p(G, \{1\}) \quad \text{and} \quad d_p(G) = d_p(G, \{1\}).$$

The number $\mathrm{def}_p(G)$ is called the **p-deficency** of G. Comparing this invariant with the deficiency of G we observe that

$$\mathrm{def}(G) \leq \mathrm{def}_p(G) \leq \dim H^1(G, \mathbb{F}_p) - \dim H^2(G, \mathbb{F}_p), \tag{2.5}$$

so

$$\mathrm{def}(G) \leq \mathrm{def}_p(G) \leq \overline{\chi}_2(G, \mathbb{F}_p) + 1.$$

When G is pro-p, we obtain from (2.4) that $\mathrm{def}_p(G) = \overline{\chi}_2(G, \mathbb{F}_p) - 1$; by [21, Corollary 5.5] $d_G(\bar{R}) = \max\{d(G), \dim H^2(G, \mathbb{F}_p)\}$ from which we deduce that

$$\mathrm{def}(G) = \min\{0, \mathrm{def}_p(G)\}.$$

Lemma 2.9. *Let G be a finitely generated profinite group and $G_{[p]}$ its maximal pro-p quotient, then $\overline{\chi}_2(G, \mathbb{F}_p) \leq \overline{\chi}_2(G_{[p]}, \mathbb{F}_p)$.*

Proof. Let N be the kernel of the natural map $G \to G_{[p]}$. Then $H^1(N, \mathbb{F}_p) = 0$. Thus, the lemma follows from Corollary 2.3(3). □

Lemma 2.10. *Let G be a profinite group and G_p its p-Sylow subgroup. Assume that G_p is not cyclic. Then we can find an open subgroup U of G such that $\overline{\chi}_1(U, \mathbb{F}_p) \geq 1$.*

Proof. Since G_p is not cyclic, we can find an open normal subgroup N of G such that G_pN/N is not cyclic. Put $U = G_pN$. Then it is clear that $\dim H^1(U, \mathbb{F}_p) > 1$. Therefore $\overline{\chi}_1(U, \mathbb{F}_p) = \dim H^1(U, \mathbb{F}_p) - 1 \geq 1$. □

Lemma 2.11. *Let G be a finitely generated profinite group, N a closed subgroup and H an open subgroup containing N. Then*

$$\mathrm{def}_p(H, N) - 1 \geq [G : H](\mathrm{def}_p(G, N) - 1)$$

and

$$d_p(H, N) - 1 \leq [G : H](d_p(G, N) - 1).$$

Proof. Let M be a finite $\mathbb{F}_p[H]$-module. By Shapiro's lemma $H^i(H, M) = H^i(G, Coind_H^G(M))$ for all i, so we obtain that

$$\overline{\chi}_i(H, M) = [G : H]\overline{\chi}_i(G, Coind_H^G(M)).$$

Thus,

$$\begin{aligned} \mathrm{def}_p(H, N) - 1 &= \inf_{M \in \mathcal{M}_p(N)}\{\overline{\chi}_2(H, M)\} \\ &= [G : H] \inf_{M \in \mathcal{M}_p(N)}\{\overline{\chi}_2(G, Coind_H^G(M))\} \\ &\geq [G : H] \inf_{M' \in \mathcal{M}_p(N)}\{\overline{\chi}_2(G, M')\} \\ &= [G : H](\mathrm{def}_p(G, N) - 1) \end{aligned}$$

and

$$\begin{aligned} d_p(H, N) - 1 &= \max_{M \in \mathcal{M}_p(N)}\{\overline{\chi}_1(H, M)\} \\ &= [G : H] \max_{M \in \mathcal{M}_p(N)}\{\overline{\chi}_2(G, Coind_H^G(M))\} \\ &\leq [G : H] \max_{M' \in \mathcal{M}_p(N)}\{\overline{\chi}_2(G, M')\} \\ &= [G : H](d_p(G, N) - 1). \end{aligned}$$

□

We shall finish the section with two general technical results about the numerical invariants introduced here.

Proposition 2.12. *Let G_1 be a finitely generated profinite group, N a normal subgroup, G_2 an open normal subgroup of index a power of p in G_1 containing N and let M be a non zero finite $\mathbb{F}_p[[G_1]]$-module. Denote by β_i $(i = 1, 2)$ the restrictions maps $H^2(G_i, M) \to H^2(N, M)$ respectively. Then*

$$\dim(\operatorname{Im}\beta_2) \geq (\overline{\chi}_2(G_1, M)\dim(M) + \dim(\operatorname{Im}\beta_1))[G_1 : G_2] - \overline{\chi}_2(G_2, M)\dim(M)$$

and, in particular,

$$\overline{\chi}_1(G_2, M) \geq [G_1 : G_2]\left(\frac{\dim(\operatorname{Im}\beta_1)}{\dim(M)} + \overline{\chi}_2(G_1, M)\right).$$

Proof. Let

$$1 \to R \to F_1 \xrightarrow{\phi} G_1 \to 1$$

be a presentation for G_1 with $d(F_1) = d(G_1)$. We have the following presentation for G_2:

$$1 \to R \to F_2 \xrightarrow{\phi} G_2 \to 1,$$

where $F_2 = \phi^{-1}(G_2)$.

From Corollary 2.3(1), we obtain the following two exact sequences $(i = 1, 2)$:

$$0 \to H^1(G_i, M) \to H^1(F_i, M) \to H^1(R, M)^{G_i} \to H^2(G_i, M) \to 0. \tag{2.6}$$

Note that $\overline{\chi}_1(F_i, M) = d(F_i) - 1$ (see Example 2.6). Therefore, using the Schreier formula for F_i and the equality $H^0(F_i, M) = H^0(G_i, M)$ one has

$$\dim H^1(R, M)^{G_i} = \dim H^1(F_i, M) - \overline{\chi}_2(G_i, M)\dim(M) - \dim H^0(G_i, M) =$$

$$(\overline{\chi}_1(F_1, M)[G_1 : G_i] - \overline{\chi}_2(G_i, M))\dim(M). \tag{2.7}$$

Let α_i be the composition of the map $H^1(R, M)^{G_i} \to H^2(G_i, M)$ from (2.6) and the restriction map $\beta_i \colon H^2(G_i, M) \to H^2(N, M)$. Put

$$M_i = \ker \alpha_i.$$

Since the following diagram is commutative

$$\begin{array}{ccc} H^1(R, M)^{G_1} & \xrightarrow{\alpha_1} & H^2(N, M) \\ \downarrow & & \| \\ H^1(R, M)^{G_2} & \xrightarrow{\alpha_2} & H^2(N, M), \end{array}$$

we obtain that $M_2^{G_1} \leq M_1$. Recall that

$$\begin{aligned} \dim(M_1) &= \dim H^1(R, M)^{G_1} - \dim(\operatorname{Im}\alpha_1) \\ &= \dim H^1(R, M)^{G_1} - \dim(\operatorname{Im}\beta_1) \\ &= (\overline{\chi}_1(F_1, M) - \overline{\chi}_2(G_1, M))\dim(M) - \dim(\operatorname{Im}\beta_1). \end{aligned}$$

Since G_1/G_2 is a finite p-group, we have $\dim(M_2) \leq \dim(M_2^{G_1})[G_1 : G_2]$ (see page 7 in [12] for G_1/G_2 cyclic, the general case follows by an obvious induction on the order of G_1/G_2). Hence we get

$$\dim(M_2) \leq \dim(M_2^{G_1})[G_1 : G_2] \leq \dim(M_1)[G_1 : G_2] =$$

$$= ((\overline{\chi}_1(F_1, M) - \overline{\chi}_2(G_1, M))\dim(M) - \dim(\operatorname{Im}\beta_1))[G_1 : G_2]).$$

Thus, combining this with (2.7) we get

$$\begin{aligned}\dim(\operatorname{Im}\beta_2) &= \dim(\operatorname{Im}\alpha_2) = \dim H^1(R,M)^{G_2} - \dim(M_2)\\ &\geq (\overline{\chi}_2(G_1,M)\dim(M) + \dim(\operatorname{Im}\beta_1))[G_1:G_2] - \overline{\chi}_2(G_2,M)\dim(M).\end{aligned}$$

Since

$$\overline{\chi}_1(G_2,M) = \tfrac{\overline{\chi}_2(G_2,M)\dim(M)+\dim H^2(G_2,M)}{\dim(M)} \text{ and } \dim H^2(G_2,M) \geq \dim \operatorname{Im}(\beta_2),$$

we obtain

$$\begin{aligned}\overline{\chi}_1(G_2,M) &\geq \tfrac{\dim(\operatorname{Im}\beta_2)+\overline{\chi}_2(G_2,M)\dim(M)}{\dim(M)}\\ &\geq (\overline{\chi}_2(G_1,M) + \tfrac{\dim(\operatorname{Im}\beta_1)}{\dim(M)})[G_1:G_2].\end{aligned}$$

□

Proposition 2.13. *Let G be a finitely generated profinite group and N a normal subgroup such that $\overline{\chi}_1(N,\mathbb{F}_p)$ is non-negative and finite. Then there exists an open subgroup V of G containing N such that for every open subgroup U of V containing N*

$$\begin{aligned}\overline{\chi}_2(U,\mathbb{F}_p) &\leq -\overline{\chi}_1(U/N,\mathbb{F}_p)\overline{\chi}_1(N,\mathbb{F}_p) - \dim H^2(U/N,\mathbb{F}_p)\\ &= -\left[\overline{\chi}_1(U,\mathbb{F}_p) - \overline{\chi}_1(N,\mathbb{F}_p) - 1\right]\overline{\chi}_1(N,\mathbb{F}_p) - \dim H^2(U/N,\mathbb{F}_p).\end{aligned}$$

Proof. Since $N^p[N,N]$ is open in N there exists an open subgroup J of G such that $J\cap N = N^p[N,N]$. Put $V = JN$ and let U be an open subgroup of V. Then, by construction, $\overline{\chi}_1(U,\mathbb{F}_p) = \overline{\chi}_1(U/N,\mathbb{F}_p) + \overline{\chi}_1(N,\mathbb{F}_p) + 1$. Thus using Corollary 2.3(5), we obtain that

$$\begin{aligned}\overline{\chi}_2(U,\mathbb{F}_p) &= \overline{\chi}_1(U,\mathbb{F}_p) - \dim H^2(U,\mathbb{F}_p)\\ &\leq (\overline{\chi}_1(U/N,\mathbb{F}_p) + \overline{\chi}_1(N,\mathbb{F}_p) + 1) -\\ &\qquad - (\overline{\chi}_1(U/N,\mathbb{F}_p)+1)(\overline{\chi}_1(N,\mathbb{F}_p)+1) - \dim H^2(U/N,\mathbb{F}_p)\\ &= -\overline{\chi}_1(U/N,\mathbb{F}_p)\overline{\chi}_1(N,\mathbb{F}_p) - \dim H^2(U/N,\mathbb{F}_p)\\ &= -[\overline{\chi}_1(U,\mathbb{F}_p) - \overline{\chi}_1(N,\mathbb{F}_p) - 1]\overline{\chi}_1(N,\mathbb{F}_p) - \dim H^2(U/N,\mathbb{F}_p).\end{aligned}$$

□

2.3 The number of generators of modules over a profinite group

In this subsection we describe a way to calculate the number of generators of a profinite G-module. This will be used several times in the paper. As an application we obtain a characterization of a profinite group to be of type p-FP_m similar to Lubotzky's characterization of a profinite group to have finite deficiency.

For any irreducible $\mathbb{Z}_p[[G]]$-module M, denote by I_M the annihilator of M in $\mathbb{Z}_p[[G]]$. If K is a $\mathbb{Z}_p[[G]]$-module, then $K/I_MK \cong (\mathbb{Z}_p[[G]]/I_M)\widehat{\otimes}_{\mathbb{Z}_p[[G]]}K$ is the maximal quotient of K isomorphic to a direct sum of copies of M. Thus, $\mathbb{Z}_p[[G]]/I_M$ is the maximal cyclic $\mathbb{Z}_p[[G]]$-module isomorphic to a direct sum of copies of M.

Note that the Jacobson radical $J(K)$ of a $\mathbb{Z}_p[[G]]$-module K is equal to the intersection of I_MK and so

$$K/J(K) \cong \prod_{M \text{ is irreducible}} K/I_MK.$$

Thus

$$d_G(K) = d_G(K/J(K)) = \max_{M \text{ is irreducible}} d_G(K/I_M K),$$

and so we obtain that

$$d_G(K) = \max_{M \text{ is irreducible}} \left\lceil \frac{\dim(K/I_M K)}{\dim(\mathbb{Z}_p[[G]]/I_M)} \right\rceil.$$

Since $K/I_M K$ and $\mathbb{Z}_p[[G]]/I_M$ are direct sums of copies of M, we conclude that

$$\left\lceil \frac{\dim(K/I_M K)}{\dim(\mathbb{Z}_p[[G]]/I_M)} \right\rceil = \left\lceil \frac{\dim \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(K/I_M K, M)}{\dim \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G]]/I_M, M)} \right\rceil.$$

Note that

$$\operatorname{Hom}_{\mathbb{Z}_p[[G]]}(K/I_M K, M) \cong \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(K, M)$$

and

$$\dim \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G]]/I_M, M) = \dim \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p[[G]], M) = \dim M.$$

Thus, we conclude that

$$d_G(K) = \max_{M \text{ is irreducible}} \left\lceil \frac{\dim \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(K, M)}{\dim M} \right\rceil. \tag{2.8}$$

The next theorem is inspired by a theorem of Lubotzky [21, Theorem 0.3] that says that a finitely generated profinite group is finitely presented if and only if there exists C such that $\dim H^2(G, M) \leq C \dim M$ for any irreducible $\hat{\mathbb{Z}}[[G]]$-module M.

Theorem 2.14. *Let G be a profinite group. Then G is of type p-FP_m if and only if there exists a constant C such that $\dim H^i(G, M) \leq C \dim M$ for any irreducible $\mathbb{Z}_p[[G]]$-module M and any $0 \leq i \leq m$.*

Proof. We prove the theorem by induction on m. The case $m = 0$ is trivial. Assume that theorem holds for $m - 1$.

Suppose, first, that G is of type p-FP_m. By induction, there exists C' such that $\dim H^i(G, M) \leq C' \dim M$ for any irreducible $\mathbb{Z}_p[[G]]$-module M and any $1 \leq i \leq m - 1$. Since G is of type p-FP_m, there exists an exact sequence of finitely generated projective modules

$$\mathcal{R}: \quad P_m \to P_{m-1} \to \ldots \to P_0 \to \mathbb{Z}_p \to 0.$$

Let M be an irreducible $\mathbb{Z}_p[[G]]$-module. If we apply $\operatorname{Hom}_{\mathbb{Z}_p[[G]]}(-, M)$ to the complex $\mathcal{R}_{del}$ obtained by suppressing $\mathbb{Z}_p$, we obtain the complex $\operatorname{Hom}_{\mathbb{Z}_p[[G]]}(\mathcal{R}_{del}, M)$:

$$0 \to \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(P_0, M) \to \ldots \to \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(P_{m-1}, M) \to^{\phi} \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(P_m, M).$$

The cohomology groups $H^i(G, M)$ for $i \leq m - 1$ are the cohomology groups of this complex and $H^m(G, M)$ is a subgroup of $\operatorname{Hom}_{\mathbb{Z}_p[[G]]}(P_m, M)/\operatorname{Im}\phi$. Hence

$$\sum_{i=0}^{m} (-1)^{m-i} \dim \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(P_i, M) \geq \sum_{i=0}^{m} (-1)^{m-i} \dim H^i(G, M), \tag{2.9}$$

Therefore

$$\begin{aligned} \dim H^m(G, M) &\leq \textstyle\sum_{i=0}^{m} \dim \operatorname{Hom}_{\mathbb{Z}_p[[G]]}(P_i, M) + \sum_{i=0}^{m-1} \dim H^i(G, M) \\ &\leq (\textstyle\sum_{i=0}^{m} d_G(P_i) + mC') \dim M. \end{aligned}$$

Thus we may put $C = \sum_{i=0}^{m} d_G(P_i) + mC'$.

Suppose now that there exists a constant C such that $\dim H^i(G, M) \leq C \dim M$ for any irreducible $\mathbb{Z}_p[[G]]$-module M and any $0 \leq i \leq m$. By inductive assumption, there exists an exact sequence

$$\mathcal{R}: \quad 0 \to A \to P_{m-1} \to \ldots \to P_0 \to \mathbb{Z}_p \to 0$$

with P_i finitely generated projective for $0 \leq i \leq m-1$. We want to show that A is finitely generated, since then we can cover it by a finitely generated free module. Let M be an irreducible $\mathbb{Z}_p[[G]]$-module. If we apply $\mathrm{Hom}_{\mathbb{Z}_p[[G]]}(-, M)$ to the complex $\mathcal{R}_{del}$ obtained by suppressing $\mathbb{Z}_p$, we obtain the complex $\mathrm{Hom}_{\mathbb{Z}_p[[G]]}(\mathcal{R}_{del}, M)$:

$$0 \to \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(P_0, M) \to \ldots \to \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(P_{m-1}, M) \to^{\phi} \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(A, M) \to 0.$$

The cohomology groups $H^i(G, M)$ for $i \leq m-1$ are the cohomology groups of this complex and $\mathrm{Hom}_{\mathbb{Z}_p[[G]]}(A, M)/\operatorname{Im}\phi$ is a subgroup of $H^m(G, M)$. Hence

$$\begin{aligned} \dim \mathrm{Hom}_{\mathbb{Z}_p[[G]])}(A, M) &\leq -\textstyle\sum_{i=0}^{m-1}(-1)^{m-i} \dim \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(P_i, M) \\ &+ \textstyle\sum_{i=0}^{m} \dim(-1)^{m-i} H^i(G, M) \\ &\leq (\textstyle\sum_{i=0}^{m-1} d_G(P_i) + (m+1)C) \dim M. \end{aligned}$$

Therefore, by (2.8), A is finitely generated. □

We will need the following application of the previous theorem.

Corollary 2.15. *Let G be a profinite group of type p-FP_2 and N a normal subgroup of G. Assume that N is finitely generated as a normal subgroup. Then G/N is of type p-FP_2.*

Proof. Let M be an irreducible $\mathbb{Z}_p[[G/N]]$-module. From Corollary 2.3(1), we obtain that

$$\dim H^1(G/N, M) \leq \dim H^1(G, M)$$

and

$$\dim H^2(G/N, M) \leq \dim H^1(N, M)^{G/N} + \dim H^2(G, M).$$

Since $M^N = M$ we have that $H^1(N, M) = \mathrm{Hom}(N, M)$ (the set of continuous homomorphisms from N to M). Recall that the action of G on $\mathrm{Hom}(N, M)$ is defined in the following way: for $x \in G$ and $f \in \mathrm{Hom}(N, M)$

$$(xf)(y) = xf(y^x), \quad y \in N.$$

Thus if $f \in \mathrm{Hom}(N, M)^{G/N}$, $f(y^x) = x^{-1}f(y)$. Let $n_1, \ldots, n_l$ be generators of N as a normal subgroup. Then N is generated by $\{n_i^x : x \in G, 1 \leq i \leq l\}$. Therefore any element of $\mathrm{Hom}(N, M)^{G/N}$ is completely determined by the values of $\{f(n_i)\}$. Thus, $\dim \mathrm{Hom}(N, M)^{G/N} \leq l \dim M$. This implies that

$$\dim H^2(G/N, M) \leq l \dim M + \dim H^2(G, M).$$

Since G is of type p-FP_2, the previous theorem implies that G/N is also of type p-FP_2. □

3 Profinite groups of positive deficiency

This section consists of main results on profinite groups of positive deficiency. We divide our results in subsections by the reverse order on deficiency.

3.1 Groups of deficiency ≥ 2

Proposition 3.1. *Let G be a finitely generated profinite group, N a normal subgroup and H and J two open subgroups containing N. If $\mathrm{def}_p(H,N) \geq 2$, then*

$$\frac{d_p(J,N)-1}{[G:J]} \geq \frac{1}{[G:H]}.$$

Proof. Using Lemma 2.11, we obtain that

$$\begin{aligned} d_p((J\cap H),N) &\geq \mathrm{def}_p((J\cap H),N) \geq [H:(J\cap H)](\mathrm{def}_p(H,N)-1)+1 \\ &\geq [H:(J\cap H)]+1 \end{aligned}$$

Hence, by Lemma 2.11,

$$d_p(J,N)-1 \geq \frac{d_p((J\cap H),N)-1}{[J:J\cap H]} \geq \frac{[H:(J\cap H)]}{[J:J\cap H]} \geq \frac{[G:J]}{[G:H]}.$$

□

Proposition 3.2. *Let G be a finitely generated profinite group and N a normal subgroup of infinite index such that $H^1(N,\mathbb{F}_p)\neq 0$. If $\mathrm{def}_p(G,N)\geq 2$, then $H^1(N,\mathbb{F}_p)$ is infinite.*

Proof. If $H^1(N,\mathbb{F}_p)$ is finite, then by Proposition 2.13 there exists an open subgroup U containing N such that

$$\overline{\chi}_2(U,\mathbb{F}_p) \leq -\overline{\chi}_1(U/N,\mathbb{F}_p)\overline{\chi}_1(N,\mathbb{F}_p) - \dim H^2(U/N,\mathbb{F}_p).$$

Since by Proposition 3.1 we have $\overline{\chi}_1(U/N,\mathbb{F}_p)$ non negative, it follows that $\overline{\chi}_2(U,\mathbb{F}_p)\leq 0$. Thus $\mathrm{def}_p(U,N)=\inf_{M\in\mathcal{M}_p(N)}\{1+\overline{\chi}_2(U,M)\}\leq 1$. By Lemma 2.11, $\mathrm{def}_p(G,N)\geq 2$ implies $\mathrm{def}_p(U,N)\geq 2$, a contradiction. □

Theorem 8.6.5 in [30] originally proved by Melnikov states that a non-trivial normal subgroup of a non-cyclic free profinite group of infinite index is infinitely generated. The next corollary extends this to profinite groups of deficiency ≥ 2.

Corollary 3.3. *Let G be a finitely generated profinite group and N a normal subgroup of infinite index such that p divides $|N|$. If $\mathrm{def}_p(G)\geq 2$, then some open subgroup of N has infinite p-abelianization. In particular, any non-trivial normal subgroup of a profinite group of deficiency ≥ 2 that has infinite index is infinitely generated.*

Proof. Find an open subgroup U of G such that $H^1(U\cap N,\mathbb{F}_p)\neq 0$ and apply the previous proposition. □

3.2 Groups of deficiency 1

The main tool of this subsection is the following result.

Theorem 3.4. *Let G be a finitely generated profinite group, $K\leq N$ two normal subgroups of G such that $|G/N|_p$ is infinite and $\mathrm{def}_p(G,K)\geq 1$. Let M be a non zero finite $\mathbb{F}_p[[G]]$-module on which K acts trivially. Suppose that*

$$\inf\left\{\frac{d_p(H,K)-1}{[G:H]_p} \mid N<H\leq_0 G\right\}=0,$$

where $[G:H]_p$ is the greatest power of p dividing $[G:H]$. Then

(1) $\mathrm{def}_p(U,K)=1$ *for any open subgroup* U *containing* N *and*

(2) $H^2(N,M)=\{0\}$.

Proof. Let U be an open subgroup of G containing N. Then by Lemma 2.11, $\mathrm{def}_p(U,K)\geq 1$ and by Proposition 3.1, $\mathrm{def}_p(U,K)\leq 1$. Thus, $\mathrm{def}_p(U,K)=1$.

Now, by way of contradiction let us assume that $H^2(N,M)\neq\{0\}$. Let $G=H_1>H_2>\dots$ be a chain of open normal subgroups such that $\cap_i H_i=N$. Thus we have that

$$N=\varprojlim H_i.$$

Hence $H^2(N,M)=\varinjlim H^2(H_i,M)$. Since $H^2(N,M)\neq 0$, we obtain that there exists j such that the image of the restriction map $\beta: H^2(H_j,M)\to H^2(N,M)$ is not zero.

Let H be an open normal subgroup of G contained in H_j which contains N and let P be a subgroup of H_j containing H such that P/H is a p-Sylow subgroup of H_j/H. By (1) we have $\mathrm{def}_p(P,K)=1$, so $\overline{\chi}_2(P,M)\geq 0$. Therefore by Proposition 2.12,

$$\overline{\chi}_1(H,M)\geq(\overline{\chi}_2(P,M)+\frac{\dim(\mathrm{Im}\,\beta)}{\dim(M)})[P:H]\geq\frac{[P:H]}{\dim(M)}=\frac{[G:H]_p}{[G:H_j]_p\dim(M)}.$$

Hence $d_p(H,K)\geq 1+\frac{[G:H]_p}{[G:H_j]_p\dim(M)}$. Now, if H is an arbitrary open subgroup containing N, applying Lemma 2.11 for H and $H\cap H_j$ and the latter inequality for $H\cap H_j$, we get

$$\frac{d_p(H,K)-1}{[G:H]_p}\geq\frac{d_p((H\cap H_j),K))-1}{[H:(H\cap H_j)][G:H]_p}\geq\frac{1}{[G:H_j]_p[H:(H\cap H_j)]_{p'}\dim(M)}.$$

Since H_j is fixed, $\frac{1}{[H:(H\cap H_j)]_{p'}}$ has positive lower bound that gives a contradiction with the hypothesis. Hence, $H^2(N,M)=0$. □

Corollary 3.5. *Let* G *be a finitely generated pro-p group with* $\mathrm{def}_p(G)>0$ *and* N *a normal subgroup of infinite index. Suppose that*

$$\inf\left\{\frac{d(H)-1}{[G:H]}\mid N<H\leq_0 G\right\}=0.$$

Then N *is a free pro-p group.*

Proof. The previous theorem implies that $H^2(N,\mathbb{F}_p)=0$. □

3.3 Finitely generated normal subgroups of profinite groups of deficiency 1

We need the following criterion for a group of positive deficiency to have cohomological p-dimension 2.

Proposition 3.6. *Let* G *be a finitely generated profinite group with* $\mathrm{def}_p(G)=1$. *Suppose that for any open subgroup* V *of* G *there exist an open subgroup* U *of* V *such that* $\overline{\chi}_2(U,\mathbb{F}_p)=0$. *Then* $\mathrm{cd}_p(G)\leq 2$.

Proof. Since $\mathrm{def}_p(G)=1$, there exists an exact sequence of modules

$$\mathcal{R}: 0\to M\xrightarrow{h}\mathbb{F}_p[[G]]^{d-1}\to\mathbb{F}_p[[G]]^d\to\mathbb{F}_p[[G]]\to\mathbb{F}_p\to 0,$$

where $d = d(G)$ (see (2.3) and Theorem 2.8). We want to show that $M = 0$. If $M \neq 0$, then there exists an open normal subgroup V of G such that the image of the composition map

$$M \xrightarrow{h} \mathbb{F}_p[[G]]^{d-1} \to \mathbb{F}_p[G/V]^{d-1}$$

is not trivial. Let $U \leq_o V$ be such that $\overline{\chi}_2(U, \mathbb{F}_p) = 0$. Then, applying the functor $\mathbb{F}_p \otimes_{\mathbb{F}_p[[U]]} -$ to $\mathcal{R}$ we obtain the complex $\mathcal{R}_U = \mathbb{F}_p \otimes_{\mathbb{F}_p[[U]]} \mathcal{R}$ given by

$$0 \to \mathbb{F}_p \otimes_{\mathbb{F}_p[[U]]} M \xrightarrow{h_U} \mathbb{F}_p[[G/U]]^{d-1} \xrightarrow{g} \mathbb{F}_p[[G/U]]^d \xrightarrow{f} \mathbb{F}_p[[G/U]] \to \mathbb{F}_p \to 0,$$

with $\operatorname{Im} h_U \neq 0$. Let $n = |G/U|$. Counting $\mathbb{F}_p$-dimension one gets

$$\begin{aligned}
\dim H_1(U, \mathbb{F}_p) &= \dim H_1(\mathcal{R}_U) \\
&= \dim(\ker f) - \dim(\operatorname{Im} g) \\
&= nd - n + 1 - [n(d-1) - (\dim(\operatorname{Im} h_U) + \dim H_2(\mathcal{R}_U)] \\
&= 1 + \dim(\operatorname{Im} h_U) + \dim H_1(U, \mathbb{F}_p) - 1.
\end{aligned}$$

It follows that $\operatorname{Im} h_U = 0$, a contradiction. Hence $M = 0$. Thus, $\operatorname{cd}_p(G) \leq 2$ (cf. Proposition 2.1 (5)). □

Remark 3.7. Note that the proof of Proposition 3.6 also shows that the hypotheses of the proposition imply that $\overline{\chi}_2(G, M) = 0$ for any non zero finite $\mathbb{F}_p[[G]]$-module M.

Now we are ready to prove Theorem 1.1.

Theorem 3.8. *Let p be a prime. Let G be a finitely generated profinite group with $\operatorname{def}_p(G) \geq 1$ and N a finitely generated normal subgroup such that $|G/N|_p$ is infinite and p divides $|N|$. Then $\operatorname{def}_p(G) = 1$ and either the p-Sylow subgroup of G/N is virtually cyclic or the p-Sylow subgroup of N is cyclic. Moreover, $\operatorname{cd}_p(G) = 2$, $\operatorname{cd}_p(N) = 1$ and $\operatorname{vcd}_p(G/N) = 1$.*

Proof. First observe that Corollary 3.3 implies $\operatorname{def}_p(G) = 1$. Note also that by Proposition 2.13 for every open subgroup J of G such that $\overline{\chi}_1(N \cap J, \mathbb{F}_p) \geq 0$ there exists an open subgroup V of J containing $J \cap N$ such that for any open subgroup U of V containing $J \cap N$,

$$\overline{\chi}_2(U, \mathbb{F}_p) \leq -\overline{\chi}_1(U/(N \cap J), \mathbb{F}_p)\overline{\chi}_1(N \cap J, \mathbb{F}_p) - \dim H^2(U/(N \cap J), \mathbb{F}_p). \tag{3.1}$$

Claim 1. Suppose that the p-Sylow subgroup of N is not cyclic. Then

(i) the p-Sylow subgroup of G/N is virtually cyclic.

(ii) $\operatorname{cd}_p(N) = 1$.

(i) By Lemma 2.10 we can find an open subgroup J of G such that $\overline{\chi}_1(J \cap N, \mathbb{F}_p) \geq 1$. Then from Equation (3.1) we get $\overline{\chi}_1(U/(N \cap J), \mathbb{F}_p) \leq 0$, because $\overline{\chi}_2(U, \mathbb{F}_p) \geq \operatorname{def}_p(U) - 1 \geq 0$. This means that the p-Sylow subgroup of G/N is virtually cyclic.

(ii) Let W be an open subgroup of G. Since $W \cap N$ is finitely generated and the p-Sylow subgroup of $W/(W \cap N)$ is virtually cyclic, W and $W \cap N$ satisfy the hypothesis of Theorem 3.4. Hence we obtain that $H^2(W \cap N, \mathbb{F}_p) = 0$, for any open subgroup W of G, and by Proposition 2.1, $\operatorname{cd}_p(N) \leq 1$. Since p divides $|N|$, we obtain that $\operatorname{cd}_p(N) = 1$.

Claim 2. Suppose that the p-Sylow subgroup of N is cyclic. Then $\mathrm{vcd}_p(G/N) = 1$.

We can find an open subgroup J of G such that $\overline{\chi}_1(J \cap N, \mathbb{F}_p) = 0$. Applying Equation (3.1) we deduce that there exists an open subgroup V of J containing $J \cap N$ such that for any open subgroup U of V containing $J \cap N$, $\dim H^2(U/(N \cap J), \mathbb{F}_p) = 0$. Hence, $\mathrm{cd}_p(V/(J \cap N)) = 1$, so $\mathrm{vcd}_p(G/N) = 1$.

Claim 3. $\mathrm{cd}_p(G) = 2$.

Let V' be open subgroup of G. Since $|N|_p$ is infinite, we can find an open subgroup J of V' such that $\overline{\chi}_1(J \cap N, \mathbb{F}_p)$ is non-negative. Hence there exists an open subgroup V of J such that for any open subgroup U of V containing $J \cap N$ Equation (3.1) holds. Since $|G/N|_p$ is infinite there exists an open subgroup U' of V containing $J \cap N$ such that $\overline{\chi}_1(U'/(J \cap N), \mathbb{F}_p) \geq 0$. Thus, Equation (3.1) implies that

$$0 \leq \mathrm{def}_p(U') - 1 \leq \overline{\chi}_2(U', \mathbb{F}_p) \leq 0.$$

Hence, by Proposition 3.6, $\mathrm{cd}_p(G) \leq 2$.

□

Theorem 3.8 is the profinite version of [10, Theorem 4], where G was assumed pro-p.

Corollary 3.9. *Let G be a finitely generated profinite group of positive deficiency whose commutator subgroup $[G, G]$ is finitely generated. Then $\mathrm{def}(G) = 1$ and $[G, G]$ is projective. Moreover, $\mathrm{cd}(G) = 2$ unless $G = \widehat{\mathbb{Z}}$.*

Proof. First let us suppose that G is abelian. We want to show that $G \cong \hat{\mathbb{Z}} \times \mathbb{Z}_\pi$ for some set of primes π (possibly empty).

If $d(G) = 1$ then positive deficiency means 0 relations, so the result is obvious in this case.

Let $G_{[p]}$ be a maximal pro-p quotient of G. Then $d(G) = d(G_{[p]})$ for some p, and so $\mathrm{def}_p(G_{[p]}) \geq \mathrm{def}(G)$ because any presentation of G serves as a presentation for $G_{[p]}$ as a pro-p group. Since $\mathrm{def}_p(G_{[p]}) = \dim H_1(G_{[p]}, \mathbb{F}_p) - \dim H_2(G_{[p]}, \mathbb{F}_p)$ we have $\mathrm{def}_p(G_{[p]}) \leq 0$ for $d(G) > 2$. Therefore $\mathrm{def}(G) \leq 0$ for $d(G) > 2$.

Suppose $d(G) = 2$. It suffices to prove that $G_{[p]}$ is non-trivial for every p. But this is clear since otherwise $0 \leq \mathrm{def}(G) \leq \mathrm{def}_p(G) \leq \mathrm{def}_p(G_{[p]}) = 0$, a contradiction. So the result follows from Theorem 3.8.

Suppose now that G is not abelian. Let $G_{[p]}$ denote again the maximal pro-p quotient of G. Then $\mathrm{def}_p(G_{[p]}) \geq \mathrm{def}(G) > 0$ and so, (see, for example, [22, Window 5, Sec.1, Lemma 3]) $G_{[p]}$ has infinite abelianization. Hence G has $\mathbb{Z}_p$ as an epimorphic image for every p and therefore has $\widehat{\mathbb{Z}}$ as a quotient. Then Theorem 3.8 implies that $\mathrm{def}(G) = 1$, $\mathrm{cd}(G) = 2$ and $[G, G]$ is projective. □

Remark 3.10. The groups considered in Subsection 5.2 show that $[G, G]$ does not have to be free profinite.

3.4 Pro-p groups of subexponential subgroup growth

Let G be a profinite group. Denote by $a_n(G)$ the number of open subgroups of G of index n. If G is finitely generated then $a_n(G)$ is finite for all n. We say that a group G is of **subexponential** subgroup growth if $\limsup_{n\to\infty} a_n(G)^{1/n} = 1$. The following characterization of pro-p groups of subexponential subgroup growth is given by Lackenby.

Proposition 3.11. *([15, Theorem 1.7]) Let G be a finitely generated pro-p group. Then G is of subexponential subgroup growth if and only if*

$$\limsup_{[G:U]\to\infty} \frac{d(U)}{[G:U]} = 0.$$

For example, since p-adic analytic profinite groups have finite rank, they are of subexponential subgroup growth (in fact, they are of polynomial subgroup growth). Also this criterion implies that the pro-p wreath product $C_p \hat{\wr} \mathbb{Z}_p$ is of exponential subgroup growth. Moreover the following holds.

Lemma 3.12. *Let G be a finitely generated pro-p group of subexponential subgroup growth and N a normal subgroup of G such that G/N is virtually cyclic. Then N is finitely generated.*

Proof. Without loss of generality we may assume that G/N is cyclic. If N is not finitely generated, then $N/\Phi(N)$ is an infinite finitely generated $\mathbb{F}_p[[G/N]]$-module. Since $\mathbb{F}_p[[G/N]]$ is isomorphic to the ring of power series over $\mathbb{F}_p$, the $\mathbb{F}_p[[G/N]]$-module $N/\Phi(N)$ is a direct sum of cyclic modules. Hence there exists a normal subgroup M of G such that $\Phi(N) \leq M < N$ and $N/M \cong \mathbb{F}_p[[G/N]]$ as $\mathbb{F}_p[[G/N]]$ modules. Therefore G/M is isomorphic to the pro-p wreath product $C_p \hat{\wr} \mathbb{Z}_p$. But the last group is of exponential subgroup growth, whence G is of exponential subgroup growth, a contradiction. □

We believe that the following conjecture holds.

Conjecture 1. *Let G be a finitely generated pro-p group of subexponential subgroup growth with $\overline{\chi}_2(G, \mathbb{F}_p) = 0$. Then G is $\mathbb{Z}_p$ or $\mathbb{Z}_p \rtimes \mathbb{Z}_p$.*

We can prove the following result.

Theorem 3.13. *Let G be a finitely generated pro-p group of subexponential subgroup growth. If $\overline{\chi}_2(G, \mathbb{F}_p) = 0$, then G is free pro-p by cyclic and all finitely generated subgroups of infinite index are free pro-p groups.*

Proof. Since G is a pro-p group $\mathrm{def}_p(G) = \overline{\chi}_2(G, \mathbb{F}_p) + 1 = 1$. Hence there exists a map of G onto $\mathbb{Z}_p$. Let N be the kernel of this map. By Lemma 3.12, N is finitely generated. Applying Corollary 3.5, we obtain that N is a finitely generated free pro-p group. In particular $\mathrm{cd}(G) = 2$ and $\overline{\chi}_2(U, \mathbb{F}_p) = 0$ for all open subgroups.

Claim Let V be an open subgroup of G and U an open normal subgroup of V. Assume that the image of $H_2(U, \mathbb{F}_p)$ in $H_2(V, \mathbb{F}_p)$ is not trivial. Then $\overline{\chi}_1(U, \mathbb{F}_p) \geq \overline{\chi}_1(V, \mathbb{F}_p) - 1 + [V:U]$.

Let $d = d(V)$. Since V is a pro-p of cohomological dimension 2 and its p-deficiency is 1, we have the following exact sequence of V-modules:

$$0 \to \mathbb{F}_p[[V]]^{d-1} \to \mathbb{F}_p[[V]]^d \to \mathbb{F}_p[[V]] \to \mathbb{F}_p \to 0. \tag{3.2}$$

Applying the functor $\mathbb{F}_p \widehat{\otimes}_{\mathbb{F}_p[[U]]} -$ we obtain the complex

$$0 \to \mathbb{F}_p[V/U]^{d-1} \xrightarrow{\alpha} \mathbb{F}_p[V/U]^d \xrightarrow{\beta} \mathbb{F}_p[V/U] \to \mathbb{F}_p \to 0, \tag{3.3}$$

where $H_2(U, \mathbb{F}_p) \cong \ker \alpha$ and $H_1(U, \mathbb{F}_p) \cong \ker \beta / \operatorname{Im} \alpha$. Note that we can calculate $H_1(V, \mathbb{F}_p)$ and $H_2(V, \mathbb{F}_p)$ either tensoring (3.3) with $\mathbb{F}_p \otimes_{\mathbb{F}_p[V/U]} -$ or tensoring (3.2) with $\mathbb{F}_p \widehat{\otimes}_{\mathbb{F}_p[[V]]} -$, once the obtained complexes are isomorphic.

Let T be a transversal of U in V. Denote by a the element $\sum_{t\in T} t$ of $\mathbb{F}_p[V]$. Since $d = d(V)$, the rank of

$$\mathbb{F}_p \widehat{\otimes}_{\mathbb{F}_p[[V]]} (\mathbb{F}_p[V/U]^d / \operatorname{Im}\alpha) \cong H_1(V, \mathbb{F}_p)$$

is d. Thus, $\operatorname{Im}\alpha$ is contained in the Jacobson radical of V-module $\mathbb{F}_p[V/U]^d$ and so $a \operatorname{Im}\alpha = 0$. Thus $a(\mathbb{F}_p[V/U]^{d-1}) \leq \ker\alpha$ which implies that $(\ker\alpha)^V$ has rank $d-1$.

Since the image of $H_2(U, \mathbb{F}_p)$ in $H_2(V, \mathbb{F}_p)$ is not trivial, the rank of $\mathbb{F}_p \widehat{\otimes}_{\mathbb{F}_p[[V]]} (\mathbb{F}_p[V/U]^{d-1} / \ker\alpha)$ is at most $d-2$. Hence $\ker\alpha$ is not in the Jacobson radical of V-module $\mathbb{F}_p[V/U]^{d-1}$. Since any element outside the Jacobson radical of $\mathbb{F}_p[V/U]^{d-1}$ generates a V-submodule isomorphic to $\mathbb{F}_p[V/U]$, we conclude that the $\mathbb{F}_p[[V]]$-module $\ker\alpha$ contains a submodule Z isomorphic to $\mathbb{F}_p[V/U]$. Thus, since $\dim Z^V = 1$ we obtain that

$$\dim H_2(U, \mathbb{F}_p) = \dim\ker\alpha \geq \dim Z + \dim(\ker\alpha)^V - \dim Z^V = |V/U| + d - 2,$$

and so

$$\overline{\chi}_1(U, \mathbb{F}_p) \geq \dim H_2(U, \mathbb{F}_p) \geq \overline{\chi}_1(V, \mathbb{F}_p) - 1 + [V : U].$$

This proves Claim.

Now, let K be a finitely generated subgroup of infinite index in G. By way of contradiction assume $H^2(K, \mathbb{F}_p) \neq 0$. Then there exists an open subgroup V of G containing K such that for any open subgroup U of V containing K the image or restriction map $H_2(K, \mathbb{F}_p) \to H_2(U, \mathbb{F}_p)$ is not trivial. Let $U_0 = V$, and let U_{i+1} $(i = 0, 1, \ldots)$ be a subgroup of index p in U_i containing K. Applying Claim, we obtain that $\overline{\chi}_1(U_i, \mathbb{F}_p) \geq \overline{\chi}_1(U_0, \mathbb{F}_p) + (p-1)i$. Hence we can find an open subgroup U of V containing K such that $\overline{\chi}_1(U, \mathbb{F}_p) - 1 = d(U)$ is arbitrary large and in particular, there exists $K < U \leq_o V$ such that

$$d(U) - 1 + d(K) < \frac{(d(U) - d(K))^2}{4}.$$

Let N be a normal subgroup of U generated by K. Then U/N does not satisfy the Golod-Shafarevich inequality (see, for example, [4, Interlude D]):

$$\dim H_2(U/N, \mathbb{F}_p) \leq d(U) - 1 + d(K) < \frac{(d(U) - d(K))^2}{4} \leq \frac{(d(U/N))^2}{4},$$

and so N is of infinite index in U. Since N is not free pro-p, Corollary 3.5 and Proposition 3.11 imply that U is not of subexponential subgroup growth, a contradiction. □

4 Poincaré duality groups of dimension 3

Let G be a profinite group of type p-FP_∞ and B a profinite $\mathbb{Z}_p[[G]]$-module. Then $B = \varprojlim B_j$, where each B_j is a finite discrete p-torsion $\mathbb{Z}_p[[G]]$-module, and so $H^i(G, B_j)$ is finite for all i and j. Thus we can define the ith continuous cohomology of G in the profinite module B by the profinite group

$$H^i_{cts}(G, B) = \varprojlim H^i(G, B_j).$$

Note that this definition coincides with the one given in [32, Thm. 3.7.2] where it is $\operatorname{Ext}^i_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, B)$ (see also [27, Corollary 2.3.5]).

As defined in [32], a profinite group G of type p-FP_∞ is called a **Poincaré duality group** at p of dimension n if $cd_p(G) = n$ and

$$H^i_{cts}(G, \mathbb{Z}_p[[G]]) = 0, \quad \text{if } i \neq n,$$
$$H^n_{cts}(G, \mathbb{Z}_p[[G]]) \cong \mathbb{Z}_p \quad \text{(as abelian groups)}.$$

Remark 4.1. In [27, page 165] a more general definition of a Poincaré duality group at p is given (without the assumption of G to be of type p-FP_∞). If G is of type p-FP_∞, then both definitions coincide. In this paper we always assume that a Poincaré duality group at p is of type p-FP_∞.

Remark 4.2. *Note that the cohomology group $H^n_{cts}(G, \mathbb{Z}_p[[G]])$ has a canonical structure of right $\mathbb{Z}_p[[G]]$-module. Indeed, the coeficient module $\mathbb{Z}_p[[G]]$, which is thought as a left module for the purpose of defining $H^n_{cts}(G, \mathbb{Z}_p[[G]])$ also admits a right $\mathbb{Z}_p[[G]]$-action which commutes with the left action; this right action induces a right action of $\mathbb{Z}_p[[G]]$ on $H^n_{cts}(G, \mathbb{Z}_p[[G]])$. We convert $H^n_{cts}(G, \mathbb{Z}_p[[G]])$ into a left $\mathbb{Z}_p[[G]]$-module, defining the action of $g \in G$ on the left as the action of g^{-1} on the right. We will denote this module by $D_p(G)$.*

We use the term profinite PD^n-group at p for a Poincaré duality profinite group G at p of dimension n. If G is a profinite group with $cd_p(G) < \infty$ and U is an open subgroup of G, then G is a profinite PD^n-group at p if and only if U is a profinite PD^n-group at p.

By a result of Lazard, compact p-adic analytic groups G are virtual Poincaré duality groups of dimension $n = \dim(G)$ at a prime p ([32, Thm. 5.9.1]). The Demushkin pro-p groups are exactly the pro-p PD^2-groups ([31, I.4.5 Example 2]) and $\mathbb{Z}_p$ is the only pro-p PD^1-group ([32, Example 4.4.4]).

Let G be PD^n at p and $I_p(G) = Hom_{\mathbb{Z}_p}(D_p(G), \mathbb{Q}_p/\mathbb{Z}_p)$ its dualizing module. Then $I_p(G)$ is isomorphic (as abelian group) to $\mathbb{Q}_p/\mathbb{Z}_p$ (note that the action of G on $I_p(G)$ is not always trivial). For any finite $\mathbb{Z}_p[[G]]$-module M, $M^\vee = \mathrm{Hom}(M, I_p(G))$ is called the **dual** of M. The action of G on $M^\vee$ is given as

$$(gf)(m) = gf(g^{-1}m),\ g \in G,\ m \in M, f \in M^\vee.$$

It is clear that $M^{\vee\vee} \cong M$. We say that $M^\vee$ is **self-dual** if $M \cong M^\vee$.

In this section we are interested in PD^3-groups at p. Some important consequences of Poincaré duality are recalled in the following proposition.

Proposition 4.3. *Let G be a PD^3-group at p. Then the following holds.*

1. *If the trivial G-module $\mathbb{F}_p$ is self-dual then*

$$\overline{\chi}_2(G, \mathbb{F}_p) = -\dim H^3(G, \mathbb{F}_p) = -1.$$

   In particular, if G is a pro-p group, then $\mathrm{def}_p(G) = 0$.

2. *Let M be a finite self-dual $\mathbb{F}_p[[G]]$-module, then* $\dim H^3(G, M) = \dim M^G$ *and*

$$\overline{\chi}_2(G, M) = -\frac{\dim M^G}{\dim M}.$$

3. *Let N be a closed subgroup of G such that $[G : N]_p$ is infinite. Then* $\mathrm{cd}_p(N) \leq 2$.

Proof. Let M be a finite $\mathbb{F}_p[[G]]$-module. By [27, Theorem 3.4.6],

$$\dim H^{n-i}(G, M) = \dim H^i(G, M^\vee).$$

This implies the first and second statements.

In order to prove (3) we have to note that if U is an open subgroup of G, then the correstriction map $H^3(U, \mathbb{F}_p) \to H^3(G, \mathbb{F}_p)$ is an isomorphism (see proof of Proposition 30, item (5), in [31, I.§4.5]). Hence the restriction map $H^3(G, \mathbb{F}_p) \to H^3(U, \mathbb{F}_p)$ is trivial if p divides $[G : U]$. Therefore if L is an open subgroup of N

$$H^3(L, \mathbb{F}_p) = \varinjlim_{L<U\leq_o G} H^3(U, \mathbb{F}_p) = 0.$$

because $[G : L]_p$ is infinite. By Proposition 2.1, $\mathrm{cd}_p(N) \leq 2$. □

In general it is not true that if G is a PD^3-group at p, then $\mathrm{def}_p(G) = 0$, even in orientable case (i.e. G acts trivially on $I_p(G)$); the example showing this was obtained in communication with Peter Symonds.

Example 4.4. *Let $p \geq 5$ be a prime number and $a \in \mathbb{Z}_p$ such that $a^2 \neq 1$ and $a^{p-1} = 1$. Let x be a generator of C_{p-1}. Consider $G = (\mathbb{Z}_p^3) \rtimes C_{p-1}$, where x acts on $\mathbb{Z}_p^3$ as multiplication by a on the first and second coordinates and as multiplication by a^{-2} on the third. So G is an orientable PD^3-profinite group at p, because x acts on $I_p(G)$ as multiplication by $(a \cdot a \cdot a^{-2})^{-1} = 1$.*

Set $H = \mathbb{Z}_p^3$ and let $M = \mathbb{F}_p$ be a G-module such that H acts trivially and x acts as multiplication by a^{-1}. By Proposition 2.3(4),

$$H^\bullet(G, M) \cong H^\bullet(H, M)^{G/H}.$$

Moreover, $H^\bullet(H, \mathbb{F}_p) \cong \bigwedge^\bullet H^1(H, \mathbb{F}_p)$. Thus $\dim H^1(H, M) = 3$ with eigenvalues a^{-2}, a^{-2} and a for the action of x and $\dim H^2(H, M) = 3$ with eigenvalues 1, 1 and a^{-3} for the action of x. Therefore,

$$\dim H^1(G, M) = \dim(H^1(H, M)^{G/H}) = 0$$

and

$$\dim H^2(G, M) = \dim(H^2(H, M)^{G/H}) \geq 2.$$

Thus,

$$\overline{\chi}_2(G, M) = \frac{-\dim H^2(G, M) + \dim H^1(G, M) - \dim H^0(G, M)}{\dim(M)} \leq -2$$

and so $\mathrm{def}_p(G) \leq 1 + \overline{\chi}_2(G, M) \leq -1$.

The following theorem is a consequence of [27, Theorem 3.7.4].

Theorem 4.5. *Let $1 \to N \to G \to G/N \to 1$ be an exact sequence of profinite groups such that*

a) G, N and G/N are p-FP_∞,

b) $\mathrm{cd}_p(G/N) < \infty$.

Then if two of three groups are $PD\bullet$-groups at p, so is the third. Moreover, $\mathrm{cd}_p(G) = \mathrm{cd}_p(G/N) + \mathrm{cd}_p(N)$.

The next result is an analog of Theorem 3.4.

Theorem 4.6. *Let G be a finitely generated profinite PD^3-group at p, $K \leq N$ two normal subgroup such that $|G/N|_p$ is infinite and M is a finite self-dual $\mathbb{F}_p[[G]]$-module on which K acts trivially. Suppose that*

$$\inf \left\{ \frac{d_p(H, K) - 1}{[G : H]_p} \;\middle|\; N < H \leq_0 G \right\} = 0.$$

Then, $\dim H^2(N, M) \leq \dim M^N$.

Moreover, either $\mathrm{cd}_p(N) \leq 1$ or a p-Sylow subgroup of G/N is virtually cyclic.

Proof. First, let us show that $\dim H^2(N, M) \leq \dim M^N$. By way of contradiction let us assume that $\dim H^2(N, M) > \dim M^N$. Let $G = H_1 > H_2 > \ldots$ be a chain of open normal subgroups such that $\cap_i H_i = N$. Thus we have that

$$N = \varprojlim H_i.$$

Hence $H^2(N, M) = \varinjlim H^2(H_i, M)$. Since $\dim H^2(N, M) > \dim M^N$, we obtain that there exists j such that if β denotes the restriction map $H^2(H_j, M) \to H^2(N, M)$, then $\dim \mathrm{Im}\, \beta \geq \dim M^N + 1$.

Let $N \leq H$ be a open normal subgroup of G contained in H_j and P a subgroup of H_j containing H such that P/H is a p-Sylow subgroup of H_j/H. Since P is also a PD^3-group at p and M is self-dual, Proposition 4.3 gives that

$$\overline{\chi}_2(P, M) \geq -\frac{\dim M^N}{\dim M}.$$

Therefore by Proposition 2.12,

$$\overline{\chi}_1(H, M) \geq (\overline{\chi}_2(P, M) + \frac{\dim(\operatorname{Im}\beta)}{\dim(M)})[P : H] \geq \frac{[P : H]}{\dim(M)} = \frac{[G : H]_p}{[G : H_j]_p \dim(M)}.$$

Hence $d_p(H, K) \geq 1 + \frac{[G:H]_p}{[G:H_j]_p \dim(M)}$. Now, if H is an arbitrary open subgroup containing N, applying Lemma 2.11 for H and $H \cap H_j$ and the latter inequality for $H \cap H_j$, we get

$$\frac{d_p(H, K) - 1}{[G : H]_p} \geq \frac{d_p((H \cap H_j), K) - 1}{[H : (H \cap H_j)][G : H]_p} \geq \frac{1}{[G : H_j]_p [H : (H \cap H_j)]_{p'} \dim(M)}.$$

Since H_j is fixed, $\frac{1}{[H:(H\cap H_j)]_{p'}}$ has positive lower bound that gives a contradiction with the hypothesis. Hence, $\dim H^2(N, M) \leq \dim M^N$.

Note that by Proposition 4.3, $\operatorname{cd}_p(N) \leq 2$. Let us now analyze the case $\operatorname{cd}_p(N) = 2$. In this case there exists an open subgroup L of N such that $H^2(L, \mathbb{F}_p) \neq 0$. Since $L = U \cap N$ for some open subgroup U of G, we conclude that $H^2(L, \mathbb{F}_p) = \mathbb{F}_p$. Now, Theorem 2.4 implies that U/L and so G/N are of virtual cohomological p-dimension 1. Without loss of generality let us assume that $\operatorname{cd}_p(U/L) = 1$.

If a p-Sylow subgroup of U/L is not cyclic then, by Lemma 2.10, there exists an open subgroup V of U containing L such that $\overline{\chi}_1(V/L, \mathbb{F}_p) \geq 1$ and V acts trivially on $H^2(L, \mathbb{F}_p)$. Thus, by Corollary 2.3(4),

$$1 = \dim H^3(V, \mathbb{F}_p) = \dim H^1(V/L, H^2(L, \mathbb{F}_p)) > 1,$$

a contradiction. Hence, the Sylow pro-p subgroup of U/L is cyclic. □

Now we need the following.

Lemma 4.7. *Let G be a profinite group which the Sylow pro-p subgroup is infinite and cyclic. Then there is an open subgroup U isomorphic to a semidirect product of a profinite pro-p' group and $\mathbb{Z}_p$. In particular, U and therefore G are profinite PD^1-groups at p.*

Proof. We can find an open subgroup U such that $H^1(U, \mathbb{F}_p) = \mathbb{F}_p$. Indeed, obviously U can be chosen with its maximal pro-p quotient $U_{[p]}$ non-trivial, and since it is a quotient of the p-Sylow of U, it is also cyclic. It remains to note that $H^1(U, \mathbb{F}_p) = \operatorname{Hom}(U, \mathbb{F}_p) = \operatorname{Hom}(U_{[p]}, \mathbb{F}_p)$. Then Lemma 2.9 implies that $U_{[p]} \cong \mathbb{Z}_p$. Therefore, the kernel N of the natural map $U \to U_{[p]}$ is a pro-p' group. □

Now we are ready to prove Theorem 1.4.

Theorem 4.8. *Let G be a finitely generated profinite PD^3-group at a prime p and N be a finitely generated normal subgroup of G such that $|G/N|_p$ is infinite and p divides $|N|$. Then either N is PD^1 at p and G/N is virtually PD^2 at p or N is PD^2 at p and G/N is virtually PD^1 at p.*

Proof. During this proof when we write PD^n we shall mean PD^n at p.

Claim 1. Assume that the p-Sylow subgroup of N is not cyclic. Then

(i) The p-Sylow subgroup of G/N is virtually cyclic;

(ii) N is of type p-FP_∞;

(iii) G/N is virtually PD^1 at p and N is PD^2 at p.

(i) By Lemma 2.10 we can find an open subgroup J of G such that $\overline{\chi}_1(J \cap N, \mathbb{F}_p) \geq 1$. Applying Proposition 2.13 and repeating the same argument as in (3.1), we obtain that there exists an open subgroup V of J containing $J \cap N$ such that for any open subgroup U of J containing $J \cap N$, $\overline{\chi}_1(U/N \cap J, \mathbb{F}_p) \leq 0$. This means that the p-Sylow subgroup of G/N is virtually cyclic.

For simplicity let us assume that the p-Sylow subgroup of G/N is cyclic, and so $\mathrm{cd}_p(G/N) = 1$.

(ii) Note that if U is an open subgroup of G containing N and M a finite $\mathbb{Z}_p[[U]]$-module, then by Corollary 2.3(1),

$$\dim H^1(U, M) \leq \dim H^1(U/N, M^N) + \dim H^1(N, M)^{U/N}.$$

Since by Lemma 4.7, U/N is PD^1 at p one has

$$\dim H^1(U/N, M^N) = \dim H^0(U/N, (M^N)^\vee) = \dim((M^N)^\vee)^{U/N}$$

and using

$$\dim((M^N)^\vee)^{U/N} + \dim H^1(N, M)^{U/N} \leq \dim M + \dim H^1(N, M)$$

one has

$$\dim H^1(U, M) \leq \dim M + \dim H^1(N, M).$$

Thus, $d_p(U) \leq 1 + d_p(N)$ and so we may apply Theorem 4.6 for $K = 1$.

Let S be an irreducible finite $\mathbb{F}_p[[N]]$-module. There exists an open subgroup V of G such that $V \cap N$ acts trivially on S. We convert S in a VN-module by assuming that the elements of V act trivially on S. Since VN is open subgroup of G, it is also PD^3 at p. By Theorem 4.6, $\dim H^2(N, S \oplus S^\vee) \leq 2 \dim S$. Hence $\dim H^2(N, S) \leq 2 \dim S$.

Therefore since, by Proposition 4.3 , N has cohomological p-dimension 2, it follows from Theorem 2.14 that N is of type p-FP_∞.

(iii) By Lemma 4.7, G/N is PD^1 and we can apply Theorem 4.5 and conclude that N is PD^2.

Claim 2. Suppose that the p-Sylow subgroup of N is cyclic. Then N is PD^1 at p and G/N is virtually PD^2 at p

Applying Corollary 2.5, we obtain that $\mathrm{vcd}_p(G/N) = 2$. Let U be an open subgroup of G containing N such that $\mathrm{cd}_p(U/N) = 2$. Since N is finitely generated, U/N is of type p-FP_2 by Corollary 2.15 and so it is also of type p-FP_∞. By Lemma 4.7, N is PD^1. Now, Theorem 4.5 implies that U/N is PD^2 and so G/N is virtually PD^2.

□

Corollary 4.9. *Let G be a finitely generated pro-p PD^3-group at p and N a normal subgroup of infinite index. Suppose that G has subexponential subgroup growth. Then N is either free pro-p or a Demushkin group.*

Proof. Proposition 3.11 implies that

$$\inf \left\{ \frac{d(H) - 1}{[G : H]} \,\middle|\, N < H \leq_0 G \right\} = 0.$$

Therefore, by Theorem 4.6, $\dim H^2(N, \mathbb{F}_p) \leq 1$. If $H^2(N, \mathbb{F}_p) = 0$ then N is free pro-p. If $H^2(N, \mathbb{F}_p) = \mathbb{F}_p$ then Theorem 4.6 implies that G/N is virtually cyclic. By Lemma 3.12, N is finitely generated. By the previous theorem, N is a Demushkin group. □

5 Applications to discrete groups

In this section we shall describe applications of our profinite results to discrete finitely generated groups.

5.1 Good groups

Let Γ be a group, $\widehat{\Gamma}$ its profinite completion. The group Γ is called **p-good** if the homomorphism of cohomology groups

$$i^n(M) : H^n(\widehat{\Gamma}, M) \longrightarrow H^n(\Gamma, M)$$

induced by the natural homomorphism $i : \Gamma \longrightarrow \widehat{\Gamma}$ is an isomorphism for every finite p-primary $\mathbb{Z}[\Gamma]$-module M and for all $n \geq 0$. The group Γ is called **good** if it is p-good for all primes p. This notion was introduced by Serre (see [31, I.2.6]) and has been studied recently in several papers (see, for example, [7]).

Let $\Gamma_{\hat{p}}$ be the pro-p completion of Γ, and let $i_p : \Gamma \to \Gamma_{\hat{p}}$ denote the canonical map. This map induces natural homomorphisms

$$i_p^n(M) : H^n(\Gamma_{\hat{p}}, M) \longrightarrow H^n(\Gamma, M)$$

for all $n \geq 0$ and for all finite $\mathbb{Z}[\Gamma]$-modules M of p-power order for which all composition factors are trivial Γ-modules. The group Γ is called **pro-p good** if $i_p(\mathbb{F}_p)$ is an isomorphism for all n. Note that this also implies that $i_p^n(M)$ is an isomorphism for all $n \geq 0$ and for all finite $\mathbb{Z}[\Gamma]$-modules M of p-power order for which all composition factors are trivial Γ-modules.

The following relation between p-goodness and pro-p goodness was discovered by Thomas Weigel [33]:

Proposition 5.1. *Let Γ be a group. Assume that all the subgroups of Γ of finite index are pro-p good. Then Γ is p-good.*

The following result is a variation of a result of Serre from [31, I.2.6].

Proposition 5.2. *Let Γ be a discrete group, G a profinite group and $\phi : \Gamma \to G$ a homomorphism with dense image. For any finite (topological) G-module M, denote by $\phi^n(M)$ the restriction map $\phi^n(M) : H^n(G, M) \to H^n(\Gamma, M)$. Then the following properties are equivalent:*

- A_k *For every finite $\mathbb{F}_p[[G]]$-module M, $\phi^n(M)$ is bijective for all $n \leq k$ and injective for $n = k+1$.*
- B_k *For every finite $\mathbb{F}_p[[G]]$-module M, $\phi^n(M)$ is surjective for all $n \leq k$.*
- D_k $\varinjlim_{U \leq_o G} H^n(\Gamma \cap U, \mathbb{F}_p) = 0$ *for all $n \leq k$.*

As an application of some results of Section 3 we obtain the following theorem.

Theorem 5.3. *Let Γ be finitely presented group of deficiency 1 and cohomological dimension 2. Assume that $\overline{\chi}_2(U, \mathbb{F}_p) \leq 0$ for any open subgroup U of $\Gamma_{\hat{p}}$. Then Γ is pro-p good.*

Proof. First observe that the condition B_1 from Proposition 5.2 always holds in the case $G = \Gamma_{\hat{p}}$. Hence A_1 and D_1 holds. In particular, for any $U \leq_o \Gamma_{\hat{p}}$,

$$\begin{aligned}
\dim H^1(U, \mathbb{F}_p) &= \dim H^1(\Gamma_{\hat{p}}, Coind_U^{\Gamma_{\hat{p}}}(\mathbb{F}_p)) \\
&= \dim H^1(\Gamma, Coind_{U\cap\Gamma}^{\Gamma}(\mathbb{F}_p)) = \dim H^1(\Gamma \cap U, \mathbb{F}_p) \\
\dim H^2(U, \mathbb{F}_p) &= \dim H^2(\Gamma_{\hat{p}}, Coind_U^{\Gamma_{\hat{p}}}(\mathbb{F}_p)) \\
&\leq \dim H^2(\Gamma, Coind_{U\cap\Gamma}^{\Gamma}(\mathbb{F}_p)) = \dim H^2(\Gamma \cap U, \mathbb{F}_p).
\end{aligned}$$

Thus, since any subgroup of Γ of finite index has positive deficiency, we obtain that

$$\begin{aligned} 1 &\leq \operatorname{def}(\Gamma \cap U) \leq \dim H^1(\Gamma \cap U, \mathbb{F}_p) - \dim H^2(\Gamma \cap U, \mathbb{F}_p) \\ &\leq \dim H^1(U, \mathbb{F}_p) - \dim H^2(U, \mathbb{F}_p) = \overline{\chi}_2(U, \mathbb{F}_p) + 1 \leq 1. \end{aligned}$$

Therefore $\dim H^2(\Gamma \cap U, \mathbb{F}_p) = \dim H^2(U, \mathbb{F}_p)$ and so

$$\varinjlim_{U \leq_o G} H^2(\Gamma \cap U, \mathbb{F}_p) = \varinjlim_{U \leq_o G} H^2(U, \mathbb{F}_p) = H^2(\{1\}, \mathbb{F}_p) = 0.$$

Hence the condition D_2 from Proposition 5.2 holds. Thus, by the condition A_2, $i_p^k(\mathbb{F}_p)$ is an isomorphism for $k \leq 2$ and $H^3(\Gamma_{\hat{p}}, \mathbb{F}_p) \leq H^3(\Gamma, \mathbb{F}_p) = 0$. Hence $\operatorname{cd}_p(\Gamma_{\hat{p}}) = 2$ by Proposition 2.1. We conclude that Γ is pro-p good. □

Corollary 5.4. *Let Γ be finitely presented group of deficiency 1 and cohomological dimension 2. Assume that $\widehat{\Gamma}$ is a semidirect product of a finitely generated profinite group P and $\hat{\mathbb{Z}}$. Then Γ is good and pro-p good for any prime p.*

Proof. Let H be a subgroup of Γ of finite index and p a prime. Note that $\operatorname{cd}(H) = 2$ and its deficiency is positive. Since $\widehat{H}$ is a subgroup of $\widehat{\Gamma}$, it is also a semidirect product of a finitely generated profinite group and $\hat{\mathbb{Z}}$. Hence $H_{\hat{p}} \cong \widehat{H}_{[p]}$ is a semidirect product of a finitely generated pro-p group and $\mathbb{Z}_p$. Thus, by Corollary 3.3, if U is an open subgroup of $H_{\hat{p}}$, then

$$\overline{\chi}_2(U, \mathbb{F}_p) = \operatorname{def}_p(U) - 1 \leq 0.$$

Applying, Theorem 5.3, we obtain that H is pro-p good. Proposition 5.1 implies that Γ is good. □

Let Γ be a finitely presented group. A **chain** in Γ is a decreasing infinite sequence $\Gamma = \Gamma_0 > \Gamma_1 > \dots$ of subgroups of finite index in Γ. The chain is **normal** if all $ca\Gamma_n$ are normal in Γ. From a result of Lück (see, for example, [23]) we know that for any normal chain $\{\Gamma_i\}$ with trivial intersection there exists

$$\lim_{i \to \infty} \frac{\dim H^1(\Gamma_i, \mathbb{Q})}{[\Gamma : \Gamma_i]}.$$

Moreover this limit does not depend on $\{\Gamma_i\}$ and if Γ is infinite it is equal to the firtst L^2- Betti number $\beta_1^{(2)}(\Gamma)$. We need the following auxiliary proposition. We say that a group Γ is **residually-p** if G is residually (a finite p-group) group

Proposition 5.5. *Let Γ be a finitely presented residually-p group. Then*
a) $\beta_1(\Gamma) = \dim_{\mathbb{Q}} H^1(\Gamma, \mathbb{Q}) \geq \operatorname{def}_p(\Gamma_{\hat{p}})$,
b) $\beta_1^{(2)}(\Gamma) \geq \overline{\chi}_2(\Gamma_{\hat{p}}, \mathbb{F}_p)$.

Proof. In order to prove a), note that $\operatorname{def}_p(\Gamma_{\hat{p}})$ is the difference between the minimal number of generators and the minimal number of relations of $\Gamma_{\hat{p}}$ as a pro-p group. Hence arguing as in [22, Window 5, Sec.1, Lemma 3], we obtain that

$$\dim_{\mathbb{Q}_p} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Gamma_{\hat{p}}/[\Gamma_{\hat{p}}, \Gamma_{\hat{p}}] \geq \operatorname{def}_p(\Gamma_{\hat{p}}).$$

Since $\dim_{\mathbb{Q}} H^1(\Gamma, \mathbb{Q}) = \dim_{\mathbb{Q}_p} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \Gamma_{\hat{p}}/[\Gamma_{\hat{p}}, \Gamma_{\hat{p}}]$ we obtain a).

Let U be open subgroup of $\Gamma_{\hat{p}}$. By Lemma 2.11 and a),

$$\frac{\dim_{\mathbb{Q}} H^1(U \cap \Gamma, \mathbb{Q}) - 1}{[\Gamma : (U \cap \Gamma)]} \geq \frac{\operatorname{def}_p(U) - 1}{[\Gamma_{\hat{p}} : U]} \geq \operatorname{def}_p(\Gamma_{\hat{p}}) - 1 = \overline{\chi}_2(\Gamma_{\hat{p}}, \mathbb{F}_p).$$

Since Γ is residually-p group, then if $\{U_i\}$ is a normal chain of $\Gamma_{\hat{p}}$ with trivial intersection, then $\{\Gamma \cap U_i\}$ is a normal chain with trivial intersection of Γ. Hence $\beta_1^{(2)}(\Gamma) \geq \overline{\chi}_2(\Gamma_{\hat{p}}, \mathbb{F}_p)$. □

Now we are ready to prove another criterion for goodness.

Theorem 5.6. *Let Γ be finitely presented group. Suppose that Γ is virtually residually-p group and $\beta_1^{(2)}(\Gamma) = 0$. Assume also that either*

a) $\mathrm{cd}(\Gamma) = 2$ *and it is of deficiency 1 or*

b) Γ *is an orientable Poincaré duality group of dimension 3.*

Then Γ is p-good. Moreover, if G is residually-p then G is pro-p good.

Proof. Assume first that Γ is residually-p group. Let U be an open subgroup of $G_{\hat{p}}$. By the previous proposition and [23, Theorem 1.35 (9)],

$$\mathrm{def}_p(U) - 1 = \overline{\chi}_2(U, \mathbb{F}_p) \leq \beta_1^{(2)}(\Gamma \cap U) = [\Gamma : (\Gamma \cap U)]\beta_1^{(2)}(\Gamma) = 0.$$

If we are in the case a), then by Theorem 5.3, Γ is pro-p good and the same argument shows that any subgroup of Γ of finite index is pro-p good. Hence by Proposition 5.1, Γ is also p-good.

Assume now that we are in the case b). Then the same conclusion follows from [14, Theorem A and B].

Now if Γ is not residually-p, there is a subgroup of finite index of Γ which is and since p-goodeness is preserved by overgroups of finite index (cf. Proposition 5.2) Γ is p-good.

□

5.2 Ascending HNN-extensions

In this section we study mapping tori of injective endomorphisms of free groups. For a free group $F_n := \langle x_1, \ldots, x_n \rangle$ ($n \in \mathbb{N}$) let $\phi : F_n \to F_n$ be an injective endomorphism. The HNN-extension

$$M_\phi := \langle\, x_1, \ldots, x_n, t \,|\, t^{-1}x_i t = \phi(x_i) \text{ for } i = 1, \ldots n \,\rangle \tag{5.1}$$

is traditionally called the mapping torus of ϕ. Sometimes we shall also say that M_ϕ is the ascending HNN-extension of ϕ or of the free group F_n. Our methods are applicable to the study of mapping tori because

- the group M_ϕ has positive deficiency,
- the cohomological dimension of M_ϕ is less than or equal to 2.

The second property follows by an application of the Mayer-Vietoris sequence of cohomology while the first property is obvious.

Let us discuss a simple example. Let the endomorphism $\phi_1 : F_1 = \langle x \rangle \to F_1$ be given by $\phi_1(x) := x^2$. It is elementary to see that the corresponding mapping torus is metabelian and in fact isomorphic to a split extension

$$M_{\phi_1} = \langle x, t \mid x^t = x^2 \rangle \cong \mathbb{Z}\left[\frac{1}{2}\right] \rtimes \mathbb{Z}.$$

Where the generator 1 of the infinite cyclic group acts by multiplication by 2 on the group $\mathbb{Z}[1/2]$ of rational numbers with a 2-power denominator. The commutator subgroup of M_{ϕ_1} is equal to $\mathbb{Z}[1/2]$ appropriately embedded in M_{ϕ_1}. From here it is easy to detect the profinite completion of M_{ϕ_1}.

Proposition 5.7. *We have*

$$\widehat{M_{\phi_1}} = \widehat{\mathbb{Z}[1/2]} \rtimes \widehat{\mathbb{Z}}.$$

The profinite completion $\widehat{\mathbb{Z}[1/2]}$ is a projective but not a free profinite group.

In fact the isomorphism

$$\widehat{\mathbb{Z}[1/2]} \cong \prod_{p\neq 2} \mathbb{Z}_p$$

shows that all p-Sylow subgroups of $\widehat{\mathbb{Z}[1/2]}$ are free pro-p groups which implies that $\widehat{\mathbb{Z}[1/2]}$ is projective. Our example also shows that the projectivity of $[G, G]$ in Corollary 3.9 cannot be replaced by freeness.

Corollary 1.2 allows us to establish the structure of the profinite completion of this important class of groups.

Theorem 5.8. *Let M_ϕ be an ascending HNN-extension of a finitely generated free group $F = F_n$ of rank $n \in \mathbb{N}$ with respect to an injective endomorphism $\phi : F \to F$. Let P be the closure of F in the profinite completion $\widehat{M_\phi}$. Then P is normal in $\widehat{M_\phi}$, the profinite completion of M_ϕ is isomorphic to the split extension $\widehat{M_\phi} = P \rtimes \widehat{\mathbb{Z}}$ and P is a projective profinite group. The group P is free profinite of rank n if and only if ϕ is an automorphism.*

Proof. Let P be the closure of the image of F in $\widehat{M_\phi}$. To see that P is normal consider any finite quotient H of M_ϕ. Then observe that the images of F and F^t in H coincide because in a finite group conjugate subgroups have to be of the same order. Thus $\widehat{M_\phi} = P \rtimes \widehat{\mathbb{Z}}$.

The group M_ϕ has positive deficiency and therefore so has its profinite completion $\widehat{M_\phi}$. Corollary 1.2 implies that P is projective.

If ϕ is an automorphism clearly $P = \widehat{F}$. Suppose now that ϕ is not an isomorphism. Let F_0 be the image of ϕ; it is a finitely generated subgroup of F distinct from F. By a result of M. Hall F_0 is not dense in the profinite completion of F (see [24, Proposition 3.10]). To see that P is not free profinite of rank n, it suffices to show that the profinite topology of M_ϕ does not induce the full profinite topology on F. But this follows from the theorem of Hall since as was just observed F coincides with F_0 in every finite image of M_ϕ. □

Remark 5.9. If f is not an automorphism the theorem in principle allows P to be a free profinite group of rank less than n. It is, however, easy to give criteria when this does not happen. Indeed, if F admits a finite quotient F/N modulo a characteristic subgroup N such that $d(F/N) = n$ and $F = f(F)N$ then $d(P) = n$. So for instance, if $f(x_i) = x_i^p$ for $i = 1, \dots n$ for some prime p then F/N can be taken to be elementary q-group of rank n where q is coprime to p.

Problem 2. *Describe the projective groups P obtained in this manner.*

The following interesting example was communicated to us by I. Kapovich.

Example 5.10. *Let $F := F_3 = \langle\, a,\, b,\, c \,\rangle$ be a free group of rank 3. Let the endomorphism $\phi : F \to F$ be given by*

$$\phi(a) = a, \quad \phi(b) = a^{-1}ca, \quad \phi(c) = a^{-1}bab^{-1}.$$

Let

$$\Gamma := M_\phi = \langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = a^{-1}ca, tct^{-1} = a^{-1}bab^{-1}\rangle$$

be the corresponding ascending HNN-extension. Then $\phi(F) = \langle a, c, bab^{-1}\rangle$ is a proper subgroup of F, so that Γ is a strictly ascending HNN-extension.

However, we can also rewrite the defining relations for G as follows:

$$a^{-1}ta = t, \quad a^{-1}ca = tbt^{-1}, \quad a^{-1}ba = tct^{-1}b.$$

Thus G is an HNN-extension of $H := \langle\, t,\, b,\, c \,\rangle$ with respect to $\psi : H \to H$ where

$$\psi(t) = t, \quad \psi(c) = tbt^{-1}, \quad \psi(b) = tct^{-1}b$$

and with stable letter a. We have $\psi(H) = \langle\, t, tbt^{-1}, tct^{-1}b \,\rangle = \langle t, b, c\rangle = H$. Thus ψ is an automorphism of H and hence H is normal in G.

This example combined with Remark 5.9 shows that the profinite completion $\widehat{\Gamma}$ can be written as semidirect product $P \rtimes \widehat{\mathbb{Z}}$ of a projective (non-free) finitely generated profinite group and as a semidirect product $F \rtimes \widehat{\mathbb{Z}}$ of a free profinite group of rank 3 and $\widehat{\mathbb{Z}}$.

The next result that comes as an application of Corollary 5.4 is that an ascending HNN-extension is good.

Theorem 5.11. *An ascending HNN-extension Γ of a finitely generated free group is good and pro-p good for every prime p.*

Proof. By Theorem 5.8, $\widehat{\Gamma} \cong P \rtimes \widehat{\mathbb{Z}}$ where P is finitely generated. Now by Corollary 5.4, Γ is good and pro-p good for every prime p. □

Remark 5.12. We note that Lorensen [16] proved that if M is a finite $\mathbb{Z}[\Gamma]$-module, then

$$i^n(M) : H^n(\widehat{\Gamma}, M) \longrightarrow H^n(\Gamma, M)$$

is an isomorphism for $n \leq 2$. Thus, the goodness of Γ also follows directly from his result and our Theorem 5.8, because $\mathrm{cd}(\Gamma) = \mathrm{cd}(\widehat{\Gamma}) = 2$.

5.3 Implications for the congruence kernel

In this subsection we will study the cohomological dimensions of the congruence kernels of certain arithmetic groups.

Recall that a lattice in $\mathrm{SL}_2(\mathbb{C})$ is a discrete group (Kleinian group) of finite covolume. One particular family of lattices is a family of arithmetic groups. We recall the definition of an arithmetic group in this case; see [5], [25] for more details. Let k be a number field with exactly one pair of complex places and let A be a quaternion algebra over k which is ramified at all real places. Let ρ be a k-embedding of A into the algebra $\mathrm{M}_2(\mathbb{C})$ of two by two matrices over $\mathbb{C}$ (using one of the complex places). Let $\mathcal{O}$ be the ring of integers of k and let $\mathcal{R}$ be a $\mathcal{O}$-order of A. Let $A^1(\mathcal{R})$ be the corresponding group of elements of norm one. It is well known that $\rho(A^1(\mathcal{R}))$ is a lattice in $\mathrm{SL}_2(\mathbb{C})$. Then a subgroup Γ of $\mathrm{SL}_2(\mathbb{C})$ is an arithmetic Kleinian group if it is commensurable with some such a $\rho(A^1(\mathcal{R}))$ (groups are commensurable if they have $\mathrm{SL}_2(\mathbb{C})$-conjugate subgroups of finite index). The quotient $\mathrm{SL}_2(\mathbb{C})/\rho(\mathrm{A}^1(\mathcal{R}))$ is not compact if k is an imaginary quadratic number field and $A = \mathrm{M}_2(k)$.

To define the congruence kernels let us (without loss of generality) in the following consider the arithmetic Kleinian group $\Gamma = \rho(A^1(\mathcal{R}))$. The congruence kernel $\mathbf{C}(A, \mathcal{R})$ is the kernel of the canonical map from the profinite completion $\widehat{\Gamma}$ of Γ to $\rho(A^1(\widehat{\mathcal{R}}))$. Here $\widehat{\mathcal{R}}$ stands for the profinite completion of the ring $\mathcal{R}$. The congruence subgroup problem (in general, for arithmetic groups) asks whether the congruence kernel is trivial. If the congruence kernel is finite, i.e. the congruence subgroup problem has almost positive solution, one says that Γ has a congruence subgroup property. It is proved by Lubotzky [19] that the congruence kernels $\mathbf{C}(A, \mathcal{R})$ of the arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ are infinite. But for some of the arithmetic Kleinian groups, like for example for the $\mathrm{SL}_2(\mathcal{O})$ ($\mathcal{O}$ the ring of integers in some imaginary quadratic number field), some further information has been obtained. For these arithmetic Kleinian groups there are subgroups of finite index which map onto non-abelian free groups. For the $\mathrm{SL}_2(\mathcal{O})$ this was proved in [8], many more cases are treated in [20]. The fact that Γ has a subgroup of finite index which maps onto non-abelian free group, leads to an embedding of the free profinite group on countably many generators $\widehat{F}_\omega$ into the corresponding congruence kernel (see [18]).

Led by a result of Melnikov [26] in the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ we ask:

Question 3. *Is the congruence kernel of an arithmetic Kleinian group isomorphic to $\widehat{F}_\omega$? Or more vaguely, what can be said about the congruence kernel in this case?*

Of course the answer to question 3 is negative if the cohomological dimension of the congruence kernel is not one. But it follows from Theorem 5.13 that it is 1 or 2. We shall describe in the following an interesting connection between question 3 and certain cohomological problems.

In [17] it is proved that if Γ is a lattice in $\mathrm{SL}_2(\mathbb{C})$, then for any chain of normal subgroups Γ_i of finite index of Γ with trivial intersection the numbers

$$\frac{\dim H^1(\Gamma_i, \mathbb{Q})}{[\Gamma : \Gamma_i]}$$

tend to zero when i tends to infinity (this means that $\beta_1^{(2)}(\Gamma) = 0$). Let us formulate the following analoguous problem for the dimensions of the first cohomology groups over $\mathbb{F}_p$.

Question 4. *Let Γ be an arithmetic Kleinian group and p a prime number. Do the numbers*

$$\frac{\dim H^1(\Gamma_i, \mathbb{F}_p)}{[\Gamma : \Gamma_i]}$$

tend to zero when i tends to infinity for any chain of normal p-power index subgroups Γ_i of Γ with trivial intersection.

We shall now describe a connection beween the problems posed in Questions 3 and 4. Our methods are applicable since the discrete subgroups of $\mathrm{SL}_2(\mathbb{C})$ have a very restrictive structure. If Γ is a finitely generated torsion free discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$, then the deficiency of Γ is 0 or 1 depending whether the quotient space $\mathrm{SL}_2(\mathbb{C})/\Gamma$ is compact or not. Moreover, if $\mathrm{SL}_2(\mathbb{C})/\Gamma$ is compact, then Γ is Poincaré duality groups of dimension 3 and if $\mathrm{SL}_2(\mathbb{C})/\Gamma$ is not compact then Γ is of cohomological dimension 2. First using Theorem 5.6, we prove that the arithmetic lattices are good. For the Bianchi groups it was shown in [7].

Theorem 5.13. *Let Γ be an arithmetic Kleinian group in $\mathrm{SL}_2(\mathbb{C})$. Then Γ is good and pro-p good for every prime p.*

Proof. Notice that for any prime p an arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ is a virtually residually-p group. Now the result follows from Theorem 5.6, because $\beta_1^{(2)}(\Gamma) = 0$ ([17]). □

Remark 5.14. Using a recent deep result of D. Wise [35] it is also possible to prove goodness for all fundamental groups of 3-manifolds. We give a sketch of the argument.

Let $\Gamma = \pi_1(M)$ be the fundamental group of a 3-manifold M (possibly with a boundary). Without loss of generality we may assume that the boundary of M is incompressible and M is irreducible, because a free product of good groups is also good. From the Geometrization Conjecture proved by Perelman, it follows that we can cut M along a finite collection of incompressible tori so that the resulting pieces $\{M_i\}$ are geometric. Wilton and Zalesski [34] proved that $\pi_1(M)$ is good if $\pi_1(M_i)$ are good. Also they proved that $\pi_1(M_i)$ is good if M_i is a Seifert manifold and it is clear that $\pi_1(M_i)$ is good if M_i is a solvmanifold. Thus, we have to show that $\pi_1(M_i)$ is good when M_i is hyperbolic. Hence assume that M is hyperbolic. In the case when M is not virtually Haken the goodness of $\pi_1(M)$ was first observed by Reznikov [29] (see also [14, 33]). If M is virtually Haken, then by [35] and [1], M is virtually fibered over circle and so $\pi_1(M)$ is also good.

Now we are ready to prove our main result.

Theorem 5.15. *Let Γ be an arithmetic Kleinian group in $\mathrm{SL}_2(\mathbb{C})$ and p be a prime number. If the answer to the Question 4 is positive for all subgroups of Γ of finite index, then the p-cohomological dimension of the congruence kernel of Γ is 1.*

Proof. By Theorem 5.13, Γ is p-good and so $G = \widehat{\Gamma}$ is either a Poincaré duality profinite group at p of dimension 3 or has p-deficiency 1.

Let $\mathbf{C}$ denote the congruence kernel corresponding to Γ. Suppose that the p-cohomological dimension of $\mathbf{C}$ is not 1. Then by Proposition 2.1 there exists an open subgroup C_0 of $\mathbf{C}$ with $H^2(C_0, \mathbb{F}_p) \neq 0$. Further, there exists a subgroup Γ_0 of finite index in Γ which congruence kernel is equal to C_0. Hence without loss of generality we may assume that $\mathbf{C} = C_0$ and $\Gamma = \Gamma_0$.

Note that

$$H^2(\mathbf{C}, \mathbb{F}_p) = \varinjlim_{\mathbf{C} \leq H \lhd_o G} H^2(H, \mathbb{F}_p).$$

Hence there exist a subgroup $\mathbf{C} \leq H \lhd_o G$ such that the image of the restriction map $H^2(H, \mathbb{F}_p) \to H^2(\mathbf{C}, \mathbb{F}_p)$ is not trivial. Note that $H \cap \Gamma$ is a congruence subgroup of Γ and $H \cong \widehat{H \cap \Gamma}$. Hence without loss of generality we may also assume that $H = G$.

Since Γ is an arithmetic Kleinian subgroup of $\mathrm{SL}_2(\mathbb{C})$ there exists a normal subgroup N of G such that $C \leq N$, G/N is a p-adic analytic group and Γ is embedded in G/N (see, for example, Lemma 4 of Window 9 in [22]). Replacing Γ by one of its congruence subgroups we may also assume that G/N is a pro-p group.

Note that the image of compositions of two restriction maps

$$H^2(G, \mathbb{F}_p) \to H^2(N, \mathbb{F}_p) \to H^2(\mathbf{C}, \mathbb{F}_p)$$

is not zero. Hence $H^2(N, \mathbb{F}_p) \neq 0$. Thus, we may apply Theorem 3.4 and Theorem 4.6 and obtain that there exists $c > 0$ such that for any open subgroup $N \leq H \leq_o G$

$$\dim H^1(\Gamma \cap H, \mathbb{F}_p) = d_p(H, N) \geq c[G : H] = c[\Gamma : (\Gamma \cap H)].$$

But this contradicts the assumption that the answer to Question 4 is positive for Γ. □

Acknowledgements. We thank Wilhelm Singhof for conversations on the subject.

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